Choice-free duality for orthocomplemented lattices by means of spectral spaces

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Duality theory and ZFC

- Assuming Alexander's Subbase Theorem, Goldblatt (1975) developed a topological representation theory for ortholattices via the clopen orthoregular subsets a Stone space that comes endowed with a binary orthogonal relation.
- Bimbó (2007) then developed a full duality for ortholattices by introducing a new class of partially ordered topological orthospaces.
- Bezhanishvili and Holliday (2020) recently developed a choice-free duality for Boolean algebras using spectral spaces. Their techniques were informed by Stone (1937), Tarski (1937,1938), and Vietoris (1922).
- Our choice-free duality for ortholattices (2021) combines the techniques of Bezhanishvili and Holliday with Goldblatt and Bimbó.

Orthocomplemented lattices (ortholattices)

Definition

An *ortholattice* is an algebra $(L, \wedge, \vee, ^{\perp}, 0, 1)$ of type $(2, 2, 1, 0, 0)$ satisfying the following conditions:

1. $(L, \wedge, \vee, 0, 1)$ is a bounded lattice
2. the operator $(-)^{\perp} : L \to L$ is an *orthocomplementation*, i.e.,
   1. $a \wedge a^{\perp} = 0$, $a \vee a^{\perp} = 1$
   2. $a \leq b \Longrightarrow b^{\perp} \leq a^{\perp}$
   3. $a^{\perp\perp} = a$

Example

Lattices of subspaces of various inner product spaces such as the lattice of closed linear subspaces of a Hilbert space.

Orthospaces and orthoregularity

Definition

An *orthospace* is pair $(X, \perp)$ such that X is a set and $\perp \subseteq X^2$ is a binary orthogonal relation which is irreflexive and symmetric. Moreover:

1. For every $x \in X$ and $Y \subseteq X$, let $x \perp Y$ iff $x \perp y$ for all $y \in Y$
2. For each $Y \subseteq X$, define $Y^* = \{x \in X \mid x \perp Y\}$
3. A subset $Y \subseteq X$ is *orthoregular* iff $Y = Y^{**}$

Example

Two vectors $x = [x_1, \ldots, x_n], y = [y_1, \ldots, y_n] \in \mathbb{R}^n$ are orthogonal if

$$x \cdot y = \sum_{i=1}^{n} x_i y_i = 0$$

Every closed linear subspace $X \subseteq \mathbb{R}^n$ is orthoregular in that $X^{\perp\perp} = X$ where

$$X^{\perp} = \{x \in \mathbb{R}^n \mid x \cdot y = 0,\ \forall y \in X\}$$

Goldblatt-Bimbó duality for ortholattices

Definition (patch space of $\mathfrak{F}(L)$)

Let L be an ortholattice, let $\mathfrak{F}(L)$ be the set of proper lattice filters of L, and let $\widehat{a} = \{x \in \mathfrak{F}(L) \mid a \in x\}$. If we define $\perp_L \subseteq \mathfrak{F}(L) \times \mathfrak{F}(L)$ by

$$x \perp_L y \iff \exists a \in L (a^\perp \in x \;\&\; a \in y)$$

then $X_L^\pm = (X_L^\pm, \perp_L)$ is the space of proper lattice filters of L whose topology is generated by $\{\widehat{a} \mid a \in L\} \cup \{\complement\widehat{a} \mid a \in L\}$ as a subbasis.

Theorem [Goldblatt]

If L is an ortholattice, then $X_L^\pm$ is a Stone space. Moreover, every ortholattice L can be represented (up to isomorphism) as $\mathrm{CLOP}\mathscr{R}(X_L^\pm)$

Theorem [Bimbó]

The category of Goldblatt-Bimbó spaces and continuous p-morphisms is dually equivalent to the category of ortholattices and ortholattice homomorphisms.

Spectral spaces

Definition (spectral space)

A topological space X is a *spectral space* if X is T_0, compact, coherent, and sober in the sense that every completely prime filter in the lattice $\mathscr{O}(X)$ can be written as $\mathscr{O}_X(x) = \{U \in \mathscr{O}(X) \mid x \in U\}$ for some point $x \in X$.

Theorem [Hoschter]

A topological space X is a spectral space iff X is homeomorphic to the spectrum of a commutative ring R, i.e., the space $(\mathrm{Spec}(R), \mathscr{T}_{\mathrm{Zariski}})$

Theorem [Stone]

Every distributive lattice can be represented (up to isomorphism) as $\mathscr{C}\mathscr{O}(X)$ for some spectral space X.

