Dualities and logical aspects of Baire functions

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It includes joint works with Antonio Di Nola, Giacomo Lenzi and Ioana Leuştean

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Summary

1. preliminary notions and background algebraic results
2. characterization of Baire functions as free objects in an infinitary variety
3. dualities for such infinitary variety

Di Nola A., Lapenta S., Leuştean I., *An infinitary logic for basically disconnected compact Hausdorff spaces*, Journal of Logic and Computation, 28(6) (2018) 1275–1292.

Di Nola A., Lapenta S., Lenzi G., *Dualities and algebraic geometry of Baire functions in Non-classical Logic*, accepted for publication in the Journal of Logic and Computation, 2021.

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A is an MV-algebra iff $A \in HSP([0, 1])$

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Take (G, u) ℓu-group, u is strong order unit:

$$\text{for any } x \text{ exists } n \text{ such that } x \vee (-x) \leq nu$$

take $x, y \in G$, define

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Categorical equivalence. Mundici, 1986

For any MV-algebra A there exists (G, u) such that

$$A \simeq [0, u]_G$$

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Some examples:

- $[0,1]^X$ for any nonempty X;
- $[0,1] \otimes A$, for any A semisimple MV-algebra;
- PWL-functions.

Functional representation of the free Riesz MV-algebra

$f : [0,1]^n \to [0,1]$ is a $\mathrm{PWL}_u(\mathbb{R})$ function if it is continuous and there is a finite set of affine functions $p_1, \ldots, p_k : \mathbb{R}^n \to \mathbb{R}$ with coefficients in $\mathbb{R}$ such that for any $\mathrm{a} \in [0,1]^n$ there exists $i \in \{1, \ldots, k\}$ with $f(\mathrm{a}) = p_i(\mathrm{a})$.

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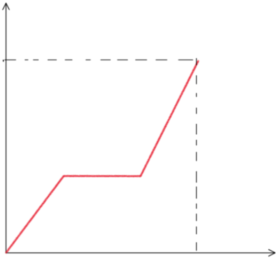

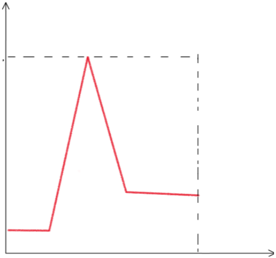

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If A is a semisimple Riesz MV-algebra, that is, $A \subseteq C(X)$, we define

$$\| \cdot \|_u : A \to [0,1], \quad \|x\|_u = \min\{r \in [0,1] \mid x \leq r1\}$$

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A Riesz MV-algebra is norm-complete if it is a complete normed space wrt to $\| \cdot \|_u$.

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Our starting point was the question:
can we axiomatize norm-complete Riesz MV-algebras and get an algebraic dual for **KHausd**?

We approached the problem adding order limits of functions *to the language*,

$(s_m)_{m\in\mathbb{N}}$ converges in order to s if there exists $(r_m)_{m\in\mathbb{N}}$ such that $\bigwedge_m r_m = 0$ and $|\, s - s_m \,| \leq r_m$ for any $m \in \mathbb{N}$

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Thus, our initial problem had already being solved!

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One example: continuous functions

When X is basically disconnected (the closure of any open F_σ is open), $C(X) = \{f : X \to [0,1] \mid f \text{ is continuous}\}$ is σ-complete.

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A Riesz tribe over X is a Riesz MV-algebra of $[0,1]$-valued functions over X that is closed under pointwise countable suprema.

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Thus, any set of bounded measurable functions on a space of finite measure is a Riesz tribe.

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Any σ-complete R Riesz MV-algebra is a homomorphic image of a Riesz tribe $\mathcal{T}$. The homomorphism preserves countable suprema.

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By standard results of UA, the free κ generated algebra is also the algebra of term functions build upon κ variables in the language of $\oplus, \neg, 0, \bigvee, \{\alpha\}_{\alpha \in [0,1]}$.

