Varieties of PBZ*–lattices

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Boolean Algebras, Lattices, Universal Algebra, Set Theory & Topology (BLAST) 2021

NEW MEXICO STATE UNIVERSITY
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Las Cruces, USA

June 09*th*–13*th*, 2021

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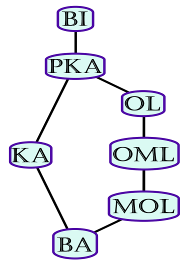
Bounded lattices with involution: type $(2,2,1,0,0)$

Notation	for the variety of:
$\mathbb{BI}$:	**bounded involution lattices**
$\mathbb{BA}$:	**Boolean algebras**
$\mathbb{PKA}$:	**pseudo–Kleene algebras**
$\mathbb{KA}$:	**Kleene algebras (Kleene lattices)**
$\mathbb{OL}$:	**ortholattices**
$\mathbb{OML}$:	**orthomodular lattices**
$\mathbb{MOL}$:	**modular ortholattices**

Meet–subsemilattice of the lattice of subvarieties of $\mathbb{BI}$:



Let $(L, \vee, \wedge, \cdot', 0, 1)$, where: $\begin{cases} (L, \vee, \wedge, \cdot', 0, 1): & \text{bounded lattice, with order } \leq \\ \cdot' : L \to L \end{cases}$

$$L \in \mathbb{BI} \iff \begin{cases} L \vDash x'' \approx x \text{ and} \\ L \vDash x \leq y \to y' \leq x' \end{cases} \iff \begin{cases} L \vDash x'' \approx x \text{ and} \\ L \vDash (x \vee y)' \approx x' \wedge y' \text{ and} \\ L \vDash (x \wedge y)' \approx x' \vee y' \end{cases}$$

then $\cdot'$: *involution*

(conditions over $\mathbb{BI}$, all equational but the second)

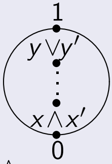
- the **Kleene condition**: $x \wedge x' \leq y \vee y'$
- *paraorthomodularity*: $(x \leq y \ \& \ x' \wedge y \approx 0) \rightarrow x \approx y$

  $\Uparrow \not\Downarrow \qquad \Updownarrow$ in $\mathbb{OL}$
- *orthomodularity*: $x \leq y \rightarrow y \approx (x' \wedge y) \vee x \quad (\Leftrightarrow \ x \vee (x' \wedge (x \vee y)) \approx x \vee y)$

  $\Uparrow$ in $\mathbb{OL} \qquad \not\Downarrow$
- *modularity* (**MOD**): $x \leq y \rightarrow x \vee (y \wedge z) \approx y \wedge (x \vee z)$
- *distributivity* (**DIST**): $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$

$$L \in \mathbb{PKA} \iff \begin{cases} L \in \mathbb{BI} \text{ and} \\ L \vDash \text{the Kleene condition} \end{cases}$$

then $\cdot'$: *Kleene complement*



$$L \in \mathbb{OL} \iff \begin{cases} L \in \mathbb{BI} \text{ and} \\ L \vDash x \wedge x' \approx 0 \end{cases} \iff \begin{cases} L \in \mathbb{BI} \text{ and} \\ L \vDash y \vee y' \approx 1 \end{cases} \implies L \in \mathbb{PKA}$$

$$L \in \mathbb{BA} \iff \begin{cases} L \in \mathbb{OL} \text{ and} \\ L \vDash \text{DIST} \end{cases} \implies L \in \mathbb{KA} \iff \begin{cases} L \in \mathbb{PKA} \text{ and} \\ L \vDash \text{DIST} \end{cases}$$

$$L \in \mathbb{MOL} \iff \begin{cases} L \in \mathbb{OL} \text{ and} \\ L \vDash \text{MOD} \end{cases} \implies L \in \mathbb{OML} \iff \begin{cases} L \in \mathbb{BI} \text{ and} \\ L \vDash \text{orthomodularity} \end{cases} \implies L \in \mathbb{OL}$$

Bounded lattices with two complements: type $(2, 2, 1, 1, 0, 0)$

(introduced by Roberto GIUNTINI, Antonio LEDDA and Francesco PAOLI for the study of Quantum Logics)

- $\mathbb{PBZL}^*$:= the variety of **PBZ*–lattices**: the paraorthomodular Brouwer–Zadeh lattices which satisfy condition $(*)$

$(L, \vee, \wedge, \cdot', \cdot^\sim, 0, 1) \in \mathbb{PBZL}^*$ iff:

