Some research directions in the theory of lattice-ordered groups

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W. C. Holland (1935–2020)

J. Martinez (1945–2020)
What does Birkhoff get us for free?

Lattice-ordered groups ($\ell$-groups) are a variety, so Birkhoff’s Subdirect Representation Theorem applies.

Congruences on $\ell$-groups are representable by $\ell$-ideals, i.e. kernels of homomorphisms.

$G - \ell$-group.

$I \subseteq G - \ell$-ideal: sublattice and normal subgroup that is convex, i.e., $x \leq y \leq z$ and $x, y \in C$ entails $y \in C$.

$\text{Idl } G$ — lattice of $\ell$-ideals of $G$. 
Subdirectly irreducible ℓ-groups can be quite complicated.

(Compare with Heyting algebras.)

Congruences are too coarse to afford a revealing structure theory.

We replace Idl \( G \) with a larger lattice.

We use convex sublattice subgroups which are not necessarily normal; for these, there is a useful notion of primeness.

Note from the outset that in subvarieties things may be—and in important cases are—different.

Obvious example: Abelian ℓ-groups: every subgroup normal, subdirectly irreducible entails totally ordered.
Prime subgroups

$\text{Cnv } G$ — lattice of convex sublattice subgroups.

Given $C \in \text{Cnv } G$, order $G/C$, the set of (right) cosets of $C$ in $G$, by $Cx \leq Cy$ iff $cx \leq y$ for some $c \in C$.

Then $G/C$ is a lattice, and the map $x \mapsto Cx$ is a lattice homomorphism.

$P \subseteq G$ — prime: convex sublattice subgroup such that $G/P$ is totally ordered (or a “chain”). Equivalently, $x \land y = 1$ entails $x \in P$ or $y \in P$.

$\text{Spec } G$ — set of all primes, or spectrum of $G$.

$p, q$ — notation for primes.
Ordered permutation groups, and primes

If $\Omega$ is any chain, write $\text{Aut} \Omega$ for the group of all automorphisms (order-preserving bijections) $\Omega \to \Omega$.

$\text{Aut} \Omega$ is an $\ell$-group under the pointwise ordering:

$$f \leq g \quad \text{iff} \quad \text{for all } x \in \Omega, \quad fx \leq gx.$$ 

A prime $p \subseteq G$ induces a permutation representation of $G$, its (right) regular representation.

$$g \in G \quad \overset{\sim}{\longrightarrow} \quad \widehat{g} \in \text{Aut} G / p$$

$$\widehat{g}(px) := pxg$$

This is a lattice-group homomorphism ($\ell$-homomorphism):

$$\widehat{\sim} : G \longrightarrow \text{Aut} \Omega.$$
If we let $P$ range in $\text{Conv} G$ and consider the induced map to the product
\[
G \longrightarrow \prod_{p \in \text{Spec } G} \text{Aut } G/p
\]
we get a faithful representation of $G$: “any $\ell$-group has enough primes”. (Observe parallel with subdirect representation.)

Can we make the codomain an $\text{Aut } \Omega$? Set
\[
\Omega := \bigsqcup_{p \in \text{Spec } G} G/p
\]
Totally order $\Omega$ so as to make each $G/P$ convex.

Note: This order on $\Omega$ has apparently very little to do with $G$. 
Holland’s Theorem

As before, we get an injective $\ell$-homomorphism

\[ G \longrightarrow \text{Aut} \, \Omega \]

which is a faithful representation of $G$.

Theorem (W.C. Holland, 1963)

For every $\ell$-group $G$ there is a chain $\Omega$ together with an injective $\ell$-homomorphism $G \rightarrow \text{Aut} \, \Omega$.

One can try to tie $\Omega$ to $G$ more closely (transitivity). This direction has been extensively pursued.

In another deeply studied direction we look at right orders on groups, generalising ordered groups.
Right orders

A total order \( \leq \) on a group \( G \) is a right order if for all \( x, y, t \in G \), \( x \leq y \) implies \( xt \leq yt \).

If \( G \) is partially ordered, we only look at those right orders that extend the partial order of \( G \).

Dropping antisymmetry, we get right preorders on \( G \); notation: \( \preceq \).

We can extend the right regular representation \( G \to \text{Aut } G/\mathcal{P} \) to the setting of right preordered groups.

The relation \( \preceq \) induces the equivalence relation: \( x \equiv y \) iff \( x \preceq y \) and \( y \preceq x \). The quotient set \( G/\preceq \) is totally ordered by \([x] \leq [y]\) iff \( x \leq y \).

We get a map

\[
\hat{\sim} : G \longrightarrow \text{Aut } G/\preceq
\]

defined by

\[
\hat{g}([x]) := [xg].
\]
Then

\[ \hat{\sim}: G \longrightarrow \text{Aut} G/\leq \]

is an order-preserving group homomorphism.

Its image \( \hat{G} \) need not be an \( \ell \)-subgroup of \( \text{Aut} G/\leq \). But look at the \( \ell \)-subgroup that \( \hat{G} \) generates in \( \text{Aut} G/\leq \), denoted \([\hat{G}]\).

