Profiniteness and spectra of Heyting algebras

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A **Heyting algebra** is a structure $\boldsymbol{A} = \langle A; \wedge, \vee, \rightarrow, 0, 1\rangle$ that comprises a bounded lattice $\langle A; \wedge, \vee, 0, 1\rangle$ and satisfies the **residuation law**

$$a \wedge b \leqslant c \Longleftrightarrow a \leqslant b \rightarrow c.$$

As a consequence, the lattice underlying $\boldsymbol{A}$ is distributive.
1974: Esakia proves that the category of Heyting algebras is dually equivalent to that of **Esakia spaces**, i.e., Priestley spaces s.t.

$$\text{if } U \text{ is open, then so is } {\downarrow}U.$$

- The prime spectrum $\boldsymbol{A}_+$ of a Heyting algebra $\boldsymbol{A}$ becomes an Esakia space, when endowed with the topology generated by

$$\{\epsilon(a), \epsilon(a)^c : a \in A\} \text{ where } \epsilon(a) = \{F \in \boldsymbol{A}_+ : a \in F\}.$$

- If X is an Esakia space, then the collection $\mathsf{ClUp}(X)$ of clopen upsets of X can be viewed as a Heyting algebra

$$\langle \mathsf{ClUp}(X); \cap, \cup, \rightarrow, \emptyset, X\rangle$$

in which $U \rightarrow V$ is defined as $X \smallsetminus {\downarrow}(U \smallsetminus V)$.

Definition

An algebra is said to be **profinite** if it is the inverse limit of an inverse system of finite algebras.

Profinite Heyting algebras admit an transparent description.

Definition

A poset X is said to be **image finite** if $\uparrow x$ is finite, for all $x \in X$.

Example. For every cardinal κ, the disjoint union of κ copies of the order dual of the binary tree is image finite.

Theorem (G. & N. Bezhanishvili 2008)

A Heyting algebra is profinite iff it is isomorphic to

$$\langle \mathsf{Up}(X); \cap, \cup, \to, \emptyset, X \rangle,$$

for some **image finite** poset X.

Definition

The **profinite completion** of an algebra $\boldsymbol{A}$ is the inverse limit of the inverse system of finite algebras of the form $\boldsymbol{A}/\theta$, where θ is a congruence of $\boldsymbol{A}$.

- It follows that every profinite completion is profinite, while the converse need not be true.

Problem. Does the converse hold for Heyting algebras?

- Given a poset X, we denote by X_{fin} its subposet with universe

$$\{x \in X : \uparrow x \text{ is finite}\}.$$

Theorem (G. Bezhanishvili, Gehrke, Mines, Morandi 2006)

A Heyting algebra $\boldsymbol{A}$ is a profinite completion iff $\boldsymbol{A} \cong \text{Up}(X_{\text{fin}})$ for some Esakia space X.

Corollary

Given a poset X, the Heyting algebra $\mathrm{Up}(X)$ is a profinite completion iff $X = Y_{\mathrm{fin}}$, for some Esakia space Y.

- Hence, to construct a profinite Heyting algebra that is not a profinite completion, it suffices to exhibit an image finite poset X different from the image finite part of every Esakia space.

We had to learn to recognize posets underlying Esakia spaces. But...

1985: Esakia asked for a description of posets isomorphic to prime spectra of Heyting algebras.

- In view of Esakia duality, these are precisely the order reducts of Esakia spaces.

Definition

A poset is said to be **Esakia representable** if it is isomorphic to the prime spectrum of a Heyting algebra.

Necessary conditions for a poset of be Esakia representable.

Definition

A poset X is **enough gaps** when

$$\text{if } x < y \text{, there are } x' \geqslant x \text{ and } y' \leqslant y \text{ s.t. } x' < y' \text{ and } [x', y'] = \{x', y'\}.$$

- Esakia representable posets have enough gaps. Consequently, no nontrivial dense linear order is Esakia representable.

A subset of a poset X is **order closed** in X if it belongs to the least family $\mathcal{C} \subseteq \mathcal{P}(X)$ s.t.

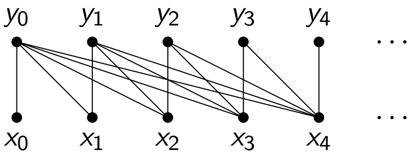
- $\uparrow x, \downarrow x \in \mathcal{C}$, for all $x \in X$;
- if $U \in \mathcal{C}$, then $\uparrow U, \downarrow U \in \mathcal{C}$;
- $\mathcal{C}$ is closed under finite unions and arbitrary intersections.

