Conjunctive Join-Semilattices

Joint Work with
Charles N. Delzell (LSU), Oghenetega Ighedo (UNISA)

James J. Madden, Louisiana State University

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The road to conjunctive join-semilattices . . . and beyond.

LSU Seminar on Ordered Algebraic Structures:

Tega Ighedo, visiting LSU from UNISA (Fall 2019–Spring 2020), focuses seminar on z-ideals.

We studied Banaschewski (Functorial maximal spectra, 2002), Johnstone (Almost maximal ideals, 1984), Isbell (subfittness, 1973), Wallman (disjunctivity, 1938) and related papers, and sought a setting in which we could unify ideas from this stream.

Fall 2020—Spring 2021. A new theme took shape in responding to suggestions and criticisms of the referee of the paper: the profound role of distributivity.

https://drive.google.com/file/d/1cLbco8h472qaAi5Wkis8qArkGiamrbrF/view
Definition. A *join-semilattice* is a set $L$ equipped with an associative, commutative and idempotent operation $\lor$ with identity $0$. The relation $\leq$ on $L$ is defined by $a \leq b \iff b = a \lor b$.

Fact. $\leq$ is a partial order in which $a \lor b$ is the l.u.b. of $a$ and $b$.

Remarks
- The theory can be developed without 0, but . . .
  the additional generality so obtained is not deep & many theorems become messy to state.
- In general, we do not require a top element 1.
- Any lattice (or frame) with 0 is a join-semilattice.

Definition. An *ideal* of $L$ is a non-empty down-set that is closed under $\lor$. An ideal is *prime* if its complement is downward directed.
Conjunctivity and Ideal Conjunctivity

**Definition.** We say a join-semilattice $L$ is *conjunctive* if:

(i) $L$ has a top element $1$, and

(ii) for every pair of elements $a, b \in L$:

either $b \leq a$, 

or there is $w \in L$ such that $b \lor w = 1$ and $a \lor w \neq 1$.

**Remarks.** If $b \lor w = 1$, we say $w$ is an upper complement of $b$. There are numerous ways to rephrase (ii), e.g., “Distinct elements of $L$ have different sets of upper complements,” or more indirectly, “For all $a, b \in L$ such that $a < b$, there is $w \in L$ such that $b \lor w = 1$ and $a \leq w < 1$.”

**Definition.** We say a join-semilattice $L$ is *ideally conjunctive* if: for every pair of elements $a, b \in L$: either $b \leq a$, or there is a proper ideal $W \subseteq L$ containing $a$ such that $b \lor W = L$.

**Remarks.** A *Yosida frame* (Martinez-Zenk) is the frame of ideals of an ideally conjunctive distributive lattice. Ideally conjunctive join-semilattices generalize Yosida frames in two ways: distributive lattice → join-semilattice. Nonetheless, much of the Martinez-Zenk theory carries over, and follows from the theory of ideally conjunctive join-semilattices. In particular, every principal ideal is an intersection of maximal ideals.
Examples

**Example.** The two-element join-semilattice $2 := \{0, 1\}$ is conjunctive.

**Proposition.** Let $X$ be set, and let $L$ be a sub-join-semilattice of $2^X$ that contains the point-complements $X \setminus \{y\}$, $y \in X$. Then $L$ is conjunctive.

*Proof.* If $A, B \in L$ and $B \subsetneq A$, then there is $b \in B \setminus A$. Let $C := X \setminus \{b\}$. Then $A \subseteq C \neq X$ and $B \cup C = X$. □

**Fact.** A product of conjunctive join-semilattices is conjunctive.

**Example.** (Deadly! See below.) A retract of a conjunctive join-semilattice *need not be conjunctive*. Let $r : 2 \times 2 \to \{(0, 0), (1, 0), (1, 1)\} \subseteq 2 \times 2$ be the identity map, except $r(0, 1) = (1, 1)$.
The Pierce Kernel

**Definition.** Let $L$ be a join-semilattice with 1.

$$R^1_L := \{ (a, b) \in L \times L \mid \forall x \in L \quad x \lor a = 1 \iff x \lor b = 1 \}.$$ 

**Fact.** $R^1_L$ is the strongest join-semilattice congruence on $L$ in which \{1\} is a class. $L/R^1_L$ is conjunctive.

Fact. The class of conjunctive join-semilattices is closed neither under limits nor colimits.

