Reflective subcategories of bounded archimedean ℓ-algebras

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Introduction

Jorge Martínez and Tony Hager each visited us at NMSU back in the early 2010's. They, along with many others including Banaschewski, Ball, and Madden, have done considerable work with the category **W** of archimedean lattice-ordered groups with a weak-order unit and the subcategory $\mathbf{W}^*$ where the weak order-unit is a strong-order unit.

The category $\mathbf{W}^*$ is the group analogue of the category ***baℓ*** of bounded archimedean lattice-ordered $\mathbb{R}$-algebras which we study. After visiting us and learning about our first paper in the subject, they were particularly interested in a result of ours, which we will discuss after some background.

Rings of continuous functions

KHaus is the category of compact Hausdorff spaces and continuous maps.

If $X \in \mathbf{KHaus}$, then the set $C(X)$ of all real-valued continuous functions is an $\mathbb{R}$-algebra. It has a partial order $\leq$ given by

$$f \leq g \text{ if } f(x) \leq g(x) \text{ for all } x \in X.$$

$C(X)$ has lattice operations, given by

$$(f \vee g)(x) = \max\{f(x), g(x)\}$$

and

$$(f \wedge g)(x) = \min\{f(x), g(x)\}.$$

In addition, $C(X)$ is an ℓ-algebra, meaning

- $a \leq b$ implies $a + c \leq b + c$.
- $a \leq b$ and $0 \leq r \in \mathbb{R}$ implies $ra \leq rb$.

$C(X)$ is **bounded** (1 is a **strong order-unit**) in that $|f| \leq n$ for some $n \in \mathbb{N}$ since X is compact.

$C(X)$ is **archmedean** ($f \leq 1/n$ for all $n \geq 1$ implies $f \leq 0$) since functions are real-valued.

Definition. $\boldsymbol{ba\ell}$ is the category of bounded archimedean ℓ-algebras and unital ℓ-algebra homomorphisms.

Uniformly complete objects of $\boldsymbol{ba\ell}$

$C(X)$ is complete with respect to the **uniform norm**

$$\|f\| = \sup\{|f(x)| : x \in X\} = \inf\{r \in \mathbb{R} : |f| \leq r\}.$$

If $A \in \boldsymbol{ba\ell}$, we set $|a| = a \vee -a$.

Each $A \in \boldsymbol{ba\ell}$ has a norm, given by $\|a\| = \inf\{r \in \mathbb{R} : |a| \leq r\}$.

Definition. $\boldsymbol{uba\ell}$ is the full subcategory of uniformly complete objects of $\boldsymbol{ba\ell}$.

C is a functor from **KHaus** to $\boldsymbol{uba\ell}$.

The Yosida space of $A \in \boldsymbol{ba\ell}$

An ℓ-ideal of $A \in \boldsymbol{ba\ell}$ is a ring ideal I such that $|a| \leq |b|$ and $b \in I$ imply $a \in I$.

Definition. The **Yosida space** Y_A of $A \in \boldsymbol{ba\ell}$ is the set of maximal ℓ-ideals with the Zariski topology.

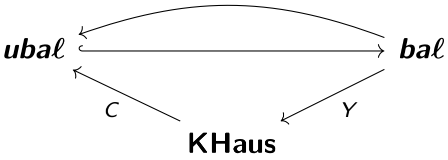
Compactness of Y_A is a standard Zariski topology argument.

Hölder's theorem gives $A/M \cong \mathbb{R}$ for each $M \in Y_A$, which implies that Y_A is Hausdorff. This and $\bigcap Y_A = 0$ yield that A embeds in $C(Y_A)$.

By the Stone-Weierstrass theorem, the image of A is dense in $C(Y_A)$.

Y is a functor from $\boldsymbol{ba\ell}$ to **KHaus**.

Theorem. The functors C and Y yield a dual adjunction between **KHaus** and $\boldsymbol{ba\ell}$ which restricts to a dual equivalence between **KHaus** and $\boldsymbol{uba\ell}$.



Bimorphisms in ***baℓ***

A morphism is a **bimorphism** if it is both monic and epic.

Bimorphisms in ***baℓ*** are 1-1 but need not be onto functions.

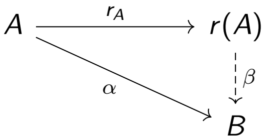
If $\alpha : A \to B$ is a bimorphism in ***baℓ*** with $A \in$ ***ubaℓ***, then α is onto, and so is an isomorphism.

Lemma. A morphism $\alpha : A \to B$ in ***baℓ*** is a bimorphism iff $Y(\alpha) : Y_B \to Y_A$ is a homeomorphism.

Reflective subcategories

Let **C** be a subcategory of **D**. Recall that **C** is a **reflective subcategory** if the inclusion functor has a left adjoint.

