The universal completion of $C(L)$ and the localic representation of Riesz spaces

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Normal semicontinuity and the Dedekind completion of pointfree function rings
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If X is completely regular, then $\mathrm{NL}(X)$ is the universal completion of $\mathrm{C}(X)$.

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- U is a universally complete Riesz space;
- $\mu \colon R \to U$ is a Riesz space embedding such that, for every $f \in U^{+}$,

$$f = \bigvee \{\mu(g) \mid g \in R, 0 \leq \mu(g) \leq f\}.$$

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(R4) $\bigvee\{(p, q) \mid p, q \in \mathbb{Q}\} = 1$.

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B. Banaschewski,
The real numbers in pointfree topology,
Textos Mat. Sér. B **12** Departamento de Matemática da Universidade de Coimbra (1997).

Continuous real functions on Boolean frames

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Proposition (Folklore?)

Let L and M be Boolean frames. If $\mathrm{C}(L)$ and $\mathrm{C}(M)$ are isomorphic, so are L and M.

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The map

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For a completely regular frame L, $\Gamma\colon \mathrm{C}(L) \to \mathrm{C}(\mathfrak{B}(L))$ is the universal completion of $\mathrm{C}(L)$.

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A ring R is *regular* if for each $a \in R$ exists $b \in R$ such that $a = aba$.

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Universally complete spaces of continuous functions
arxiv:2105.04810 (preprint)

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  Ordered by inclusion, $\mathcal{B}(R)$ forms a complete Boolean algebra.
- A *weak unit* of R is an element $0 < e \in R$ such that the band generated by e is R itself.

The **classical** Maeda-Ogasawara-Vulikh representation theorem

R

Riesz space

The **classical** Maeda-Ogasawara-Vulikh representation theorem

R → $\mathcal{B}(R)$

Riesz space   Boolean algebra

$R \longrightarrow \mathcal{B}(R) \longrightarrow \Omega$

Riesz space — Boolean algebra — Stone space

The **classical** Maeda-Ogasawara-Vulikh representation theorem

R	→	$\mathcal{B}(R)$	→	Ω	→	$\mathrm{C}_{nf}(\Omega)$
Riesz space		Boolean algebra		Stone space		Universaly completion

The **classical** Maeda-Ogasawara-Vulikh representation theorem

R	$\mathcal{B}(R)$	Ω	$C_{nf}(\Omega)$
Riesz space	Boolean algebra	Stone space	Universaly completion

Theorem (Maeda-Ogasawara, 1942; Vulikh, 1947)

Let R be an Archimedean Riesz space with weak unit and Ω the Stone space of its Boolean algebra of bands $\mathcal{B}(R)$. R embeds into the space $C_{nf}(\Omega)$ of nearly finite continuous real functions on Ω. This embedding constitutes its universal completion.

$R \longrightarrow \mathcal{B}(R)$

Riesz space → Boolean algebra

The **localic** Maeda-Ogasawara-Vulikh representation theorem

R → $\mathcal{B}(R)$ → $\mathrm{C}(\mathcal{B}(R))$

Riesz space — Boolean algebra — Universal completion

The **localic** Maeda-Ogasawara-Vulikh representation theorem

R (Riesz space) $\longrightarrow$ $\mathcal{B}(R)$ (Boolean algebra) $\longrightarrow$ $\mathrm{C}(\mathcal{B}(R))$ (Universal completion)

Theorem

Let R be an Archimedean Riesz space with weak unit e. R embeds into $\mathrm{C}(\mathcal{B}(R))$ and this embedding constitutes the universal completion of R.

Eskerrik asko.

Thank you.