On the Riesz structures of a lattice ordered abelian group

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The origin of this research

In his lattice theory 1940 book, Birkhoff says erroneously that every map between two Riesz spaces which is a homomorphism of lattice ordered abelian groups, respects also the Riesz space structure. A similar error occurs in the seminal Di Nola-Leustean 2014 paper on Riesz MV-algebras.
The error is corrected and counterexamples are given in Conrad's 1975 Australian paper and, later, in Schechter's 1997 handbook.
The research presented in this talk elaborates on Conrad's paper.

l-groups

By l-group we intend an abelian group with a lattice order.

It is required that if $x \leq y$ then $x + z \leq y + z$.

A strong unit of an l-group G is an element u of G such that for every $x \in G$ there is n with $x \leq nu$.

The absolute value of an element $x \in G$ is $|x| = x \vee -x$.

x, y are othogonal if $|x| \wedge |y| = 0$.

Some relations

$x << y$ (x is well below y) if $n|x| \leq |y|$ for every $n \in N$

yDx (y dominates x) if $|x| \leq n|y|$ for some $n \in N$

xEy (x, y are Archimedean equivalent) if xDy and yDx.

Riesz spaces

By Riesz space we intend a vector space (over the real field R) with a lattice order.

The scalar multiplication on a Riesz space is denoted by $\rho : R \times G \to G$.

It is required that if $r, g \geq 0$ then $\rho(r, g) \geq 0$.

From l-groups to Riesz spaces

Note that every Riesz space gives an l-group if we forget the vector space structure.

However, an l-group can be expanded to several Riesz spaces (or none). This has been observed in Conrad's 1975 paper even for totally ordered abelian groups.

The main result

The previously mentioned paper of Conrad asks, at page 333, whether two Riesz structures on the same l-group are necessarily isomorphic.
In the present talk we answer in the negative by providing a counterexample.

Uniformities

A uniformity in an l-group G is a set U of subsets of $G \times G$ such that:

U1 If $D \in U$ then $\{(g,g)|g \in G\} \subseteq D$;

U2 if $D_1, D_2 \in U$ then $D_1 \cap D_2 \in U$;

U3 if $D \in U$ then $E \circ E \subseteq D$ for some $E \in U$;

U4 If $D \in U$ then $E^{-1} \subseteq D$ for some $E \in U$;

U5 if $D \in U$ and $D \subseteq E \subseteq G \times G$ then $E \in U$.

U-convergence and U-Cauchyness

Let U be a uniformity on an l-group G.

A sequence $x_\alpha, \alpha < \theta$ U-converges to x of for every $D \in U$ there is $\theta_0 < \theta$ such that for every α from θ_0 to θ we have $(x, x_\alpha) \in D$.

The sequence is U-Cauchy if for every $D \in U$ there is $\theta_0 < \theta$ such that for every α, β from θ_0 to θ we have $(x_\alpha, x_\beta) \in D$.

Partial isomorphism

A partial isomorphism between two structures S, S' is a nonempty set I of isomorphisms of substructures of S onto substructures of S' with the following back and forth property:

(1) for every $f \in I$ and $s \in S$ there is $g \in I$ with $f \subseteq g$ and $s \in dom(g)$;

(2) for every $f \in I$ and $s' \in S'$ there is $g \in I$ such that $f \subseteq g$ and $s' \in ran(g)$.

On totally ordered abelian groups

Theorem: Any two Riesz structures on the same totally ordered abelian group are partially isomorphic.

Proof: A partial isomorphism is the set of all isomorphisms $f : W \to Z$ where W is a finite dimensional subspace of (G, ρ), Z is a finite dimensional subspace of (G, ρ'), $e_1 >> e_2 \ldots >> e_n$ is a basis of W and $e_i E f(e_i)$ for all i.

A corollary

Corollary: if the archimedean classes of a totally ordered abelian group G are wellordered, then all Riesz space structures of G are isomorphic.

Proof: The isomorphism can be built by induction.

This generalizes the well known fact that an archimedean l-group has at most one Riesz space structure.

Many Riesz structures

Theorem: for every cardinal k there is a totally ordered abelian group S (actually a field) with at least k Riesz space structures (all of them are isomorphic).

Proof sketch: let $\eta : R \to S$ be a strongly ω-homogeneous elementary extension of R of size greater than $max(k, 2^{\aleph_0})$. S has at least k infinitesimals. For every infinitesimal ϵ of S we have a field automorphism a_ϵ of S sending $\eta(e)$ to $\eta(e) + \epsilon$ $(e = 2.71828\ldots)$. Now consider

$$\rho_\epsilon(r, x) = a_\epsilon(\eta(r)x).$$

The main theorem

Theorem: there is an l-group G with strong unit having two nonisomorphic Riesz structures.

Proof sketch, beginning

We build an l-group with two asymmetric Riesz structures.

The field K

Let R^a the field of the real algebraic numbers. By Frayne's Theorem there is an embedding $j_1 : R \to^* (R^a)$, where the latter is an ultrapower $(R^a)^I/\mathcal{U}$. Let

$$K = R^I/\mathcal{U}.$$

Let $j_2 : R \to K = R^I/\mathcal{U}$ be the diagonal embedding.