Theorem [Bezhanishvili and Holliday]

Every Boolean algebra can be represented (up to isomorphism) as $\mathscr{C}\mathscr{O}\mathrm{RO}(X)$ for some spectral space X.

Representation of ortholattices via spectral spaces

Definition (spectral space of $\mathfrak{F}(L)$)

Let L be an ortholattice, $X_L^+ = (X_L^+, \perp_L)$ is the space of proper lattice filters of L whose topology is generated by $\{\widehat{a} \mid a \in L\}$ as a basis.

Lemma

For every ortholattice L, the space X_L^+ is a spectral space whose specialization order is given by set-theoretic inclusion.

For compactness, we show that every basic open cover $\widehat{a} \subseteq \bigcup_{i \in I} \widehat{b_i}$ of X_L^+ admits of a finite subcover, thereby avoiding use of AST.

Theorem (representation theorem for L)

Every ortholattice L can be represented up to isomorphism as $(\mathscr{C}\mathscr{O}\mathscr{R}(X_L^+), \cap, ^*, \emptyset)$ ordered by set-theoretic inclusion.

We show that the map $\widehat{\bullet} \colon L \to \mathscr{C}\mathscr{O}\mathscr{R}(X_L^+)$ gives the desired isomorphism where $\widehat{a \wedge b} = \widehat{a} \cap \widehat{b}$, $\widehat{a^\perp} = (\widehat{a})^*$, and $\widehat{0} = \emptyset$.

Upper Vietoris Orthospaces

Definition (UVO-space)

Let $X = (X, \leqslant, \perp, \mathscr{T})$ be an ordered topological space endowed with an orthogonal binary relation $\perp \subseteq X^2$ and whose specialization order is $\leqslant$, then X is a *UVO-space* whenever the following conditions are satisfied:

1. X is a T_0 space
2. $\mathscr{COR}(X)$ is closed under $\cap$, *, and is a basis for X
3. Every proper filter in $\mathscr{COR}(X)$ for some point $x \in X$ is of the form:

$$\mathscr{COR}_X(x) = \{U \in \mathscr{COR}(X) \mid x \in U\}$$

4. $x \perp y \Longrightarrow \exists U \in \mathscr{COR}(X) : x \in U \ \& \ y \in U^*$

Remark

Note that the requirement that the compact open orthoregular subsets of a UVO-space X form a basis implies our analogue of the Priestley separation axiom: $x \not\leqslant y \Longrightarrow \exists U \in \mathscr{COR}(X) : x \in U \ \& \ y \notin U$

The topological characterization theorem for X_L^+

Lemma

If X is a UVO-space, then $(\mathscr{COR}(X), \cap, ^*, \emptyset)$ is an ortholattice. Moreover, if L is an ortholattice, then X_L^+ is a UVO-space.

Theorem (Characterization of X_L^+)

For each UVO-space X, the map $X \longrightarrow X^+_{\mathscr{COR}(X)}$ is a homeomorphism and an isomorphism with respect to $(X, \perp)$ and $(X^+_{\mathscr{COR}(X)}, \perp)$.

We prove that $x \mapsto \mathscr{COR}_X(x) = \{U \in \mathscr{COR}(X) \mid \exists x \in X : x \in U\}$ gives the desired homeomorphism and isomorphism from X to $X^+_{\mathscr{COR}(X)}$.

Corollary

If X is a UVO-space, then X is a spectral space. Moreover, every element in $\mathscr{CO}(X)$ is a finite union of elements in $\mathscr{COR}(X)$.

UVO-mappings

Definition (spectral map)

If X and X' are spectral spaces, then a mapping $f: X \to X'$ is a *spectral map* if $f^{-1}[U] \in \mathscr{C}\mathscr{O}(X)$ for every $U \in \mathscr{C}\mathscr{O}(X')$.

Definition (p-morphism)

If (X, R) and (X', R') are Kripke frames, then a function $f: (X, R) \to (X', R')$ is a *p-morphism* if:

1. $xRy \Longrightarrow f(x)R'f(y)$
2. $f(x)R'y' \Longrightarrow \exists y \in X : xRy \;\&\; f(y) = y'$

Definition (UVO-map)

If X and X' are UVO-spaces, then a mapping $f: X \to X'$ is a *UVO-map* if f is a spectral map and also a p-morphism for $(X, \not\perp)$ and $(X', \not\perp')$, i.e.,

1. $x \not\perp y \Longrightarrow f(x) \not\perp' f(y)$
2. $f(x) \not\perp' y' \Longrightarrow \exists y \in X : x \not\perp y \;\&\; f(y) = y'$

The main result

Theorem (duality theorem)

The category of UVO-spaces and UVO-mappings is dually equivalent to the category of ortholattices and ortholattice homomorphisms.