Definition

A poset X is said to be **order compact** when, for every family $\{U_i : i \in I\}$ of order closed sets,

$$\text{if } \bigcap_{i \in I} U_i = \emptyset, \text{ there exists } U_1, \ldots, U_n \text{ s.t. } U_1 \cap \cdots \cap U_n = \emptyset.$$

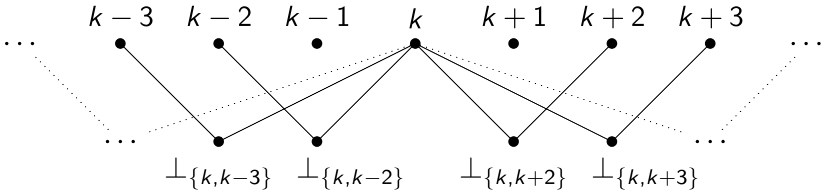
- Esakia representable posets are order compact.



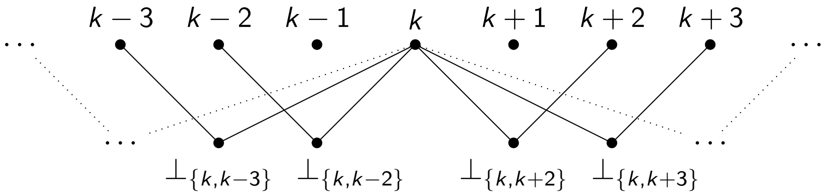
Remark. If a poset X is order compact, then **infima** and **suprema** of nonempty chains exist in X.

Recap. We need to exhibit an image finite poset X that is not the image finite part of any Esakia representable poset.

- Let X be the following poset.



- Suppose $X = Y_{\text{fin}}$ for some Esakia space Y.
- A trick based on depth and width shows that X is the order reduct of an Esakia space.
- Thus, X is Esakia representable, whence order compact.



- Observe that

$$\bigcap_{k \in Z} \uparrow\downarrow k = \emptyset.$$

- There there must be $k_1, \ldots, k_n \in Z$ such that

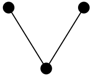
$$\uparrow\downarrow k_1 \cap \cdots \cap \uparrow\downarrow k_n = \emptyset,$$

which is impossible, because $\max\{k_1, \ldots, k_n\} + 2$ belongs to the above intersection. QED

Corollary

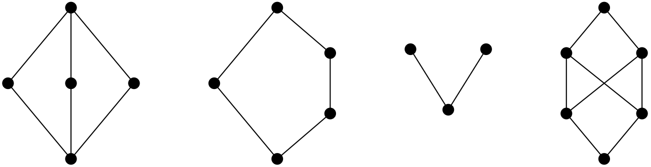
There are profinite Heyting algebras that are not profinite completions, e.g., $\mathrm{Up}(X)$.

Remark. The algebra $\mathsf{Up}(X)$ is a embeds into a direct power of the Heyting algebra of upsets of the **two-fork** F_2.

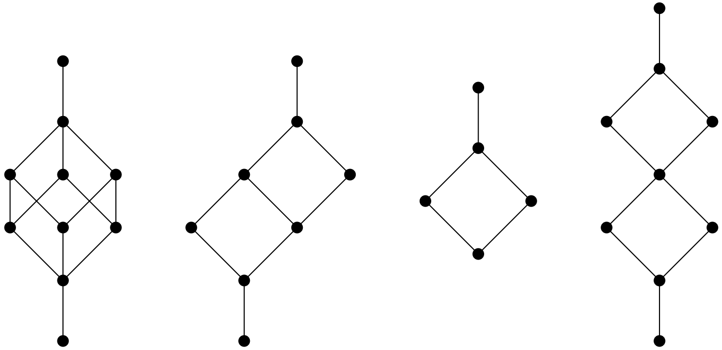


- Since varieties are closed under I, S and P, if the profinite members of a variety K of Heyting algebras are profinite completions, then K omits $\mathsf{Up}(F_2)$.

Even more is true: if the profinite members of a variety K are profinite completions, K omits $\mathsf{Up}(Z)$ for the posets Z below.



That is, K omits the finite subdirectly irreducible algebras below.

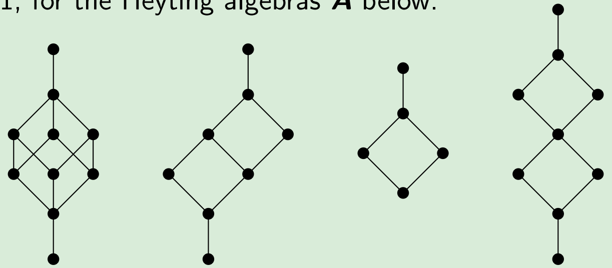


Theorem (Jankov 1963)

Every finite subdirectly irreducible Heyting algebra $\boldsymbol{A}$ is **splitting**: there exists the largest variety K of Heyting algebras omitting $\boldsymbol{A}$. K is axiomatized by the equation $J(\boldsymbol{A}) \approx 1$, where $J(\boldsymbol{A})$ is the **Jankov formula** of $\boldsymbol{A}$.

