Some research directions in the theory of lattice-ordered groups

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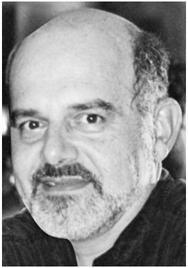
W. C. Holland (1935–2020)



J. Martinez (1945–2020)

What does Birkhoff get us for free?

Lattice-ordered groups (ℓ-groups) are a variety, so Birkhoff's Subdirect Representation Theorem applies.

Congruences on ℓ-groups are representable by ℓ-ideals, i.e. kernels of homomorphisms.

G – ℓ-group.

$I \subseteq G$ – ℓ-ideal: sublattice and normal subgroup that is convex, i.e., $x \leqslant y \leqslant z$ and $x, y \in C$ entails $y \in C$.

$\operatorname{Idl} G$ — lattice of ℓ-ideals of G.

Not that much

Subdirectly irreducible ℓ-groups can be quite complicated.

(Compare with Heyting algebras.)

Congruences are too coarse to afford a revealing structure theory.

We replace $\mathrm{Idl}\, G$ with a larger lattice.

We use convex sublattice subgroups which are not necessarily normal; for these, there is a useful notion of primeness.

Note from the outset that in subvarieties things may be—and in important cases are—different.

Obvious example: Abelian ℓ-groups: every subgroup normal, subdirectly irreducible entails totally ordered.

Prime subgroups

$\mathrm{Cnv}\, G$ — lattice of convex sublattice subgroups.

Given $C \in \mathrm{Cnv}\, G$, order G/C, the set of (right) cosets of C in G, by $Cx \leqslant Cy$ iff $cx \leqslant y$ for some $c \in C$.

Then G/C is a lattice, and the map $x \mapsto Cx$ is a lattice homomorphism.

$P \subseteq G$ – prime: convex sublattice subgroup such that G/P is totally ordered (or a "chain"). Equivalently, $x \wedge y = 1$ entails $x \in P$ or $y \in P$.

$\mathrm{Spec}\, G$ — set of all primes, or spectrum of G.

$\mathfrak{p}, \mathfrak{q}$ — notation for primes.

Ordered permutation groups, and primes

If Ω is any chain, write $\mathrm{Aut}\,\Omega$ for the group of all automorphisms (order-preserving bijections) $\Omega \to \Omega$.

$\mathrm{Aut}\,\Omega$ is an ℓ-group under the pointwise ordering:

$$f \leqslant g \qquad \text{iff} \qquad \text{for all } x \in \Omega, \quad fx \leqslant gx.$$

A prime $\mathfrak{p} \subseteq G$ induces a permutation representation of G, its (right) regular representation.

$$g \in G \longmapsto \widehat{g} \in \mathrm{Aut}\, G/\mathfrak{p}$$
$$\widehat{g}(\mathfrak{p}x) := \mathfrak{p}xg$$

This is a lattice-group homomorphism (ℓ-homomorphism):

$$\widehat{-}\colon G \longrightarrow \mathrm{Aut}\,\Omega.$$

If we let P range in $\operatorname{Cnv} G$ and consider the induced map to the product

$$G \longrightarrow \prod_{\mathfrak{p} \in \operatorname{Spec} G} \operatorname{Aut} G/\mathfrak{p}$$

we get a faithful representation of G: "any ℓ-group has enough primes". (Observe parallel with subdirect representation.)

Can we make the codomain an $\operatorname{Aut} \Omega$? Set

$$\Omega := \bigsqcup_{\mathfrak{p} \in \operatorname{Spec} G} G/\mathfrak{p}$$

Totally order Ω so as to make each G/P convex.

Note: This order on Ω has apparently very little to do with G.

Holland's Theorem

As before, we get an injective ℓ-homomorphism

$$G \longrightarrow \operatorname{Aut}\Omega$$

which is a faithful representation of G.

Theorem (W.C. Holland, 1963)

For every ℓ-group G there is a chain Ω together with an injective ℓ-homomorphism $G \to \operatorname{Aut}\Omega$.

One can try to tie Ω to G more closely (transitivity). This direction has been extensively pursued.

In another deeply studied direction we look at right orders on groups, generalising ordered groups.

Right orders

A total order $\leqslant$ on a group G is a right order if for all $x, y, t \in G$, $x \leqslant y$ implies $xt \leqslant yt$.

If G is partially ordered, we only look at those right orders that extend the partial order of G.

Dropping antisymmetry, we get right preorders on G; notation: $\preceq$.

We can extend the right regular representation $G \to \operatorname{Aut} G/\mathfrak{p}$ to the setting of right preordered groups.

