Conjunctive Join-Semilattices

Joint Work with
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The road to conjunctive join-semilattices . . . and beyond.

LSU Seminar on Ordered Algebraic Structures:

Tega Ighedo, visiting LSU from UNISA (Fall 2019–Spring 2020), focuses seminar on z-ideals.

We studied Banaschewski (Functorial maximal spectra, 2002), Johnstone (Almost maximal ideals, 1984), Isbell (subfittness, 1973), Wallman (disjunctivity, 1938) and related papers, and sought a setting in which we could unify ideas from this stream.

Fall 2020—Spring 2021. A new theme took shape in responding to suggestions and criticisms of the referee of the paper: the profound role of distributivity.

https://drive.google.com/file/d/1cLbco8h472qaAi5Wkis8qArkGiamrbrF/v

Join-semilattices, Ideals, Prime Ideals

Definition. A *join-semilattice* is a set L equipped with an associative, commutative and idempotent operation $\vee$ with identity 0. The relation $\leq$ on L is defined by $a \leq b \iff b = a \vee b$.

Fact. $\leq$ is a partial order in which $a \vee b$ is the l.u.b. of a and b.

Remarks

- The theory can be developed without 0, but ...
  the additional generality so obtained is not deep & many theorems become messy to state.
- In general, we do not require a top element 1.
- Any lattice (or frame) with 0 is a join-semilattice.

Definition. An *ideal* of L is a non-empty down-set that is closed under $\vee$. An ideal is *prime* if its complement is downward directed.

Conjunctivity and Ideal Conjunctivity

Definition. We say a join-semilattice L is *conjunctive* if:

(i) L has a top element 1, and

(ii) for every pair of elements $a, b \in L$:

either $b \leq a$,

or there is $w \in L$ such that $b \vee w = 1$ and $a \vee w \neq 1$.

Remarks. If $b \vee w = 1$, we say *w is an upper complement of b*. There are numerous ways to rephrase (ii), e.g., "Distinct elements of L have different sets of upper complements," or more indirectly, "For all $a, b \in L$ such that $a < b$, there is $w \in L$ such that $b \vee w = 1$ and $a \leq w < 1$."

Definition. We say a join-semilattice L is *ideally conjunctive* if: for every pair of elements $a, b \in L$: **either** $b \leq a$, **or** there is a proper ideal $W \subseteq L$ containing a such that $b \vee W = L$.

Remarks. A *Yosida frame* (Martinez-Zenk) is the frame of ideals of an ideally conjunctive distributive lattice. Ideally conjunctive join-semilattices generalize Yosida frames in two ways: ~~distributive lattice~~ $\rightarrow$ join-semilattice. Nonetheless, much of the Martinez-Zenk theory carries over, and follows from the theory of ideally conjunctive join-semilattices. In particular, every principal ideal is an intersection of maximal ideals.

Examples

Example. The two-element join-semilattice $2 := \{0, 1\}$ is conjunctive.

Proposition. Let X be set, and let L be a sub-join-semilattice of 2^X that contains the point-complements $X \setminus \{y\}$, $y \in X$. Then L is conjunctive.

Proof. If $A, B \in L$ and $B \not\subseteq A$, then there is $b \in B \setminus A$. Let $C := X \setminus \{b\}$. Then $A \subseteq C \neq X$ and $B \cup C = X$. □

Fact. A product of conjunctive join-semilattices is conjunctive.

Example. (Deadly! See below.) A retract of a conjunctive join-semilattice *need not be conjunctive*. Let $r : 2 \times 2 \to \{(0,0), (1,0), (1,1)\} \subseteq 2 \times 2$ be the identity map, except $r(0,1) = (1,1)$.

The Pierce Kernel

Definition. Let L be a join-semilattice with 1.

$$R_L^1 := \{\,(a,b) \in L \times L \mid \forall x \in L \quad x \vee a = 1 \iff x \vee b = 1\,\}.$$

Fact. R_L^1 is the strongest join-semilattice congruence on L in which $\{1\}$ is a class. L/R_L^1 is conjunctive.

See R S Pierce. *Ann. of Math.* 59 (1954), 287-291.

Category-Theoretic Deficiencies

Fact. The class of conjunctive join-semilattices is closed **neither** under limits **nor** colimits.

