Modal Logics of Locally Compact Ordered Spaces

Joel Lucero-Bryan, Khalifa University

Joint work:
Guram Bezhanishvili, New Mexico State University
Nick Bezhanishvili, University of Amsterdam
Jan van Mill, University of Amsterdam

BLAST 2021, NMSU, Las Cruces, NM, USA
9–13 June 2021

Modal Logic

- In the early 1910's, C. I. Lewis introduced several modal systems while investigating 'material' vs. 'strict' implication.
- In this talk, modal logic extends classical propositional logic via the unary connectives $\Box$ and $\Diamond$.
- Because of the many different interpretations of $\Box$ and $\Diamond$, the relatively simple modal language is quite expressive. Some examples include:
  - necessity and possibility,
  - obligation and permission,
  - knowledge and consistency with a knowledge base,
  - belief and consistency with a belief system.

Possible World Semantics

- We can interpret the modal language on a state transition system or digraph; i.e., a pair (W, R) where $W \neq \varnothing$ is a set of worlds and $R \subseteq W \times W$ an accessibility relation.
    1. $\Box p$ is true at a world w provided p is true in all worlds accessible from w.
    2. $\Diamond p$ is true at a world w provided p is true in some world accessible from w.
- Say a formula is valid in (W, R) if it is true at every world under any interpretation of the basic propositions.

$(W, R) \models \Diamond\bot \leftrightarrow \bot$	always
$(W, R) \models \Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$	always
$(W, R) \models p \rightarrow \Diamond p$	R is reflexive
$(W, R) \models \Diamond\Diamond p \rightarrow \Diamond p$	R is transitive

Lewis's Fourth System: S4

- S4 is axiomatized by
  - $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$,
  - $p \to \Diamond p$,
  - $\Diamond\Diamond p \to \Diamond p$.
- S4 is complete with respect to its possible world semantics; that is, those (W, R) in which R is reflexive and transitive.
- Recall Kuratowski's closure axioms hold in any top. space:
  - $\mathrm{c}(A \cup B) = \mathrm{c}(A) \cup \mathrm{c}(B)$,
  - $A \subseteq \mathrm{c}(A)$,
  - $\mathrm{c}(\mathrm{c}(A)) \subseteq \mathrm{c}(A)$,
  - $\mathrm{c}(\varnothing) = \varnothing$.

Topological Semantics

Topological Semantics (an algebraic perspective)

In a topological space X, interpret $\Diamond$ as closure in X, and hence $\Box$ as interior.

Topological Semantics (local truth perspective)

1. $\Diamond p$ is true at a point x iff for each open neighborhood U_x of x, there is $y \in U_x$ such that p is true.
2. $\Box p$ is true at a point x iff there is an open neighborhood U_x of x such that p is true at y for each $y \in U_x$.

Remarks

- Topological semantics generalizes possible world semantics for logics containing S4.
- The role of U_x is analogous to the role of R successors of x.

Topological Completeness

- A formula φ is valid in X if it is true at every point under any interpretation of the basic propositions, written $X \vDash \varphi$.
- The logic $\mathsf{L}(X)$ of a space X is the set of formulas valid in X; that is, $\mathsf{L}(X) = \{\varphi \mid X \vDash \varphi\}$.
- $\mathsf{S4} \subseteq \mathsf{L}(X)$ for each space X.

Important Results Connecting Logic and Topology

1. **[M&T 1944]** The logic of the class of all topological spaces is S4.
2. **[M&T 1944]** Let X be a nonempty separable crowded (a.k.a. dense-in-itself) metrizable space. Then $\mathsf{L}(X) = \mathsf{S4}$.
3. **[R&S 1963]** By utilizing the Axiom of Choice, the separability hypothesis can be dropped.

Proving the McKinsey-Tarski Theorem

Let X be a nonempty crowded metrizable space.

The Original Algebraic Proof

Key Idea: embed each finite subdirectly irreducible closure algebra into the closure algebra $(\wp(X), \mathsf{c})$.

A Modern Proof Based on Possible World Semantics

Key Ideas:

1. Restrict to a 'nice' class of structures, known as quasi-trees, whose logic is S4.
   Quasi-trees replace the role of finite subdirectly irreducible closure algebras.
2. Exhibit a validity preserving mapping of X onto an arbitrary quasi-tree.
   These maps replace the aforementioned embeddings.

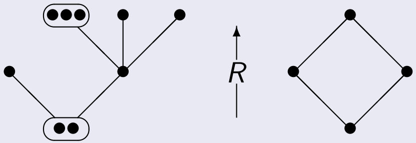
Quasi-trees and S4

Definition

(W, R) is a quasi-tree provided:

1. W is finite and R is reflexive and transitive,
2. $\exists r \in W\ \forall w \in W : rRw$, call r a root of $\mathfrak{F}$,
3. there is no 'downward forking'; meaning that $\forall w, v, u \in W : vRw \wedge uRw \rightarrow vRu \vee uRv$.

quasi-tree with 2 roots not a quasi-tree



Well-Known Result

S4 is the logic of the class of all quasi-trees.

Modern Proof: Key Ingredient I

Mapping Lemma

Let X be a nonempty crowded metrizable space, (W, R) a quasi-tree, and $R^{-1}(w) = \{v \in W \mid vRw\}$ for $w \in W$. There is an onto mapping $f : X \to W$ such that $\mathrm{c}(f^{-1}(w)) = f^{-1}(R^{-1}(w))$; i.e. f is an interior mapping.