The relation $\preceq$ induces the equivalence relation: $x \equiv y$ iff $x \preceq y$ and $y \preceq x$. The quotient set $G/{\preceq}$ is totally ordered by $[x] \leqslant [y]$ iff $x \preceq y$.

We get a map

$$\widehat{-}\colon G \longrightarrow \operatorname{Aut} G/{\preceq}$$

defined by

$$\widehat{g}([x]) \coloneqq [xg].$$

Then

$$\widehat{-}\colon G \longrightarrow \operatorname{Aut} G/{\preceq}$$

is an order-preserving group homomorphism.

Its image $\widehat{G}$ need not be an ℓ-subgroup of $\operatorname{Aut} G/{\preceq}$. But look at the ℓ-subgroup that $\widehat{G}$ generates in $\operatorname{Aut} G/{\preceq}$, denoted $[\![\widehat{G}]\!]$.

Remark

Important portions of the theory of ℓ-groups are tied to the passage

$$G \longmapsto [\![\widehat{G}]\!]$$

from right-ordered groups to lattice-ordered ones.

What is the relationships between right orders on groups and ℓ-groups? And what about right orders in their own right?

Free ℓ-groups

V a variety of ℓ-groups, G a (partially ordered) group, $\mathrm{F}_{\mathsf{V}}\, G$ the free ℓ- group generated by G in V; always exists, in some sources called “universal”.

Universal arrow

$$\eta\colon G \longrightarrow \mathrm{F}_{\mathsf{V}}\, G$$

characterised by universal property

$$\begin{tikzcd} G \arrow[r, "\eta"] \arrow[dr, "f"'] & \mathrm{F}_{\mathsf{V}}\, G \arrow[d, dashed, "h"] \\ & H \end{tikzcd}$$

with f an order-preserving group homomorphism to ℓ-group H in variety V, and h an ℓ-homomorphism.

$$\begin{tikzcd} G \arrow[r, "\eta"] \arrow[dr, "f"'] & \mathrm{F}_{\mathsf{V}}\, G \arrow[d, dashed, "h"] \\ & H \end{tikzcd}$$

- η injective $\Leftrightarrow$ there exists injective f to some H in V.
- η order-embedding $\Leftrightarrow$ there exists order-embedding f to some H in V.

Research suggestion

Study η as the mathematically tangible obstruction to embedding G into some ℓ-group in V.

P. Conrad, *Free lattice-ordered groups*, J. Algebra, 1970.

Right orders, again

Important portions of the theory of ℓ-groups are tied to the passage

$$G \longmapsto [\![\widehat{G}]\!]$$

from right-ordered groups to lattice-ordered ones.

A right preorder $\leq$ on G is a right V-preorder if the ℓ-group $[\![\widehat{G}]\!]$ lies in V.

- For V=representable ℓ-groups, these are the "representable right preorders":

$$\forall a \in G \begin{cases} \text{either} & \forall b \in G \quad 1 \leq bab^{-1}, \\ \text{or} & \forall b \in G \quad bab^{-1} \leq 1 \end{cases}$$

- For V=Abelian ℓ-groups, these are the "Abelian right preorders":

$$\forall a, b \in G \quad 1 \leq [a, b] \leq 1.$$

Right orders in their own right

A. Clay and D. Rolfsen, *Ordered groups and topology*, American Mathematical Society, 2016.

E. Ghys, *Groups acting on the circle*, Enseign. Math. 47, 329–407, 2001.

Folklore: A countable group is right-orderable precisely when it has a faithful representation as a group of orientation-preserving homeomorphisms of the real line.

Space of right preorders

A. Sikora, *Topology on the spaces of orderings of groups*, Bull. London Math. Soc. 36, 519–526, 2004.

Write $\mathscr{P}_{\mathsf{V}}\, G$ for the set of all right V-preorders on G that extend the partial order of G. Given $g \in G$, define

$$\mathbb{P}g := \left\{ \leq \in \mathscr{P}_{\mathsf{V}}\, G \;\middle|\; 1 \leq g \text{ and } 1 \not\leq g^{-1} \right\}.$$

We equip $\mathscr{P}_{\mathsf{V}}\, G$ with the smallest topology containing all sets $\mathbb{P}g$'s as g ranges in G.

Then $\mathscr{P}_{\mathsf{V}}\, G$ is the space of right V-preorders on G. It is is a poset, too: order it by specialisation of right preorders.