Proof. If r is a retraction of an algebra A onto a subalgebra $S \subseteq A$, then $r : A \to S$ is the coequalizer of the pair (id_A, r) and the inclusion map $S \subseteq A$ is the equalizer of the same pair. □

Corollary. The full subcategory of conjunctive join-semilattices (within the category of join-semilattices with 1 and 1-preserving morphisms) is **neither** reflective **nor** coreflective.

Some Basic Unanswered Questions

Fact. If L is ideally conjunctive and $a \in L$, then $\downarrow a$ is conjunctive. (Thus, if L has 1 and is ideally conjunctive, then L is conjunctive.)

Question. Suppose L is a join-semilattice, $a, b \in L$ and $\downarrow a$ and $\downarrow b$ are conjunctive. Is $\downarrow(a \vee b)$ conjunctive?

Question. (My favorite!) Suppose L is a join-semilattice without 1 and $\downarrow a$ is conjunctive for all $a \in L$. Is L ideally conjuctive?

Question. How many non-isomorphic conjunctive join-semilattices of cardinality n are there? How may different sub-join-semilattices of 2^n are there that contain the point-complements? Same question for the sub-join-semilattices stable under permutations of n (or your favorite subgroup of S_n).

Conjunctive Means Enough Maximal Ideals

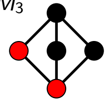
Proposition. Let L be a join-semilattice with 1. Then L is conjunctive if and only if, for every $a, b \in L$, if $b \not\leq a$, there is a maximal ideal that contains a and does not contain b.

Proof. ($\Rightarrow$) Suppose $b \not\leq a$. Select w so that $b \vee w = 1$ and $a \vee w \neq 1$. Applying Zorn's lemma, we obtain a maximal ideal $\mathfrak{m} \subseteq L$ containing $a \vee w$. Then $\mathfrak{m}$ contains a but not b.

($\Leftarrow$) Suppose $\mathfrak{m} \subseteq L$ is a maximal ideal that contains a but not b. Then $w \vee b = 1$ for some $w \in \mathfrak{m}$, while $w \vee a \in \mathfrak{m}$. □

The maximal ideals of a conjunctive join-semilattice *need not be prime.*
M_3 is conjunctive,
but the maximal ideals (e.g., the one in red) are not prime:

M_3



Representations of Arbitrary Join Semilattices

Lemma/Definition. Suppose $\phi : L \to 2 = \{0, 1\}$ is a 0-$\vee$-morphism. Then $\phi^{-1}(0_L)$ is an ideal. Conversely, if $I \subset L$ is an ideal, then

$$\phi_I : L \to 2, \text{ defined by } \phi_I(a) = 0 \iff a \in I,$$

is a 0-$\vee$-morphism.

Definition. Suppose X is a set of ideals of L.

- The *canonical representation on X*, denoted $\phi_X : L \to 2^X$, is defined by $\phi_X(a)(I) = \phi_I(a)$.
- $coz_X\, a := (\phi_X(a))^{-1}(1) = \{ I \in X \mid a \notin I \}$.

Remark. The map $a \to coz_X\, a$ is a 0-$\vee$-morphism from L to the set of subsets of X. Evidently, ϕ_X and coz_X are notational variants of one another. The *coz* notation is used widely in other representation theories e.g., Yosida Theorem.

Example. The *principal representation* π is the canonical representation on the set of principal ideals: $\pi(a)(\downarrow b) = 0 \iff a \leq b$.

Spectra

Definition. Let X be a set of ideals of L. Then $Spec_X\, L$ denotes X with the topology generated by $\{\, coz_X\, a \mid a \in L \,\}$.

Some Elementary Facts about $Spec_X\, L$:

1. For all L and all sets X of ideals of L, $Spec_X\, L$ is T_0.
2. $Spec_X\, L$ is T_1 if and only if X is an antichain.
3. $\{\, coz_X\, a \mid a \in L \,\}$ is a base (not merely a subbase) for $Spec_X\, L$ if and only if every element of X is prime.
4. Let $a \in L$ and let $B \subseteq L$. Then $coz\, a \subseteq \bigcup\{\, coz\, b \mid b \in B \,\}$ if and only if $a \in \bigcap\{\, I \in X \mid B \subseteq I \,\}$ (= the "X-radical of B").

Representations on the Set of Maximal Ideals

Theorem. Assume L has 1. Let M_L denote the set of maximal ideals of L. Then:

(*i*) $Spec_{M_L} L$ is a compact T_1 space.