Proof of MT (sketch)

- $\mathrm{S4} \subseteq \mathrm{L}(X)$.
- To see $\mathrm{L}(X) \subseteq \mathrm{S4}$:
  - Let $\varphi \notin \mathrm{S4}$.
  - Then φ is not valid in some quasi-tree (W, R).
  - **By the Mapping Lemma**, there is an interior mapping $f : X \to W$.
  - As interior mappings preserve validity, φ is not valid in X.
  - $\varphi \notin \mathrm{L}(X)$.

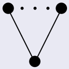
Modern Proof: Key Ingredient II

Partition Lemma

There is a partition $\{N, U_1, \ldots, U_n\}$ of X where

1. N is crowded, closed, and nowhere dense (N has empty interior) and
2. each U_i is open such that $\mathrm{c}(U_i) = U_i \cup N$.

Consequence

There is an interior mapping of X onto the quasi-tree  with n maximal points.

NOTE: Metrizable spaces can be thought of as a generalization of the the real line $\mathbb{R}$. As such we demonstrate how to iterate the Partition Lemma and its consequence to prove the Mapping Lemma for $(0, 1) \cong \mathbb{R}$.

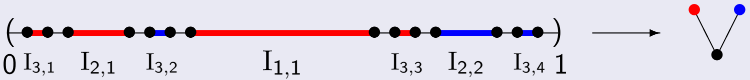
Modern Proof: A Demonstration with $(0,1)$

Applying the Partition Lemma

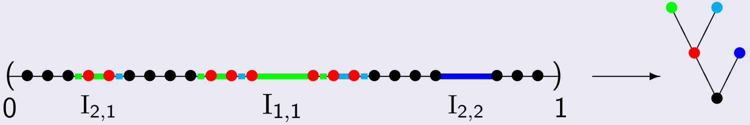
N = Cantor set (minus $\{0,1\}$)

U_1 = union of red intervals

U_2 = union of blue intervals



Iterating the above



Linearly Ordered Topological Spaces

Just as metrizable spaces generalize $\mathbb{R}$, ordered spaces also generalize the real line via the order structure $\leq$.

Definition

A topological space (X, τ) is a linearly ordered topological space (LOTS) provided there is a linear ordering $\leq$ of X such that the set of intervals $\mathcal{B}_{\leq} = \{(x, \rightarrow), (x, y), (\leftarrow, y) \mid x, y \in X\}$ forms a basis for τ. Call τ the interval topology induced by $\leq$.

Examples

1. Each of $\mathbb{R}$, $\mathbb{Q}$, and the Cantor space is a LOTS.
2. Let A be the set of all countable ordinals. The long line obtained via interval topology induced by the lexicographic ordering of $A \times [0, 1)$ is a LOTS that is not metrizable.

Generalized Ordered Spaces

Observation

The class of LOTS is not closed under subspaces.

Definition

A generalized ordered space (GO-space) is homeomorphic to a subspace of a LOTS.

Remark

Thinking of $\mathbb{R}$ as a typical example of a metrizable space, one may consider the long line as a typical example of a GO-space.

The McKinsey-Tarski Theorem for GO-spaces

Recall

A space X is locally compact provided each $x \in X$ has an open neighborhood U whose closure $\mathsf{c}(U)$ is a compact Hausdorff space.

Theorem

Let X be a nonempty crowded locally compact GO-space. Then $\mathsf{L}(X) = \mathsf{S4}$.

Proof Sketch

Same proof as above provided we have analogues of the Mapping and Partition Lemmas.

The Partition Lemma for GO-Spaces

How to Modify the Partition Lemma

- The Cantor space C makes the partitioning $(0, 1)$ easy.
- But for an arbitrary crowded metrizable space, there is no convenient analogue of C, which makes the Partition Lemma highly nontrivial.
- For a crowded locally compact GO-space, utilizing the machinery of irreducible mappings, there is an analogue of C which allows for a proof of the Partition Lemma.

The Partition Lemma

Let X be a nonempty crowded locally compact GO-space, N a nonempty crowded compact separable nowhere dense subset of X, and n a positive integer. There is a partition $\{N, U_1, \ldots, U_n\}$ of X such that each U_i is open and $\mathrm{c}(U_i) = U_i \cup N$.

The Mapping Lemma for GO-Spaces

The Mapping Lemma

Let X be a nonempty crowded locally compact GO-space and (W, R) a quasi-tree. Then there is an interior mapping $f : X \to W$.

Key Observations

- Each nonempty convex open subset of X contains a nonempty crowded compact separable nowhere dense subspace.
- Following the above demonstration for $(0, 1)$, iterate the use of the Partition Lemma for GO-spaces to obtain the desired mapping.

Concluding Remarks

- The McKinsey-Tarski theorem can be used to characterize exactly which modal logics arise from arbitrary metrizable spaces (in collaboration with Gabelaia, 2015).
- Similarly, but using different techniques than above, the McKinsey-Tarski theorem for crowded locally compact GO-spaces can be utilized to classify exactly those modal logics arising from arbitrary locally compact GO-spaces.
- It turns out that the two above sets of logics coincide! Thus, despite being quite different classes of spaces, they give rise to the same logical invariants.

Open Problems

- Are there other interesting/useful classes of spaces for which the McKinsey-Tarski theorem holds?
- The McKinsey-Tarski theorem does not require local compactness whereas our version for GO-spaces does.
  - Is there some property of spaces that would allow one to prove the McKinsey-Tarski theorem for both classes simultaneously?
  - Can locally compact be omitted from our result? And if so, would such allow one to determine the logic of an arbitrary GO-space?

Thank you (both organizers and audience) ...

any questions ...