Spectrum of an ℓ-group

If H is an ℓ-group, $A \subseteq H$, define

$$\mathbb{S}A := \{\mathfrak{p} \in \operatorname{Spec} H \mid A \nsubseteq \mathfrak{p}\},$$

whose complements in $\operatorname{Spec} H$ are

$$\mathbb{V}A := \{\mathfrak{p} \in \operatorname{Spec} H \mid A \subseteq \mathfrak{p}\}.$$

Endow $\operatorname{Spec} H$ with the topology whose open sets are precisely the supports $\mathbb{S}A$, as A ranges overs subsets of H.

Then $\operatorname{Spec} H$ is the spectrum, or spectral space, of H.

Given ℓ-group H, write $\operatorname{Min} H$ for the subspace of $\operatorname{Spec} H$ consisting of minimal primes.

$\operatorname{Spec} H$ is a completely normal generalised spectral space.

Units, and compactness of spectra

P. Conrad and J. Martinez, *Complemented lattice-ordered groups*, Indag. Math., N.S., 1, 281–298, 1990

For any ℓ-group H:

- $\operatorname{Spec} H$ is compact precisely when H has a strong unit.
- $\operatorname{Min} H$ compact precisely when H is complemented: for every $x \in H^+$ there is $y \in H^+$ with $x \wedge y = 1$ and $x \vee y$ a weak unit.

Can we spectrally characterise ℓ-groups that admit a weak unit? (Cf. the case of strong units.)

A prime $\mathfrak{p} \in \operatorname{Spec} H$ is minimal precisely when

$$\mathfrak{p} = \bigcup \left\{ x^{\perp} \mid x \notin \mathfrak{p} \right\}.$$

(Here, $x^{\perp}$ is the set of elements y orthogonal to x, i.e such that $|x| \wedge |y| = 1$.) A prime $\mathfrak{p} \in \operatorname{Spec} H$ is quasiminimal if

$$\mathfrak{p} = \bigcup \left\{ x^{\perp\perp} \mid x \in \mathfrak{p} \right\}.$$

Write $\operatorname{Qin} H$ for the subspace $\operatorname{Spec} H$ consisting of quasiminimal primes.

$$\operatorname{Min} H \subseteq \operatorname{Qin} H \subseteq \operatorname{Spec} H.$$

Inclusions are proper, in general.

A. Colacito and VM, *Orders on groups, and spectral spaces of lattice-groups*, Algebra Universalis 81, 30 pp., 2020

Theorem (A. Colacito and VM, 2020)

An ℓ-group H has a weak unit precisely when $\mathrm{Qin}\, H$ is compact.

The nature of these results on units and compactness of spectra is essentially lattice-theoretic. For instance, for weak units:

T. Speed, *Some remarks on a class of distributive lattices*, J. Aust. Math. Soc. 9, 289–296, 1969.

Research suggestion

Abstract the theorem above to distributive lattices with bottom.

From right preorders to ℓ-groups…

$$\eta\colon G \longrightarrow \mathrm{F}_{\vee}\, G$$
$$\mathscr{P}_{\vee}G \qquad\qquad \operatorname{Spec} \mathrm{F}_{\vee}\, G.$$

We send each $\leq\, \in \mathscr{P}_{\vee}G$ to a prime $\mathfrak{p} \in \operatorname{Spec} \mathrm{F}_{\vee}\, G$.

Start from $G \to [\![\widehat{G}]\!]$; the universal arrow η yields an ℓ-group homomorphism $h\colon\ \mathrm{F}_{\vee}\, G \to [\![\widehat{G}]\!]$.

Let $\mathfrak{q}$ be the stabiliser of the identity in $[\![\widehat{G}]\!]$, which is a prime; pull it back along h:

$$\mathfrak{p} := h^{-1}[\mathfrak{q}].$$

This gives a map

$$\kappa\colon\ \mathscr{P}_{\vee}G \longrightarrow \operatorname{Spec} \mathrm{F}_{\vee}\, G.$$

...and back

$$\eta\colon G \longrightarrow \mathrm{F}_{\vee}\, G$$
$$\mathscr{P}_{\vee}G \qquad\qquad \operatorname{Spec}\mathrm{F}_{\vee}\, G.$$

We send each $\mathfrak{p} \in \operatorname{Spec}\mathrm{F}_{\vee}\, G$ to a right preorder $\preceq \in \mathscr{P}_{\vee}G$.

Start from $\mathfrak{p}$, and use η to define the following relation $\preceq$ on G:

$$x \preceq y \qquad \text{iff} \qquad \mathfrak{p}\eta a \leqslant \mathfrak{p}\eta b.$$

Then it can be proved that $\preceq \in \mathscr{P}_{\vee}G$.