We prove that every OL homomorphism $h : L \to L'$ defined by $h_+(x') = h^{-1}[x']$ induces a UVO-map $h_+ : X^+_{L'} \to X^+_L$ and thus a contrvariant functor $(-)^+ : \mathbf{OrthLatt} \to \mathbf{UVO}$. Moreover, we prove that every UVO map $f : X \to X'$ defined by $f^+(U') = f^{-1}[U']$ induces an ortholattice homomorphism $f^+ : \mathscr{COR}(X') \to \mathscr{COR}(X)$ and thus a contravariant functor $\mathscr{COR}(-) : \mathbf{UVO} \to \mathbf{OrthLatt}$. Since every ortholattice L is isomorphic to $\mathscr{COR}(X^+_L)$ and every UVO-space X is homeomorphic to $X^+_{\mathscr{COR}(X)}$, it is easy to check that the following diagrams commute:

$$\begin{array}{ccc} L & \xrightarrow{h} & L' \\ \downarrow & & \downarrow \\ \mathscr{COR}(X^+_L) & \xrightarrow[(h_+)^+]{} & \mathscr{COR}(X^+_{L'}) \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ X^+_{\mathscr{COR}(X)} & \xrightarrow[(f^+)_+]{} & X^+_{\mathscr{COR}(X')} \end{array}$$

Duality dictionary

OrthLatt	**UVO**
ortholattice	UVO-space
homomorphism	UVO-map
complete lattice	complete UVO-space
atom	isolated point
atomless lattice	$X_{\text{iso}} = \emptyset$
atomic lattice	$\mathrm{Cl}(X_{\text{iso}}) = X$
injective homomorphism	surjective UVO-map
surjective homomorphism	UVO-embedding
subalgebra	image under UVO-map
direct product	UVO-sum
canonical extension	$\mathscr{R}(X)$
MacNeille completion	$\mathscr{R}(\mathfrak{P}(X))$

UVO-sums

Definition (UVO-sum)

If X and Y are UVO-spaces, then their *UVO-sum* $X+Y$ is the space whose underlying carrier set is $X+Y := X \cup Y \cup (X \times Y)$ and whose topology is generated by sets of the form $U \cup V \cup (U \times V)$ for $U \in \mathscr{COR}(X)$ and $V \in \mathscr{COR}(Y)$, together with the orthogonality relation $\perp_{X+Y}$, which is defined as the symmetric closure of:

$$\begin{aligned}\perp_X \cup \perp_Y \cup (X \times Y) \\ &\cup \{\langle\langle x,y\rangle, x'\rangle \mid x \perp_X x'\} \cup \{\langle\langle x,y\rangle, y'\rangle \mid y \perp_Y y'\} \\ &\cup \{\langle x,y\rangle, \langle x',y'\rangle \mid x \perp_X x', y \perp_Y y'\}.\end{aligned}$$

Proposition (specializaiton order of UVO-sums)

Let X and Y be UVO-spaces. Then, the specialization order $\leqslant_{X+Y}$ of their UVO-sum $X+Y$ is given by:

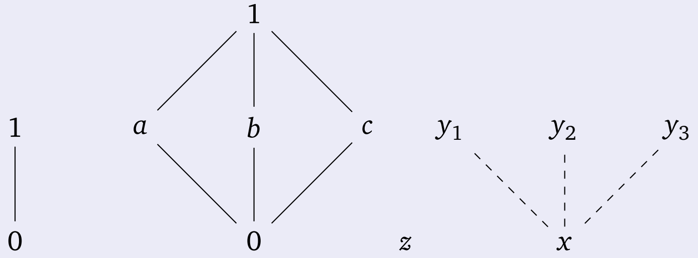
$$\begin{aligned}\Omega_{\leqslant} := \leqslant_X \cup \leqslant_Y \cup \{\langle\langle x,y\rangle, x'\rangle \mid x \leqslant_X x'\} \cup \{\langle\langle x,y\rangle, y'\rangle \mid y \leqslant_Y y'\} \\ \cup \{\langle x,y\rangle, \langle x',y'\rangle \mid x \leqslant_X x', y \leqslant_Y y'\}\end{aligned}$$