Proof. If $r$ is a retraction of an algebra $A$ onto a subalgebra $S \subseteq A$, then $r : A \to S$ is the coequalizer of the pair $(\text{id}_A, r)$ and the inclusion map $S \subseteq A$ is the equalizer of the same pair. □

Corollary. The full subcategory of conjunctive join-semilattices (within the category of join-semilattices with 1 and 1-preserving morphisms) is neither reflective nor coreflective.
Some Basic Unanswered Questions

**Fact.** If \( L \) is ideally conjunctive and \( a \in L \), then \( \downarrow a \) is conjunctive. (Thus, if \( L \) has 1 and is ideally conjunctive, then \( L \) is conjunctive.)

**Question.** Suppose \( L \) is a join-semilattice, \( a, b \in L \) and \( \downarrow a \) and \( \downarrow b \) are conjunctive. Is \( \downarrow (a \lor b) \) conjunctive?

**Question.** (My favorite!) Suppose \( L \) is a join-semilattice without 1 and \( \downarrow a \) is conjunctive for all \( a \in L \). Is \( L \) ideally conjunctive?

**Question.** How many non-isomorphic conjunctive join-semilattices of cardinality \( n \) are there? How may different sub-join-semilattices of \( 2^n \) are there that contain the point-complements? Same question for the sub-join-semilattices stable under permutations of \( n \) (or your favorite subgroup of \( S_n \)).
Proposition. Let $L$ be a join-semilattice with 1. Then $L$ is conjunctive if and only if, for every $a, b \in L$, if $b \not\leq a$, there is a maximal ideal that contains $a$ and does not contain $b$.

Proof. ($\Rightarrow$) Suppose $b \not\leq a$. Select $w$ so that $b \lor w = 1$ and $a \lor w \neq 1$. Applying Zorn's lemma, we obtain a maximal ideal $m \subseteq L$ containing $a \lor w$. Then $m$ contains $a$ but not $b$.

($\Leftarrow$) Suppose $m \subseteq L$ is a maximal ideal that contains $a$ but not $b$. Then $w \lor b = 1$ for some $w \in m$, while $w \lor a \in m$.

The maximal ideals of a conjunctive join-semilattice need not be prime. $M_3$ is conjunctive, but the maximal ideals (e.g., the one in red) are not prime:
Representations of Arbitrary Join Semilattices

Lemma/Definition. Suppose \( \phi : L \to 2 = \{0, 1\} \) is a 0-\( \lor \)-morphism. Then \( \phi^{-1}(0_L) \) is an ideal. Conversely, if \( I \subset L \) is an ideal, then

\[
\phi_I : L \to 2, \text{ defined by } \phi_I(a) = 0 \iff a \in I,
\]

is a 0-\( \lor \)-morphism.

Definition. Suppose \( X \) is a set of ideals of \( L \).

➤ The canonical representation on \( X \), denoted \( \phi_X : L \to 2^X \), is defined by \( \phi_X(a)(I) = \phi_I(a) \).

➤ \( \text{coz}_X a := (\phi_X(a))^{-1}(1) = \{ I \in X \mid a \notin I \} \).

Remark. The map \( a \to \text{coz}_X a \) is a 0-\( \lor \)-morphism from \( L \) to the set of subsets of \( X \). Evidently, \( \phi_X \) and \( \text{coz}_X \) are notational variants of one another. The \( \text{coz} \) notation is used widely in other representation theories e.g., Yosida Theorem.

Example. The principal representation \( \pi \) is the canonical representation on the set of principal ideals: \( \pi(a)(\downarrow b) = 0 \iff a \leq b \).
Definition. Let $X$ be a set of ideals of $L$. Then $\text{Spec}_X L$ denotes $X$ with the topology generated by $\{ \text{coz}_X a \mid a \in L \}$.

Some Elementary Facts about $\text{Spec}_X L$:

1. For all $L$ and all sets $X$ of ideals of $L$, $\text{Spec}_X L$ is $T_0$.
2. $\text{Spec}_X L$ is $T_1$ if and only if $X$ is an antichain.
3. $\{ \text{coz}_X a \mid a \in L \}$ is a base (not merely a subbase) for $\text{Spec}_X L$ if and only if every element of $X$ is prime.
4. Let $a \in L$ and let $B \subseteq L$. Then $\text{coz} a \subseteq \bigcup \{ \text{coz} b \mid b \in B \}$ if and only if $a \in \bigcap \{ I \in X \mid B \subseteq I \}$ (＝ the “$X$-radical of $B$”).
Representations on the Set of Maximal Ideals

**Theorem.** Assume $L$ has 1. Let $M_L$ denote the set of maximal ideals of $L$. Then:

(i) $\text{Spec}_{M_L} L$ is a compact $T_1$ space.