This gives a map

$$\pi\colon\ \operatorname{Spec}\mathrm{F}_{\vee}\, G \longrightarrow \mathscr{P}_{\vee}G.$$

Free ℓ-groups and right preorders

Theorem (A. Colacito and V.M., 2020)

The maps κ and π are mutually inverse homeomorphisms and order-isomorphisms.

$$\begin{tikzcd} \mathscr{P}_{\mathsf{V}} G \arrow[rr, bend right, "\kappa"'] & & \mathrm{Spec}\, \mathrm{F}_{\mathsf{V}}\, G \arrow[ll, bend right, "\pi"'] \end{tikzcd}$$

S. McCleary, *An even better representation for free lattice-ordered groups*, Trans. Am. Math. Soc. 290, 81–100, 1985.

Would-be correspondence theory

- Representable right preorders:

$$\forall a \in G \begin{cases} \text{either} & \forall b \in G \quad 1 \preceq bab^{-1}, \\ \text{or} & \forall b \in G \quad bab^{-1} \preceq 1 \end{cases}$$

- Abelian right preorders:

$$\forall a, b \in G \quad 1 \preceq [a, b] \preceq 1.$$

Research suggestion

For which V is the class of right V-preorders first-order definable in the language of groups with a binary relation $\preceq$?

TAKING SUBVARIETIES SERIOUSLY

$\operatorname{Spec}^* H := \operatorname{Idl} H \cap \operatorname{Spec} H$ is the normal spectrum of H.

A right preorder on G is just a preorder if it is antisymmetric. Write

$$\mathscr{B}G$$

for the set of preorders on G that extend the partial order of G.

Then if V=representable ℓ-groups, the maps κ and π descend as follows:

$$\begin{tikzcd}
\mathscr{P}_{\mathsf{V}} G \arrow[rr, bend right, "\kappa"'] & & \operatorname{Spec} \mathrm{F}_{\mathsf{V}} G \arrow[ll, bend right, "\pi"'] \\
\mathscr{B}G \arrow[u, "\subseteq"'] \arrow[rr, bend right] & & \operatorname{Spec}^* \mathrm{F}_{\mathsf{V}} G \arrow[ll, bend right] \arrow[u, "\subseteq"']
\end{tikzcd}$$

Taking subvarieties seriously

$\operatorname{Spec}^* H := \operatorname{Idl} H \cap \operatorname{Spec} H$ is the normal spectrum of H.

A right preorder on G is just a preorder if it is antisymmetric. Write

$$\mathscr{B}G$$

for the set of preorders on G that extend the partial order of G.

Research suggestion

For significant V's, identify "V-convex sublattice subgroups" and develop their theory.

Would-be geometry of representable ℓ-groups

G_n —the free n-generated group (no order).

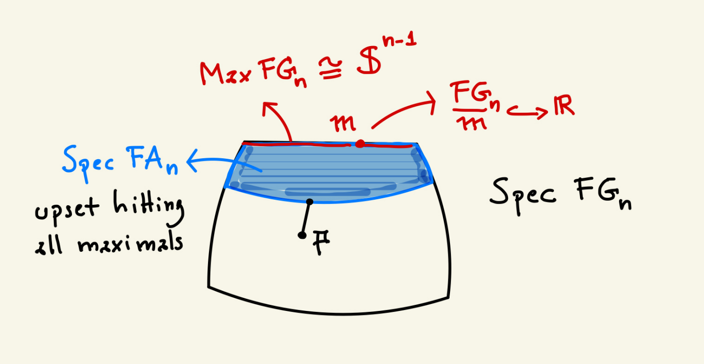
A_n —the free n-generated Abelian group ($\mathbb{Z}^n$).

$\mathrm{F}\,G_n$ —the free n-generated representable ℓ-group.

$\mathrm{F}\,A_n$ —the free n-generated Abelian ℓ-group.

There is a very significant theory of the geometric representation of $\mathrm{F}\,A_n$ by piecewise-linear functions.

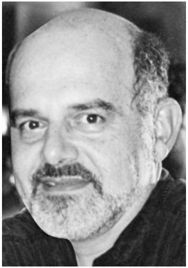
This is known as Baker-Beynon Duality, and applies to all finitely presented Abelian ℓ-groups.



Research suggestion

Can $\mathrm{F}\,G_n$ be represented by piecewise-linear actions on $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$?

W. C. Holland (1935–2020)



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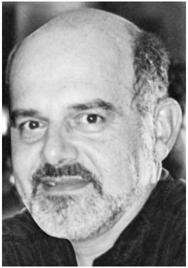

J. Martinez (1945–2020)

Thank you for your attention.