Finite UVO-spaces

Proposition

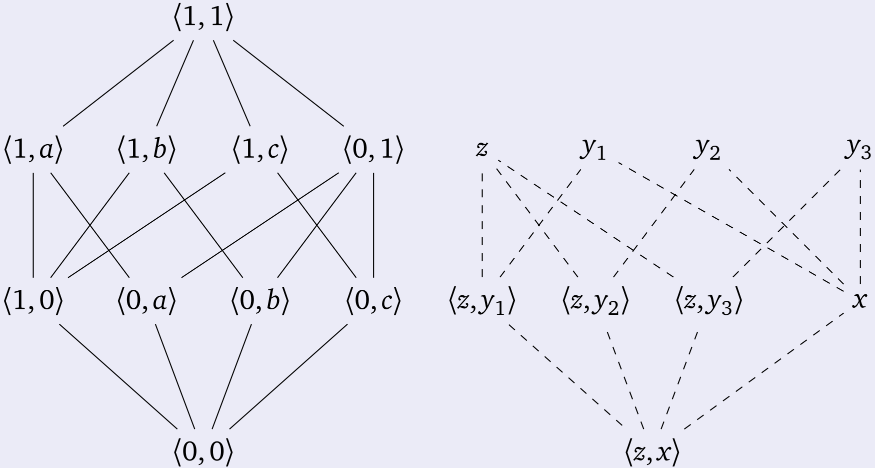
If X is a finite UVO-space, then $(X, \leqslant)$ can be constructed from an ortholattice by deleting its bottom universal bound and taking its order-theoretic dual.

Ortholattices O_2 and M_3 and UVO-spaces $X^+_{O_2}$ and $X^+_{M_3}$



$$\perp_{X^+_{M_3}} = \{\langle y_1, y_2\rangle, \langle y_2, y_3\rangle, \langle y_1, y_3\rangle\}, \qquad \perp_{X^+_{O_2}} = \emptyset$$

The direct product $O_2 \times M_3$ and UVO-sum $X^+_{O_2} + X^+_{M_3}$



$$\perp_{X^+_{O_2}+X^+_{M_3}} = \{\perp_{X^+_{M_3}}, \perp_{X^+_{O_2}}, \langle\langle z,y_1\rangle,y_2\rangle, \langle\langle z,y_1\rangle,y_3\rangle,$$
$$\langle\langle z,y_2\rangle,y_1\rangle, \langle\langle z,y_2\rangle,y_3\rangle, \langle\langle z,y_3\rangle,y_1\rangle, \langle\langle z,y_3\rangle,y_2\rangle\}$$

UVO-sums

Theorem

If L and L' are ortholattices and X_L^+ and $X_{L'}^+$ are their respective dual UVO-spaces, then there is a homeomorphism $f : X_{L\times L'}^+ \to X_L^+ + X_{L'}^+$ that is an isomorphism with respect to their orthospace reducts.

For each point $x \in X_{L\times L'}^+$, let $x_L = \{a \in L \mid \exists b \in L' : \langle a, b\rangle \in x\}$ and $x_{L'} = \{b \in L' \mid \exists a \in L : \langle a, b\rangle \in x\}$. We prove that f defined by

$$f(x) = \begin{cases} x_L & \text{if } x_{L'} \text{ is improper,} \\ x_{L'} & \text{if } x_L \text{ is improper,} \\ \langle x_L, x_{L'}\rangle & \text{otherwise.} \end{cases}$$

gives the desired homeomorphism and isomorphism.

Corollary

If X and Y are UVO-spaces, then their UVO-sum $X + Y$ is a UVO-space. Moreover, the map $\mathscr{COR}(X+Y) \to \mathscr{COR}(X) \times \mathscr{COR}(Y)$ is an ortholattice isomorphism.

Possible applications and future work

- Characterize the subclass of UVO-spaces which arise as the choice-free dual spaces of the modular and orthomodular lattices.
- Develop a theory of topological models based on UVO-spaces for which various quantum logics are complete.
- Study UVO-spaces within the context of ZFC by making explicit how Goldblatt-Bimbó spaces arise by taking certain patch spaces of UVO-spaces.
- Investigate the connections between lattices of varieties of ortholattices and lattices of varieties of modal algebras corresponding to KTB (the normal modal logic of reflexive symmetric Kripke frames) and its variants.

THANKS!

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