Associative Posets

J. Kuperman A. Petrovich P. Sánchez Terraf

Universidad Nacional de Córdoba

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Summary

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Pogroupoids

Motivation

$$\begin{array}{ccc} \langle \mathcal{P}(X), \subseteq \rangle & \longleftrightarrow & \langle \mathcal{P}(X), \cap \rangle \\ a \subseteq b & \iff & a \cap b = a. \end{array}$$

$$\begin{array}{ccc} \langle A, \leqslant \rangle & \longleftrightarrow & \langle A, \cdot \rangle \\ a \leqslant b & \iff & a \cdot b = a. \end{array}$$

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Definition

- $\langle A, \cdot \rangle$ **pogroupoid** $\iff$ $x \cdot y = x$ is a partial order on A.
- A **posemigroup** is an associative pogroupoid.

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Definition

- $\langle A, \cdot \rangle$ **pogroupoid** $\iff$ $x \cdot y = x$ is a partial order on A.
- A **posemigroup** is an associative pogroupoid.
- An **associative poset** is a poset which admits posemigroup structure.

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Posemigroups

Lemma

For every posemigroup,

1. $a \cdot b \leqslant b$; *in particular, if* b *is a minimal element,* $a \cdot b = b$.
2. $a \cdot b \cdot a = b \cdot a$.
3. $c \leqslant a, b \Rightarrow c \leqslant a \cdot b, b \cdot a$.

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Proof.

1. $(a \cdot b) \cdot b = a \cdot (b \cdot b) = a \cdot b$.
2. By item 1, $a \cdot (b \cdot a) \leqslant b \cdot a \qquad b \cdot a = b \cdot (a \cdot b \cdot a) \leqslant a \cdot b \cdot a$.
   By antisymmetry, $a \cdot b \cdot a = b \cdot a$.
3. $c \cdot (a \cdot b) = (c \cdot a) \cdot b = c \cdot b = c$. The same holds for $b \cdot a$.

Posemigroups

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Posemigroups

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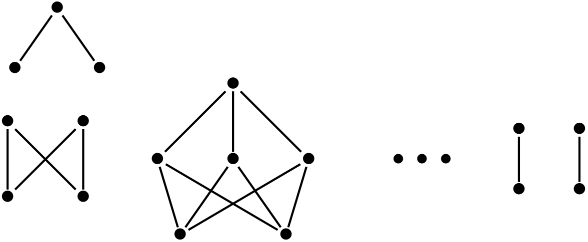
Corollary

For every posemigroup, if b *is a minimal element, then so is* $b \cdot a$*, for every* a*.*

Corollary

Let P *be an associative poset. For every* $a, b \in P$*, there exist* $a' \leqslant a$ *and* $b' \leqslant b$ *above every lower bound of* $\{a, b\}$ *such that* $a'\downarrow \approx b'\downarrow$

Examples (I)

- Every meet-semilattice.
- 

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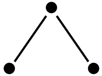
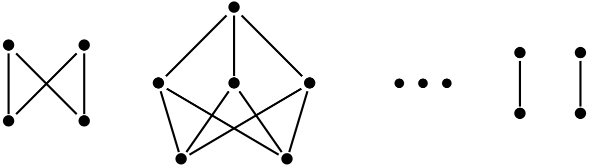
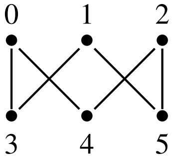
- 



The 6-crown



Counterexample

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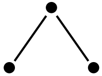
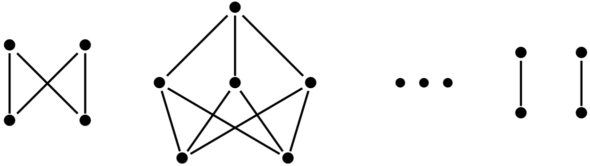
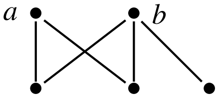
- 



The Hummingbird



- $a' = a$
- $b' = b$
- $a \downarrow \not\approx b \downarrow$

Axiomatization

Lemma

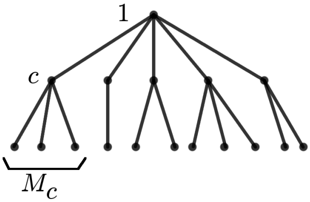
The class of associative posets is not a first-order class.