(ii) $\ker \text{coz}_{M_L}$ is the Pierce congruence, $R^1_L$.

(iii) If $K$ is a $\lor$-closed subbase for a compact $T_1$ topological space $Y$, then $y \mapsto \{ a \in K \mid y \notin a \} : Y \to \text{Spec}_{M_K} K$ is a homeomorphism, and $\text{coz}_{M_K}$ is 0-$\lor$-isomorphism.

**Comments**

- By (ii), the representation of $L$ on $M_L$ is injective iff $L$ is conjunctive.
- The hardest part of the proof is compactness, which boils down to showing that, for any subset $B \subseteq L$, $1 \in \bigcap \{ m \in M_L \mid B \subseteq m \}$ if and only if 1 is in the ideal generated by $B$. Note that $\text{coz}_{M_L} a$ need not be compact when $a \neq 1$.
- Research problem: Investigate the representation of $L$ on the set of “values of $L$,” i.e., $\{ q \in \text{Id} L \mid \exists a \in L$ such that $q$ is maximal missing $A \}$. 

Representation: Functoriality

Suppose \( \psi : L \to S \) is a morphism of join-semilattices with 1. We say that \( \psi \) is a \textit{conjunctive morphism} if \( \psi(1_L) = 1_S \), and \( \psi^{-1}(s) \) is an intersection of maximal ideals of \( L \) whenever \( s \) is a maximal ideal of \( S \).

**Definition.** \( Q_\psi : \text{Spec}_{M_S} S \Rightarrow \text{Spec}_{M_L} L \) is the multi-valued function

\[
\{(s, l) \in \text{Spec}_{M_S} S \times \text{Spec}_{M_L} L \mid \psi^{-1}(s) \subseteq l\}.
\]

For \( a \in L \), we write \( \hat{a} := \phi_{M_L}(a) \).

**Proposition.** If \( \psi : L \to S \) is a conjunctive morphism between conjunctive join-semilattices, then for all \( a \in L \) and all \( s \in \text{Spec}_{M_S} S \)

\[
\bigvee \hat{a} \circ Q_\psi(s) = \hat{\psi(a)}(s).
\]

If all the maximal ideals of \( L \) are prime, then \( Q_\psi \) is lower semicontinuous in the sense that \( (Q_\psi)^{-1}(U) \) is open in \( \text{Spec}_{M_S} S \) for all open \( U \subset \text{Spec}_{M_L} L \).
**Distributivity**

**Definition.** Suppose $L$ is a join-semilattice with 0 and $u \in L$.

$$D_L(u) : \equiv \forall a, b \in L : u \leq a \lor b \implies u = a' \lor b' \text{ for some } a' \leq a, b' \leq b.$$  

$$V_L(u) : \equiv \forall a \in L : \{ x \in L | u \leq x \lor a \} \text{ is a filter.}$$

Note that $L$ is *distributive* (in the senses of Grätzer-Schmidt and of Katriňák, which coincide if $0 \in L$) if and only if: $\forall u \in L D_L(u)$.

**Theorem.**

(i) $V_L(u)$ if and only if every $u$-maximal ideal of $L$ is prime.

(ii) $\forall v \leq u D_L(v)$ implies $V_L(u)$

(ii) $V_L(u)$ implies $D_L(u)$

There is a non-distributive conjunctive join-semilattice in which all maximal ideals are prime.

Let $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$, with $\infty \notin \mathbb{N}$. Let $K$ be the collection of all subsets of $\mathbb{N}^*$ of the following kinds: (1) the finite subsets of $\mathbb{N}$; (2) the complements in $\mathbb{N}^*$ of the finite subsets of $\mathbb{N}$; (3) $\mathbb{N}$ itself. Then $K$ is conjunctive and satisfies $V_K(1)$. The ideal $q \subseteq K$ of all finite subsets of $\mathbb{N}$ is maximal missing $\mathbb{N}$. But let $b := (\mathbb{N}^* \setminus \{0\})$. Then, $\downarrow \mathbb{N} \cap \downarrow b \subseteq q$, but neither $\downarrow \mathbb{N}$ nor $\downarrow b$ is contained in $q$, so $q$ fails to be prime.