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**Remark**

Important portions of the theory of \( \ell \)-groups are tied to the passage

\[ G \longrightarrow [\hat{G}] \]

from right-ordered groups to lattice-ordered ones.

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What is the relationships between right orders on groups and \( \ell \)-groups? And what about right orders in their own right?
**Free ℓ-groups**

$V$ a variety of ℓ-groups, $G$ a (partially ordered) group, $F_V G$ the free ℓ-group generated by $G$ in $V$; always exists, in some sources called “universal”.

Universal arrow

$$\eta : G \rightarrow F_V G$$

characterised by universal property

$$
\begin{array}{ccc}
G & \xrightarrow{\eta} & F_V G \\
\downarrow{f} & & \downarrow{h} \\
H & & H
\end{array}
$$

with $f$ an order-preserving group homomorphism to ℓ-group $H$ in variety $V$, and $h$ an ℓ-homomorphism.
Some research suggestions

\[ G \xrightarrow{\eta} F_V G \]

\[ \downarrow \quad \downarrow h \]

\[ f \quad H \]

- \( \eta \) injective \( \iff \) there exists injective \( f \) to some \( H \) in \( V \).
- \( \eta \) order-embedding \( \iff \) there exists order-embedding \( f \) to some \( H \) in \( V \).

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**Research suggestion**

Study \( \eta \) as the mathematically tangible obstruction to embedding \( G \) into some \( \ell \)-group in \( V \).

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**Right orders, again**

Important portions of the theory of ℓ-groups are tied to the passage

\[ G \rightarrow \langle \hat{G} \rangle \]

from right-ordered groups to lattice-ordered ones.

A right preorder \( \leq \) on \( G \) is a right V-preorder if the ℓ-group \( \langle \hat{G} \rangle \) lies in \( V \).

- For \( V= \text{representable} \) ℓ-groups, these are the “representable right preorders”:

  \[ \forall a \in G \left\{ \begin{array}{l}
  \text{either} \\
  \text{or}
  \end{array} \right. \forall b \in G \ 1 \leq bab^{-1},
  \forall b \in G \ bab^{-1} \leq 1 \]

- For \( V= \text{Abelian} \) ℓ-groups, these are the “Abelian right preorders”:

  \[ \forall a, b \in G \ 1 \leq [a, b] \leq 1. \]
Right orders in their own right


Folklore: A countable group is right-orderable precisely when it has a faithful representation as a group of orientation-preserving homeomorphisms of the real line.
SPACE OF RIGHT PREORDERS


Write $\mathcal{P}_V G$ for the set of all right $V$-preorders on $G$ that extend the partial order of $G$. Given $g \in G$, define

$$\mathcal{P}_g := \left\{ \leq \in \mathcal{P}_V G \mid 1 \leq g \text{ and } 1 \not\leq g^{-1} \right\}.$$  

We equip $\mathcal{P}_V G$ with the smallest topology containing all sets $\mathcal{P}_g$'s as $g$ ranges in $G$.

Then $\mathcal{P}_V G$ is the space of right $V$-preorders on $G$. It is is a poset, too: order it by specialisation of right preorders.
Spectrum of an $\ell$-group

If $H$ is an $\ell$-group, $A \subseteq H$, define

$$\mathcal{S}A := \{ p \in \text{Spec } H \mid A \not\subseteq p \} ,$$

whose complements in $\text{Spec } H$ are

$$\mathcal{V}A := \{ p \in \text{Spec } H \mid A \subseteq p \} .$$

Endow $\text{Spec } H$ with the topology whose open sets are precisely the supports $\mathcal{S}A$, as $A$ ranges over subsets of $H$.

Then $\text{Spec } H$ is the spectrum, or spectral space, of $H$.

Given $\ell$-group $H$, write $\text{Min } H$ for the subspace of $\text{Spec } H$ consisting of minimal primes.

$\text{Spec } H$ is a completely normal generalised spectral space.
UNITS, AND COMPACTNESS OF SPECTRA


For any ℓ-group $H$:

- Spec $H$ is compact precisely when $H$ has a strong unit.
- Min $H$ compact precisely when $H$ is complemented: for every $x \in H^+$ there is $y \in H^+$ with $x \wedge y = 1$ and $x \vee y$ a weak unit.
Can we spectrally characterise ℓ-groups that admit a weak unit? (Cf. the case of strong units.)

A prime \( p \in \text{Spec } H \) is minimal precisely when

\[
p = \bigcup \{ x^\perp \mid x \notin p \}.
\]

(Here, \( x^\perp \) is the set of elements \( y \) orthogonal to \( x \), i.e. such that \( |x| \wedge |y| = 1 \).) A prime \( p \in \text{Spec } H \) is quasiminimal if

\[
p = \bigcup \{ x^{\perp \perp} \mid x \in p \}.
\]

Write \( \text{Qin} H \) for the subspace \( \text{Spec } H \) consisting of quasiminimal primes.

\[\text{Min } H \subseteq \text{Qin } H \subseteq \text{Spec } H.\]

Inclusions are proper, in general.