Proof.

$(\mathbb{R} \sqcup \mathbb{R}, \leqslant)$ is associative. $(\mathbb{Q} \sqcup \mathbb{R}, \leqslant) \preceq (\mathbb{R} \sqcup \mathbb{R}, \leqslant)$. But $(\mathbb{Q} \sqcup \mathbb{R}, \leqslant)$ is not associative as we can choose any $a \in \mathbb{Q}$, $b \in \mathbb{R}$ and there do not exist $a' \leqslant a$ and $b' \leqslant b$ such that $a' \downarrow \approx b' \downarrow$. $\square$

Well-founded trees

- $T = \{1\} \sqcup C \sqcup \bigsqcup_{c \in C} M_c$.
- $\nexists c, z.\ c \in C \wedge c < z < 1$,
- M_c are minimal elements under c.



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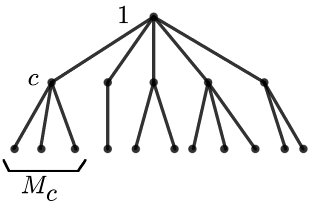

Theorem

The following are equivalent:

1. *Every well-founded tree is associative*
2. *Every tree of height* 3 *is associative.*
3. *The axiom of choice.*

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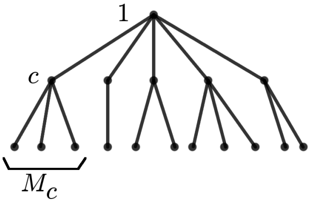

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$1 \Rightarrow 2$ is trivial.

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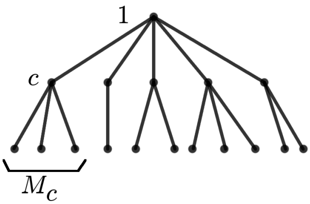

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Proof.

$(2 \Rightarrow 3)$ Fix a minimal a, the function $c \mapsto a \cdot c$ is a choice function.

Well-founded trees

Theorem

The following are equivalent:

1. *Every well-founded tree is associative*
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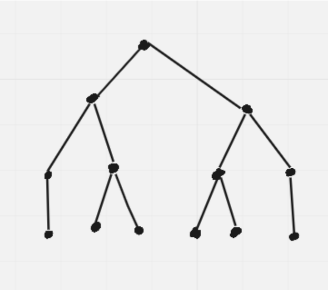
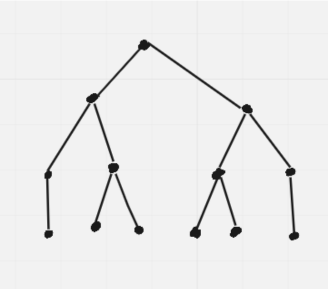
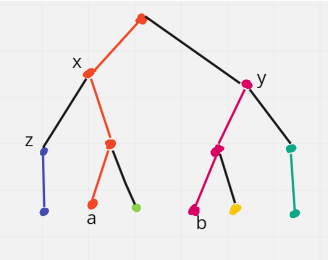
Proof.

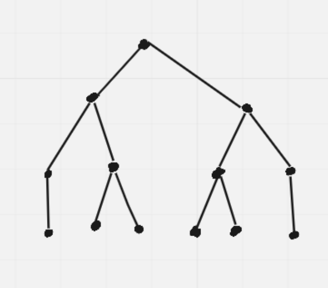
$(3 \Rightarrow 1)$ Given a well-founded tree T we can define an associative structure over T in the following way. We define a function F recursively. For every minimal element x, $F(x) = x$. If x is not minimal, then $F(x) = F(ch(M_x))$ where $M_x := \{y \in T | y \prec x\}$. We now define $\cdot$: $x \cdot y = min\{x, y\}$ if it exists and $x \cdot y = F(y)$ if they are uncomparable. Now $(T, \cdot)$ is a posemigroup.

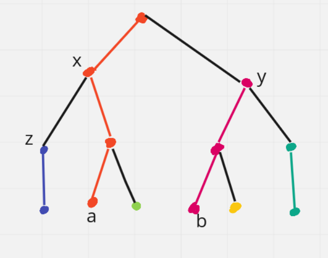






We use the axiom of choice to decompose the tree in disjoint, convex chains.