If $r : \mathbf{D} \to \mathbf{C}$ is a reflector then, for each $A \in \mathbf{D}$, there is a morphism $r_A : A \to r(A)$ in **D** such that for each $B \in \mathbf{C}$ and each morphism $\alpha : A \to B$ in **D**, there is a unique morphism $\beta : r(A) \to B$ for which $\beta \circ r_A = \alpha$.



C is a **monoreflective** subcategory if r_A is monic for each $A \in \mathbf{C}$.

C is a **bireflective** subcategory if r_A is a bimorphism for each $A \in \mathbf{C}$.

The lemma

We will assume all subcategories of $\boldsymbol{ba\ell}$ in this talk are full, replete, and nontrivial (meaning they consist of more than just 1 object and its isomorphic copies).

In our paper we characterized how $\boldsymbol{uba\ell}$ sits in $\boldsymbol{ba\ell}$. To do this we needed the following result in which Hager and Martínez were particularly interested.

Lemma. Every reflective subcategory of $\boldsymbol{ba\ell}$ is bireflective.

Lemma. Every reflective subcategory of $\boldsymbol{ba\ell}$ is bireflective.

Proof. Let $\mathbf{R}$ be a reflective subcategory of $\boldsymbol{ba\ell}$ and let $r : \boldsymbol{ba\ell} \to \mathbf{R}$ be the reflector.

Set $R := r(\mathbb{R}) \in \mathbf{R}$. Since $\mathbb{R}$ is an initial object in $\boldsymbol{ba\ell}$, R is an initial object in $\mathbf{R}$. Thus, $\hom_{\mathbf{R}}(R, R)$ is a singleton.

As $\mathbf{R}$ is a full subcategory of $\boldsymbol{ba\ell}$, we see that $\hom_{\boldsymbol{ba\ell}}(R, R)$ is a singleton.

If $M \in Y_R$, we have the canonical map $\pi : R \to \mathbb{R}$ with kernel M. Therefore, $r_{\mathbb{R}} \circ \pi \in \hom_{\boldsymbol{ba\ell}}(R, R)$, and its kernel is M.

If Y_R has two points, then we get two maps from R to R in this way, and these maps are different since their kernels are not equal. Thus, Y_R is a singleton.

Therefore, $C(Y_R, \mathbb{R}) \cong \mathbb{R}$, and since R embeds in $C(Y_R, \mathbb{R})$, it follows that $R \cong \mathbb{R}$.

We now prove **R** is monoreflective.

Let $A \in \boldsymbol{ba\ell}$. Then for each $M \in Y_A$, there is an onto morphism $\alpha : A \to R$ with kernel M since $R \cong \mathbb{R}$.

Because **R** is reflective, there is a morphism $\beta : r(A) \to R$ with $\alpha = \beta \circ r_A$.

Let $N = \ker(\beta)$. Then $N \in Y_{r(A)}$. Also, $\beta(r_A(M)) = \alpha(M) = 0$, so that $M \subseteq r_A^{-1}(N)$.

Since M is maximal, $M = r_A^{-1}(N)$. It follows that $Y(r_A) = r_A^{-1} : Y_{r(A)} \to Y_A$ is onto.

Therefore, r_A is monic. Consequently, **R** is monoreflective.

A result of category theory shows that a monoreflective subcategory is necessarily bireflective. Thus, **R** is bireflective.

Characterization of $\boldsymbol{ubal}$

Theorem. $\boldsymbol{ubal}$ is the smallest reflective subcategory of $\boldsymbol{bal}$.

Proof By the lemma $\boldsymbol{ubal}$ is a bireflective subcategory of $\boldsymbol{bal}$.

Let $\mathbf{B}$ be a reflective subcategory of $\boldsymbol{bal}$ with reflector r. We claim that $\boldsymbol{ubal} \subseteq \mathbf{B}$.

Let $C \in \boldsymbol{ubal}$. By the lemma, $\mathbf{B}$ is a bireflective subcategory of $\boldsymbol{bal}$.

Therefore, there exists a bimorphism $r_C : C \to r(C)$. Since $C \in \boldsymbol{ubal}$, this is an isomorphism. Since $\mathbf{B}$ is full replete by assumption, we have $C \in \mathbf{B}$.

The previous theorem distinguishes $\boldsymbol{ubal\ell}$ as the smallest reflective subcategory of $\boldsymbol{bal\ell}$.

$A \in \boldsymbol{bal\ell}$ is **epicomplete** if each epimorphism $\alpha : A \to B$ is onto.

Theorem. Let $A \in \boldsymbol{bal\ell}$. Then $A \in \boldsymbol{ubal\ell}$ iff A is epicomplete.

A subcategory of $\boldsymbol{bal\ell}$ is epicomplete if each of its objects is epicomplete.

Corollary. $\boldsymbol{ubal\ell}$ is the unique reflective epicomplete subcategory of $\boldsymbol{bal\ell}$.

A result of Hager and Martínez

Hager and Martínez, in a 2014 paper, recognized that our lemma can be stated in more general terms.

For this they introduced the notion of a Hölder category.

They proved that reflectors are bireflectors in a Hölder category.