Free algebras in RMV_σ

Theorem (Di Nola, Lapenta, Lenzi, 2021)

The free κ-generated algebra in RMV_σ is the the algebra

$$\mathit{Baire}([0,1]^\kappa) = \{a : [0,1]^\kappa \to [0,1] \mid a \text{ is Baire-measurable}\},$$

generated by the projections π_i, $i \in \kappa$.

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- A Baire subset of X is an element of the σ-algebra generated by the zerosets of continuous functions $f : X \to [0,1]$.
- A function $a : X \to Y$ is Baire measurable if preimages of Baire sets of Y are Baire sets of X.

Free algebras in RMV_σ: an idea of the proof

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generated by $\pi_1, \ldots, \pi_n$.

- A Borel set in (X, τ) is an element of the σ-algebra generated by the open set of τ.
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- A function $a : X \to Y$ is Borel measurable if preimages of Borel sets of Y are Borel sets of X.
- In general $\mathcal{B}a(X) \subset \mathcal{B}o(X)$.
  When $\kappa \leq \omega$, $\mathcal{B}a([0,1]^\kappa) = \mathcal{B}o([0,1]^\kappa)$.

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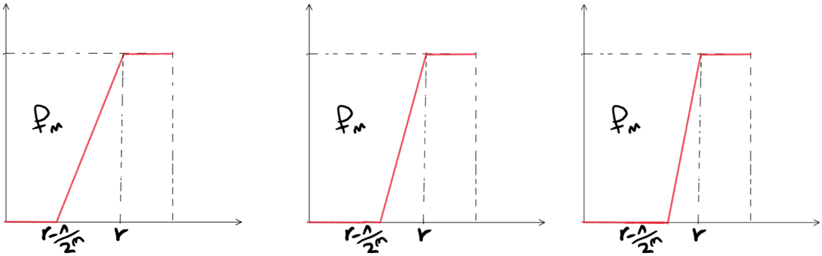

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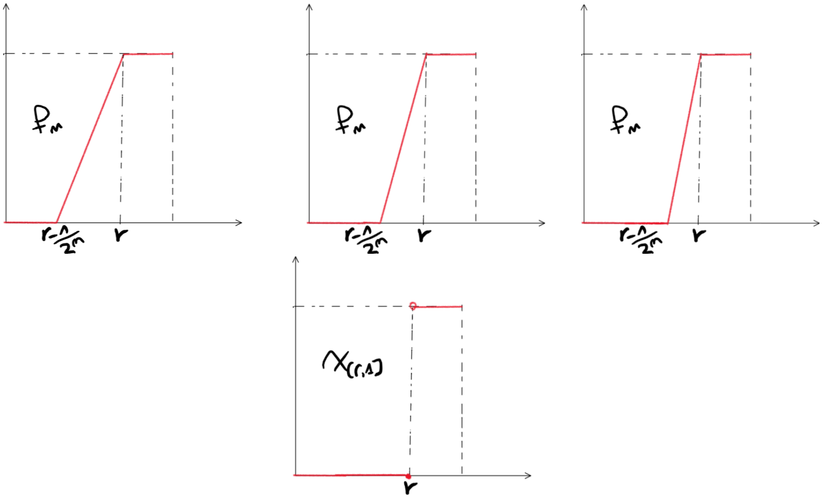

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Caramello O., Marra V., Spada L., *General affine adjunctions, Nullstellensätze, and dualities*, Journal of Pure and Applied Algebra. 225(1), 2021.

Operators on hypercubes and formulas

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The operators $\mathbb{I} - \mathbb{V}$ form a Galois connection

$$S \subseteq \mathbb{V}(J) \Leftrightarrow J \subseteq \mathbb{I}(S)$$

From a connection to a duality

- $\mathsf{RMV}^{\mathsf{p}}_{\sigma}$:
  - objects: pairs $(\text{Baire}([0,1]^X), I)$, where I is an ideal in the free algebra $\text{Baire}([0,1]^X)$.
  - morphism: $h : (\text{Baire}([0,1]^X), I) \to (\text{Baire}([0,1]^Y), J)$ is induced by a unique homomorphism $h^p : \text{Baire}([0,1]^X) \to \text{Baire}([0,1]^Y)$ such that $h^p(I) \subseteq J$.