We have in K two Riesz structures $\rho_i(r, x) = j_i(r)x$ $(i = 1, 2)$. Actually, j_1 and j_2 behave quite differently.

The ring K_0

Let K_0 be the set of all finite sums $\Sigma_{k=1}^{n} j_1(r_k) j_2(s_k)$, where $r_k, s_k \in R$.

Note that K_0 is a Riesz subspace both of (K, ρ_1) and (K, ρ_2).

The group G

Let G be set of all the sequences $g : \aleph_1 \to K_0$ whose components are finitely generated in the vector space (K_0, ρ_1) and are bounded in absolute value.

Note that the constant sequence 1 is a strong unit of G.

Atoms and uniformities

We call atom of G an element $a > 0$ such that whenever $0 < g, h \leq a$, we have that g, h are not orthogonal.

Intuitively, an atom is a sequence with only one nonzero component.

We denote by $A^\perp$ the space of all elements orthogonal to all elements of A.

The spaces $A^\perp$, with A finite set of atoms, generate a uniformity $U_\perp$.

Key idea

Lemma: Let Γ be the set of $\aleph_1$-sequences of elements $j_1(r)$, where r may vary but $|r|$ is bounded.
Then Γ generates the vector space (G, ρ_2).

Our idea is to find some properties of Γ which no generating set of (G, ρ_1) may have.

Properties of Γ

P0 (notation different from the paper) Γ is a generating set for (G, ρ_2)

P1 For every $g \in \Gamma$ and every countable set A of atoms, Γ contains the restriction $g|_A$ of g, such that $g|_A - g \in A^\perp$ and $g|_A \perp b$ for every atom $b \in A^\perp$;

P2 Every bounded $U_\perp$-Cauchy ordinal sequence of elements of Γ has a $U_\perp$-limit in Γ;

P3 Γ is a vector space over the rationals (in particular, it is a group).

Proof sketch, end

The proof of the main theorem is concluded as follows. Suppose that a set Δ of generators of (G, ρ_1) exists verifying P0-P3 (up to exchanging ρ_1 and ρ_2). Then we draw a contradiction, by constructing a suitable Cauchy-sequence g_n in Δ whose limit is not in G, since it has infinite dimension in the vector space (K_0, ρ_1).

Construction of g_n

Let us choose a sequence (t_n) of real transcendental numbers linearly independent over the subfield R^a of R. The sequence $(j_2(t_n))$ is linearly independent in the vector space (K_0, ρ_1). Let $\sigma_n \subseteq \Delta$ be a countable generating set in (G, ρ_1) for the constant $\aleph_1$-sequences $j_2(t_n)$.

Construction of g_n, continued

Lemma: there is a sequence (g_n) of short elements of Δ, one of countable ordinals (α_n) and one of nonnegative integers (k_n) so that:

- $\alpha_{n+1} > \alpha_n$,
- $g_n(\alpha_n) = \sigma_{k_n}(\alpha_n)$
- g_{n+1} agrees with g_n up to the length of g_n
- the set $\{\sigma_{k_1}(\alpha_1), \ldots, \sigma_{k_n}(\alpha_n)\}$ is linearly independent in the vector space (K_0, ρ_1).

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Many examples

Corollary: There is a proper class of l-groups with two nonisomorphic Riesz structures.

This follows by replacing $\aleph_1$ with any uncountable regular cardinal.

Applications to MV-algebras

The relation between l-groups and MV-algebras has been clarified by the results of Mundici in 1986 and later developments.
Here we take advantages of this relation.

MV-algebra axioms

$A = (X, \oplus, \neg, 0)$

$(X, \oplus, 0)$ is a commutative monoid

$\neg\neg x = x$

$x \oplus \neg 0 = \neg 0$

$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

auxiliary notation $1 = \neg 0$, $x \ominus y = \neg(\neg x \oplus y)$

The MV-algebra $[0, 1]$

$$x \oplus y = min(x + y, 1)$$

$$\neg x = 1 - x$$

Riesz MV-algebra axioms

(A, ρ) where A is an MV-algebra and $\rho : [0, 1] \times A \to A$

$\rho_r(x \ominus y) = \rho_r(x) \ominus \rho_r(y)$

$\rho_{r \ominus s}(x) = \rho_r(x) \ominus \rho_s(x)$

$\rho_r(\rho_s(x)) = \rho_{rs}(x)$

$\rho_1(x) = x.$

Note the analogy with vector spaces.

Corollary: There is an MV-algebra with two nonisomorphic Riesz MV-algebra structures.

This follows by applying the well-known categorical equivalences between MV-algebras and l-groups with strong unit (Mundici) and between Riesz MV-algebras and Riesz spaces with strong unit (Di Nola-Leustean).

A totally ordered example

A totally ordered abelian group with two nonisomorphic Riesz structures has been described in a paper with Antonio Di Nola and Gaetano Vitale, accepted on Mathematica Slovaca.

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THANK YOU!