Definition

A Heyting algebra is said to be **diamond** if it satisfies the equations $J(\boldsymbol{A}) \approx 1$, for the Heyting algebras $\boldsymbol{A}$ below.



Proposition

If the profinite members of a variety K of Heyting algebras are profinite completions, then K consists of diamond Heyting algebras.

Aim. Prove the converse of the Proposition.

Diamond Heyting algebras can be recognized from the shape of their prime spectra.

Theorem

A Heyting algebra $\boldsymbol{A}$ is diamond iff its prime spectrum $\boldsymbol{A}_+$ is a diamond system.

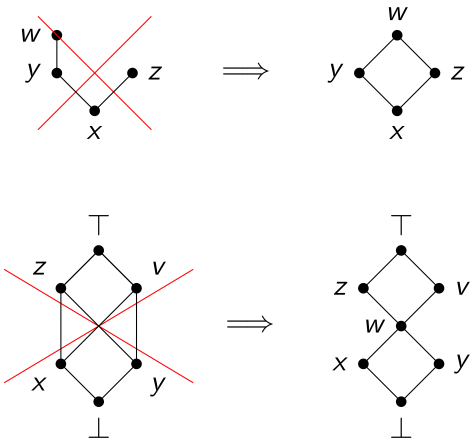
Definition

A poset X is said to be a **diamond system** if it satisfies the following conditions:

- Each principal upset $\uparrow x$ satisfies **Esakia's three point rule**: for each triple y, z, w of distinct elements of $\uparrow x$,

$$\text{if } y \parallel z \text{ and } y \leqslant w, \text{ then } z \leqslant w;$$

- Antichains in principal upsets have size $\leqslant 2$;
- Each principal upset is upward directed;
- For every $\bot, x, y, z, v, \top \in P$, if $\bot \leqslant x, y \leqslant z, v \leqslant \top$, there is a $w \in P$ such that $x, y \leqslant w \leqslant z, v$.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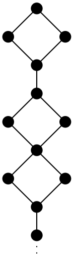
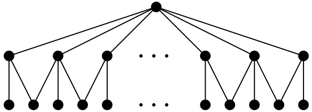
- Image finite downward directed diamond systems are linear sums of diamonds and lines (wherefrom their name).



- But in general things might get more complicated...



Theorem

A diamond system is Esakia representable iff it has enough gaps and infinima and suprema of its nonempty chains exist.

Main Theorem

The profinite members of a variety K of Heyting algebras are profinite completions iff K is a variety of diamond Heyting algebras.

Proof sketch. Consider a variety K of diamond Heyting algebras and a profinite member $\boldsymbol{A}$ of K.

- There exists an image finite poset X such that $\boldsymbol{A} \cong \mathrm{Up}(X)$.
- As $\boldsymbol{A}$ is a diamond Heyting algebra, X is a diamond system.
- For each infinite downdirected upset U of X, pick a new element $\bot_U$.
- Let X^+ be the extension of X obtained adding the various $\bot_U$ as follows:

$$x \leqslant^{X^+} y \iff x = y \text{ or } x \leqslant^X y \text{ or } x = \bot_U \text{ and } y \in U.$$

- The resulting poset X^+ is a diamond system with enough gaps and in which infinima and suprema of nonempty chains exist.
- Hence, there exists a topology τ such that $Y := \langle X^+, \leqslant, \tau\rangle$ is an Esakia space.
- As $X = Y_{\text{fin}}$, $\mathrm{Up}(X)$ is a profinite completion. QED

Some consequences.

- The problem of determining whether the profinite members of a variety of Heyting algebras (presented by a finite set of equations or by a finite set of algebras) are profinite completions is **decidable**.
- There exists the **largest** variety of Heyting algebras whose profinite members are profinite completions, namely, that of diamond Heyting algebras.
- There are denumerably many such varieties, all locally finite and finitely based.
- These varieties are primitive and have surjective epimorphisms.

Corollary

Intermediate logics algebraized by varieties of Heyting algebras whose profinite members are profinite completions are locally tabular, finitely axiomatizable, have the infinite Beth definability property, and are hereditarily structurally complete.

Thank you very much for your attention!