A. Colacito and VM, *Orders on groups, and spectral spaces of lattice-groups*, Algebra Universalis 81, 30 pp., 2020
Theorem (A. Colacito and VM, 2020)

An $\ell$-group $H$ has a weak unit precisely when $Q_{\text{in}} H$ is compact.

The nature of these results on units and compactness of spectra is essentially lattice-theoretic. For instance, for weak units:


Research suggestion

Abstract the theorem above to distributive lattices with bottom.
FROM RIGHT PREORDERS TO $\ell$-GROUPS...

$$\eta: G \longrightarrow F_V G$$

$$P_V G \quad \text{Spec } F_V G.$$

We send each $\leq \in P_V G$ to a prime $p \in \text{Spec } F_V G$.

Start from $G \rightarrow \llbracket \hat{G} \rrbracket$; the universal arrow $\eta$ yields an $\ell$-group homomorphism $h: F_V G \rightarrow \llbracket \hat{G} \rrbracket$.

Let $q$ be the stabiliser of the identity in $\llbracket \hat{G} \rrbracket$, which is a prime; pull it back along $h$:

$$p := h^{-1}[q].$$

This gives a map

$$\kappa: P_V G \longrightarrow \text{Spec } F_V G.$$
...AND BACK

$$\eta: G \longrightarrow F_V G$$

$$\mathcal{P}_V G \quad \text{Spec} \ F_V G.$$  

We send each $$p \in \text{Spec} \ F_V G$$ to a right preorder $$\leq \in \mathcal{P}_V G.$$  

Start from $$p,$$ and use $$\eta$$ to define the following relation $$\leq$$ on $$G:$$

$$x \leq y \quad \text{iff} \quad \forall \eta a \leq \forall \eta b.$$  

Then it can be proved that $$\leq \in \mathcal{P}_V G.$$  

This gives a map

$$\pi: \text{Spec} \ F_V G \longrightarrow \mathcal{P}_V G.$$
Free $\ell$-groups and right preorders

Theorem (A. Colacito and V.M., 2020)

The maps $\kappa$ and $\pi$ are mutually inverse homeomorphisms and order-isomorphisms.

WOULD-BE CORRESPONDENCE THEORY

- Representable right preorders:
  \[
  \forall a \in G \begin{cases} 
  \text{either} & \forall b \in G \ 1 \leq bab^{-1}, \\
  \text{or} & \forall b \in G \ bab^{-1} \leq 1 
  \end{cases}
  \]

- Abelian right preorders:
  \[
  \forall a, b \in G \ 1 \leq [a, b] \leq 1.
  \]

Research suggestion

For which $V$ is the class of right $V$-preorders first-order definable in the language of groups with a binary relation $\leq$?
**Taking subvarieties seriously**

Spec\(^*\) \(H := \text{Idl } H \cap \text{Spec } H\) is the **normal spectrum** of \(H\).

A right preorder on \(G\) is just a **preorder** if it is antisymmetric. Write

\[
\mathcal{B}G
\]

for the set of preorders on \(G\) that extend the partial order of \(G\).

Then if \(V\)=representable \(\ell\)-groups, the maps \(\kappa\) and \(\pi\) descend as follows:

\[
\begin{array}{c}
\mathcal{P}_V G \xleftarrow{\kappa} \text{Spec } F_V G \\
\uparrow \subseteq \uparrow \\
\mathcal{B}G \xleftarrow{\pi} \text{Spec } F_V G
\end{array}
\]

\[
\subseteq
\]

\[
\mathcal{B}G \xrightarrow{\pi} \text{Spec } F_V G
\]
TAKING SUBVARIETIES SERIOUSLY

\[ \text{Spec}^* H := \text{Idl } H \cap \text{Spec } H \] is the normal spectrum of \( H \).

A right preorder on \( G \) is just a preorder if it is antisymmetric. Write \( \mathcal{B}G \) for the set of preorders on \( G \) that extend the partial order of \( G \).

Research suggestion

For significant \( V \)’s, identify “\( V \)-convex sublattice subgroups” and develop their theory.
Would-be geometry of representable ℓ-groups

- $G_n$ — the free $n$-generated group (no order).
- $A_n$ — the free $n$-generated Abelian group ($\mathbb{Z}^n$).
- $FG_n$ — the free $n$-generated representable ℓ-group.
- $FA_n$ — the free $n$-generated Abelian ℓ-group.

There is a very significant theory of the geometric representation of $FA_n$ by piecewise-linear functions.

This is known as Baker-Beynon Duality, and applies to all finitely presented Abelian ℓ-groups.
Research suggestion

Can $FG_n$ be represented by piecewise-linear actions on $S^{n-1} \subseteq \mathbb{R}^n$?
Thank you for your attention.

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J. Martinez (1945–2020)

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