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- IRL:
  - objects: subsets of hypercubes of type $[0,1]^\kappa$ which are intersection of Baire subsets,
  - arrows: tuple of term functions $\eta = (\eta_y)_{y \in Y} : S \subseteq [0,1]^X \to T \subseteq [0,1]^Y$, with $\eta_y \in \text{Baire}([0,1]^X)$.

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  - for $S \subseteq [0,1]^X$, $\mathcal{I}(S) = (\mathrm{Baire}([0,1]^X), \mathbb{I}(S))$,
  - for $\eta = (\eta_y)_{y \in Y} \colon S \subseteq [0,1]^X \to T \subseteq [0,1]^Y$, $\mathcal{I}(\eta) \colon \mathcal{I}(T) \to \mathcal{I}(S)$ is the map given by $f \in \mathrm{Baire}([0,1]^Y) \mapsto f \circ \eta \in \mathrm{Baire}([0,1]^X)$.

$\mathbf{RMV}^{\mathsf{p}}_{\sigma}$ and $\mathbf{IRL}$ are adjoint categories

we get this for free from

Marra, Spada, Caramello, 2021.

- The functor $\mathcal{V} : \mathbf{RMV}^{\mathsf{p}}_{\sigma} \to \mathbf{Hyper}$, defined by
  - $\mathcal{V}\left(\mathrm{Baire}([0,1]^X), J\right) = \mathbb{V}(J)$;
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Which is the most general duality we can get?

Definition

$A \in \mathsf{RMV}_\sigma$ is called σ-semisimple iff

$$\bigcap\{M \subseteq A \mid M \in Max(A) \cap Id_\sigma(A)\} = \{0\}$$

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$$\mathcal{M}_\sigma = Max(\mathrm{Baire}([0,1]^X)) \cap Id_\sigma(\mathrm{Baire}([0,1]^X))$$

- $\mathbb{I}(\mathbb{V}(J))$ is the intersection of all ideals in $\mathcal{M}_\sigma$ containing J;
- $J = \mathbb{I}(\mathbb{V}(J))$ if, and only if, J is the intersection of the elements of $\mathcal{M}_\sigma$ that contain it.

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This follows from the fact that $\mathcal{M}_\sigma \leftrightarrow [0,1]^X$, via $x \mapsto \mathbb{I}(x)$.

More results on σ-semisimple algebras

An algebra $\mathrm{Baire}([0,1]^X)/_J \in \mathsf{RMV}_\sigma$ is σ-semisimple if, and only if, J is the intersection of the MV-maximal σ-ideals of $\mathrm{Baire}([0,1]^X)$ that contain J

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Any σ-semisimple algebra is (isomorphic to) a Riesz tribe!
$\text{Baire}([0,1]^X)/_J$, σ-semisimple, is isomorphic to

$$\text{Baire}([0,1]^X)\mid_{\mathbb{V}(I)} = \{g \in [0,1]^{\mathbb{V}(I)} \mid g = f|_{\mathbb{V}(I)} \text{ for some } f \in \text{Baire}([0,1]^X)\}$$

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An algebra $A \in \mathsf{RMV}_\sigma$ is σ-semisimple if, and only if, $A \in ISP([0,1])$.

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The duality further restricts

- Finitely presented algebras of RMV_σ correspond to Baire subsets of finite-dimensional hypercubes.
- Countably presented algebras of RMV_σ correspond to Baire subsets of the Hilbert cube.

$A \simeq \mathrm{Baire}([0,1]^X)/_I$ is finitely presented if X is finite and I principal. It is countably presented if X is countable and I principal.

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Let $\{A_i\}_{i\in I}$ be a collection of σ-semisimple algebras in RMV_σ, I arbitrary.