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- $x \cdot z = z \cdot x = z$
- $x \cdot y = b$
- $y \cdot x = a$

Well-Founded Trees

Theorem

Let P be a poset. If there exists a well-founded tree T such that there exists a surjective homomorphism $f : P \to T$ such that:

- $f(x) < f(y) \Rightarrow x < y$
- *For every $x \in T$, $f^{-1}(x)$ is an associative poset.*
- *If $x \in T$ is a minimal element, then $f^{-1}(x)$ has a minimum.*

then P is associative.

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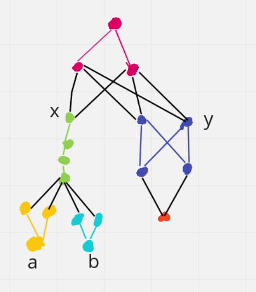

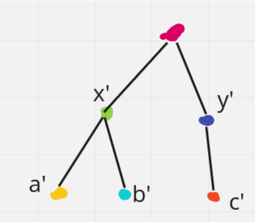

Direct products

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Let $x = x_1 +_c x_2$ be the conjunction of the following conditions

comm $x \cdot x_1 = x_1 \cdot x$ and $x \cdot x_2 = x_2 \cdot x$.

dist $x = (x \cdot x_1) \vee (x \cdot x_2)$.

p₁ $x_1 = (x \cdot x_1) \vee (c \cdot x_1)$.

p₂ $x_2 = (x \cdot x_2) \vee (c \cdot x_2)$.

prod $x_1 \cdot x_2 = x \cdot c$.

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Assume I_1, I_2 are subposemigroups of A. We will say that A is the c-direct sum of I_1 and I_2 (notation: $A = I_1 \oplus_c I_2$) if and only if the following hold:

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Mod1 For all $x, y \in A$, $x_1 \in I_1$ and $x_2 \in I_2$, if $x \cdot c \leqslant x_1 \cdot x_2$ then

$$\big((x \cdot x_1) \vee (x \cdot x_2)\big) \cdot y = (x \cdot x_1 \cdot y) \vee (x \cdot x_2 \cdot y),$$
$$y \cdot \big((x \cdot x_1) \vee (x \cdot x_2)\big) = (y \cdot x \cdot x_1) \vee (y \cdot x \cdot x_2).$$

Mod2 For all $x, y \in A$, $x_1 \in I_1$ and $x_2 \in I_2$, if $x \geq x_1 \cdot x_2$ then

$$\big((x \cdot x_i) \vee (c \cdot x_i)\big) \cdot y = (x \cdot x_i \cdot y) \vee (c \cdot x_i \cdot y),$$
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for $i = 1, 2.$.

Abs For all $x_1, y_1 \in I_1$ and $z_2 \in I_2$, we have: $x_1 \vee (y_1 \cdot z_2) = x_1 \vee (y_1 \cdot c)$ (and interchanging I_1 and I_2).

Direct products

Lemma

Assume $A = I_1 \oplus_c I_2$. *Then* $x = x_1 +_c x_2$ *defines an isomorphism* $\langle x_1, x_2 \rangle \overset{\varphi}{\mapsto} x$ *between* $I_1 \times I_2$ *and* A.

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Theorem

Let A *be a posemigroup and* $c \in A$ *a commuting element. The mappings*

$$\begin{array}{ccc} \langle \theta, \delta \rangle & \overset{\mathsf{I}}{\longmapsto} & \langle I_\theta, I_\delta \rangle \\ \langle \ker \pi_2, \ker \pi_1 \rangle & \overset{\mathsf{K}}{\longleftarrow\!\!|} & \langle I_1, I_2 \rangle \end{array}$$

where $I_\theta := \{a \in A : a \, \theta \, c\}$, *are mutually inverse maps defined between the direct product representations of* A *and the set of pairs of subposemigroups* I_1, I_2 *of* A *such that* $A = I_1 \oplus_c I_2$.

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Future Work

What other posets can we guarantee to be associative?

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Take PS and AP the categories of posemigroups and associative posets respectively. We have the forgetful functor $F : PS \to AP$ which maps a posemigroup to its asociated associative poset.

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Problem 1 Determine a left adjoint for F.

Problem 2 Determine the *associative hull* of an arbitrary poset.

Thank you!