

The universal completion of $C(L)$ and the localic representation of Riesz spaces

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- $\varphi: P \rightarrow C$ is a join- and meet-dense order embedding.

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Forum Mathematicum **27** (2015), 2551-.2585

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Normal semicontinuity and the Dedekind completion of pointfree function rings
Algebra Universalis **75** (2016), 301–330



J. H. van der Walt

The universal completion of $C(X)$ and unbounded order convergence
Journal of Mathematical Analysis and Applications **460** (2018), 76–97



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Theorem (van der Walt, 2018)

If X is completely regular, then $NL(X)$ is the universal completion of $C(X)$.



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If X is completely regular, then $NL(X)$ is the universal completion of $C(X)$.

$NL(X)$ = the Riesz space of nearly finite normal lower semicontinuous functions on X .



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On the universal completion of pointfree function spaces,
Journal of Pure and Applied Algebra **225** (2021)

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- U is a universally complete Riesz space;
- $\mu: R \rightarrow U$ is a Riesz space embedding such that, for every $f \in U^+$,

$$f = \bigvee \{ \mu(g) \mid g \in R, 0 \leq \mu(g) \leq f \}.$$

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$$(R4) \quad \bigvee \{(p, q) \mid p, q \in \mathbb{Q}\} = 1.$$

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B. Banaschewski,

The real numbers in pointfree topology,

Textos Mat. Sér. B 12 Departamento de Matemática da Universidade de Coimbra (1997).

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Proposition (Folklore?)

Let L and M be Boolean frames. If $C(L)$ and $C(M)$ are isomorphic, so are L and M .

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$$\mathfrak{B}(L) = \{a \in L \mid a = a^{**}\}$$

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For a completely regular frame L , $\Gamma: C(L) \rightarrow C(\mathfrak{B}(L))$ is the universal completion of $C(L)$.

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A ring R is *regular* if for each $a \in R$ exists $b \in R$ such that $a = aba$.

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For the more classically minded. . .

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Universally complete spaces of continuous functions

[arxiv:2105.04810](https://arxiv.org/abs/2105.04810) (preprint)

The **classical** Maeda-Ogasawara-Vulikh representation theorem

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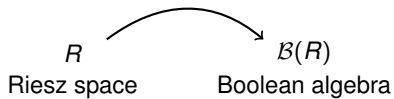
- A *weak unit* of R is an element $0 < e \in R$ such that the band generated by e is R itself.

The **classical** Maeda-Ogasawara-Vulikh representation theorem

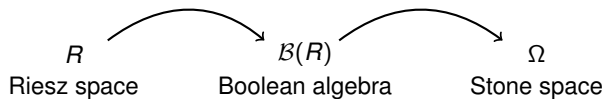
R

Riesz space

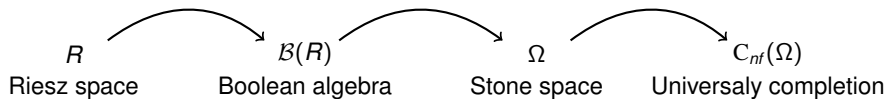
The **classical** Maeda-Ogasawara-Vulikh representation theorem



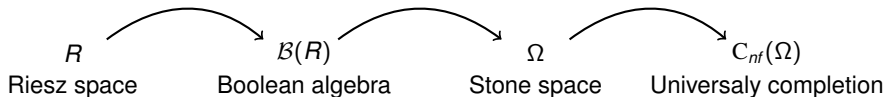
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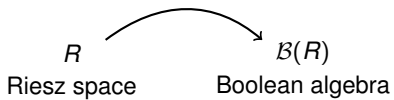
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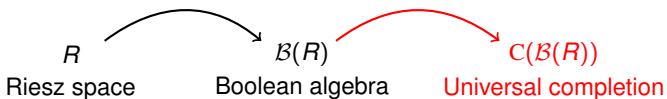
Theorem (Maeda-Ogasawara, 1942; Vulikh, 1947)

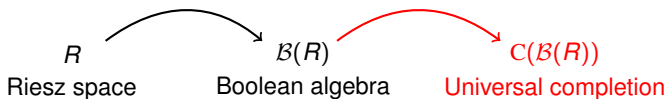
Let R be an Archimedean Riesz space with weak unit and Ω the Stone space of its Boolean algebra of bands $\mathcal{B}(R)$. R embeds into the space $C_{nf}(\Omega)$ of nearly finite continuous real functions on Ω . This embedding constitutes its universal completion.

The **localic** Maeda-Ogasawara-Vulikh representation theorem



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Eskerrik asko.
Thank you.