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Also assume that the sets X_i are pairwise disjoint.

Then the free product $\bigoplus_i A_i$ exists in RMV_σ and

$$\bigoplus_i A_i = {}^{\textstyle \mathrm{Baire}([0,1]^{\bigcup_i X_i})} \big/ {}_{J},$$

with $J = \langle \bigcup_i I_i \rangle_\sigma$, the σ-ideal generated by $\bigcup_i I_i$.

Coproducts, an application

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Moreover, in

S. Lapenta, G. Lenzi, *Models, coproducts and exchangeability: notes on states on Baire function*, submitted for publication.

we used the coproduct and the duality with Baire sets to define a MV-algebraic probability measure on the coproduct using a product measure the corresponding spaces $\mathbb{V}(I_i)$.

More on RMV_σ, via continuous functions

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From well known results, for any $R \in \mathrm{RMV}_\sigma$ there exists a basically disconnected compact Hausdorff space X space such that $R \simeq C(X)$.

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Can this correspondence be made into a duality?

Of course, we aim at restricting the well known Gelfand-Naimark-Yosida-Krein-Kakutani duality:

$$X \in \mathrm{KH} \mapsto C(X)\,\text{M-space}, \quad f : X \to Y \mapsto \tilde{f} : C(Y) \to C(Y), \tilde{f}(g) = f \circ g.$$

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Theorem

Let $f : X \to Y$ be a continuous function, and $\tilde{f} : C(Y) \to C(X)$ the dual arrow between Riesz MV-algebras. Then $\tilde{f}$ is a σ-homomorphism on the Boolean part of $C(Y)$ if, and only if, it is cozero-closed.

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via some easy computation, we get

$$\chi_{f^{-1}(\overline{U})} = \bigvee_n (\chi_{C_n} \circ f) = \bigvee_n \chi_{f^{-1}(C_n)} = \chi_{\overline{\bigcup_n f^{-1}(C_n)}} = \chi_{\overline{f^{-1}(\bigcup_n C_n)}} = \chi_{\overline{f^{-1}(U)}}.$$

From the Boolean center to the algebra

If $\tilde{f}$ is a σ-homomorphism on $\mathcal{C}(Y)^b$, then it is a σ-homomorphism on $C(Y)$.

Crucial point: every function of $C(Y)$ is the supremum of all rational multiples of Boolean elements below it.

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Theorem

The algebraic category RMV_σ*, whose objects are* σ*-complete Riesz MV-algebras, is*

- *dual to the category* BDKH *whose objects are basically disconnected, compact, Hausdorff spaces and whose morphisms are continuous and cozero-closed functions.*
- *equivalent to the algebraic category* BA_σ*, of* σ*-complete Boolean algebras.*

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From Sikorski's Boolean algebras

The algebraic category $\mathbf{BA}_\sigma$, of σ-complete Boolean algebras, is dual to the category **BDKH**, whose objects are basically disconnected, compact, Hausdorff spaces and whose morphisms are σ-continuous functions.

A straightforward consequence

A second duality (without details)

Using Gelfand duality, RMV_σ is also dual to a subcategory of C^*-algebras whose objects are called Rickart C^*-algebras and whose arrows preserves countable joins of projections.

(Di Nola, Lapenta, Lenzi, JLC 2021)

Finally... where do we go from here?

- We have a well behaved infinitary variety of algebras;
- There is a conservative extension of Łukasiewicz logic that has exactly these algebras as model. Such logic introduced in [Di Nola, Lapenta, Leuştean, 2018] and it is standard complete w.t.r. $[0, 1]$.

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- this variety has in its language (almost) all operations needed to discuss probability;
- Indeed, this work is an intermediate step towards a more ambitious metamathematics of probability

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A κ-dimensional observable on A is any σ-homomorphism of Riesz MV-algebras

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A. Di Nola, A. Dvurečenskij, S. Lapenta, *An approach to stochastic processes via non-classical logic*, accepted for publication in the Annals of Pure and Applied Logic.

Thank you!