(*ii*) $\ker coz_{M_L}$ is the Pierce congruence, R^1_L.

(*iii*) If K is a $\vee$-closed subbase for a compact T_1 topological space Y, then $y \mapsto \{ a \in K \mid y \notin a \} : Y \to Spec_{M_K} K$ is a homeomorphism, and coz_{M_K} is 0-$\vee$-isomorphism.

Comments

- By (*ii*), the representation of L on M_L is injective iff L is conjunctive.
- The hardest part of the proof is compactness, which boils down to showing that, for any subset $B \subseteq L$, $1 \in \bigcap \{ \mathfrak{m} \in M_L \mid B \subseteq \mathfrak{m} \}$ if and only if 1 is in the ideal generated by B. Note that $coz_{M_L} a$ need not be compact when $a \neq 1$.
- Research problem: Investigate the representation of L on the set of "values of L," i.e., $\{ \mathfrak{q} \in \mathrm{Id}\, L \mid \exists a \in L \text{ such that } \mathfrak{q} \text{ is maximal missing } A \}$.

Representation: Functoriality

Suppose $\psi : L \to S$ is a morphism of join-semilattices with 1. We say that ψ is a *conjunctive morphism* if $\psi(1_L) = 1_S$, and $\psi^{-1}(\mathfrak{s})$ is an intersection of maximal ideals of L whenever $\mathfrak{s}$ is a maximal ideal of S.

Definition. $Q_\psi : Spec_{M_S}\, S \rightrightarrows Spec_{M_L}\, L$ is the multi-valued function

$$\{\, (\mathfrak{s}, \mathfrak{l}) \in Spec_{M_S}\, S \times Spec_{M_L}\, L \mid \psi^{-1}(\mathfrak{s}) \subseteq \mathfrak{l} \,\}.$$

For $a \in L$, we write $\widehat{a} := \phi_{M_L}(a)$.

Proposition. If $\psi : L \to S$ is a conjunctive morphism between conjunctive join-semilattices, then for all $a \in L$ and all $\mathfrak{s} \in Spec_{M_S}\, S$

$$\bigvee \widehat{a} \circ Q_\psi(\mathfrak{s}) = \widehat{\psi(a)}(\mathfrak{s}).$$

If all the maximal ideals of L are prime, then Q_ψ is lower semicontinuous in the sense that $(Q_\psi)^{-1}(U)$ is open in $Spec_{M_S}\, S$ for all open $U \subset Spec_{M_L}\, L$.

Distributivity

Definition. Suppose L is a join-semilattice with 0 and $u \in L$.

$$D_L(u) \quad :\equiv \quad \forall a, b \in L: \; u \leq a \vee b \implies u = a' \vee b' \text{ for some } a' \leq a,\ b' \leq b.$$

$$V_L(u) \quad :\equiv \quad \forall a \in L: \; \{\, x \in L \mid u \leq x \vee a \,\} \text{ is a filter.}$$

Note that L is *distributive* (in the senses of Grätzer-Schmidt and of Katriňák, which coincide if $0 \in L$) if and only if: $\forall u \in L \; D_L(u)$.

Theorem.

(i) $V_L(u)$ if and only if every u-maximal ideal of L is prime.

(ii) $\forall v \leq u \; D_L(v)$ implies $V_L(u)$

(ii) $V_L(u)$ implies $D_L(u)$

There is a non-distributive conjunctive join-semilattice in which all maximal ideals are prime.

Let $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$, with $\infty \notin \mathbb{N}$. Let K be the collection of all subsets of $\mathbb{N}^*$ of the following kinds: (1) the finite subsets of $\mathbb{N}$; (2) the complements in $\mathbb{N}^*$ of the finite subsets of $\mathbb{N}$; (3) $\mathbb{N}$ itself. Then K is conjunctive and satisfies $V_K(1)$. The ideal $\mathfrak{q} \subseteq K$ of all finite subsets of $\mathbb{N}$ is maximal missing $\mathbb{N}$. But let $b := (\mathbb{N}^* \setminus \{0\})$. Then, $\downarrow\mathbb{N} \cap \downarrow b \subseteq \mathfrak{q}$, but neither $\downarrow\mathbb{N}$ nor $\downarrow b$ is contained in $\mathfrak{q}$, so $\mathfrak{q}$ fails to be prime.