- $(L, \vee, \wedge, \cdot', 0, 1) \in \mathbb{PKA}$ and $L \vDash$ paraorthomodularity (becomes equational) $\nearrow\!\!\!\nearrow$
- $\cdot^\sim : L \to L$ and L satisfies:

$$x \wedge x^\sim \approx 0 \qquad x \leq x^{\sim\sim}$$
$$x \leq y \to y^\sim \leq x^\sim \qquad x^{\sim\prime} \approx x^{\sim\sim}$$
$$(*): \quad (x \wedge x')^\sim \approx x^\sim \vee x'^\sim$$

then $\cdot^\sim$: *Brouwer complement*

- *Strong De Morgan* (**SDM**): $(x \wedge y)^\sim \approx x^\sim \vee y^\sim$

- $\mathbb{SDM} := \{L \in \mathbb{PBZL}^* \mid L \vDash \text{SDM}\}$
- $\mathbb{DIST} := \{L \in \mathbb{PBZL}^* \mid L \vDash \text{DIST}\}$

$\mathcal{V}(\cdot) \overset{\text{abbreviation}}{=} \mathcal{HSP}(\cdot)$, or $\mathcal{V}_{\mathbb{V}}(\cdot)$ in a variety $\mathbb{V}$

Orthomodular lattices as PBZ*–lattices, and antiortholattices

If $L \in \mathbb{PBZL}^*$, then:

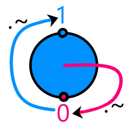
$L \vDash x \wedge x' \approx 0 \Leftrightarrow (L, \vee, \wedge, \cdot', 0, 1) \in \mathbb{OL} \Leftrightarrow (L, \vee, \wedge, \cdot', 0, 1) \in \mathbb{OML} \Leftrightarrow L \vDash x^{\sim} \approx x'$

thus:

(class identification)

$\mathbb{OML} \equiv \{L \in \mathbb{PBZL}^* \mid L \vDash x^{\sim} \approx x'\}$

$\mathbb{PBZL}^* \supset \mathbb{AOL} =$ the positive proper universal class of **antiortholattices**:

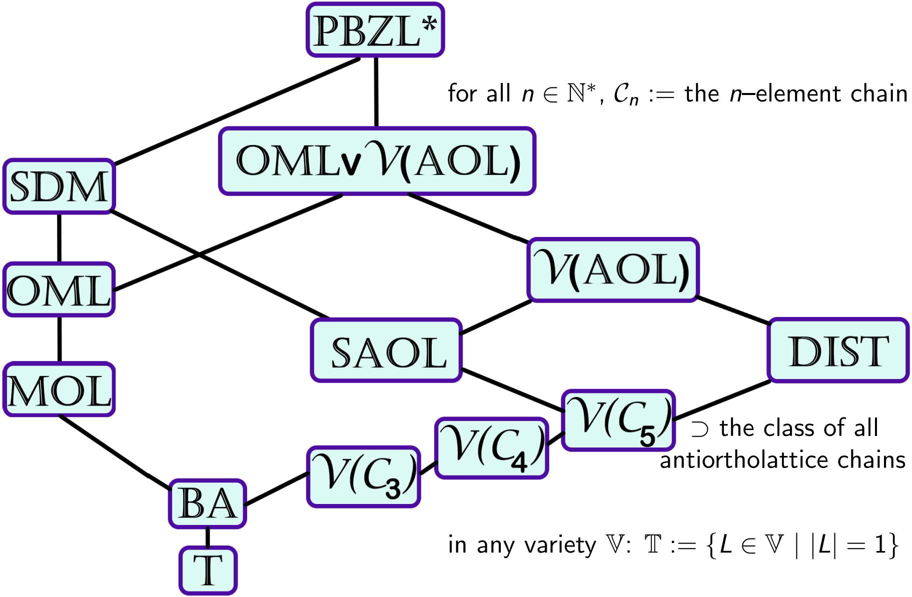


for any $L \in \mathbb{PBZL}^*$:

- $L \in \mathbb{AOL} \Longleftrightarrow \{x \in L \mid x \wedge x' = 0\} = \{0, 1\}\} \Longleftrightarrow$
$\Longleftrightarrow L \vDash x > 0 \rightarrow x^{\sim} \approx 0 \Longleftrightarrow L \vDash x \approx 0 \underline{\vee} x^{\sim} \approx 0$
- $L \in \mathbb{DIST} \cap \mathbb{AOL} \Longrightarrow$ the only complemented elements of L are 0 and 1

- $\mathcal{H}(\mathbb{AOL}) = \mathcal{S}(\mathbb{AOL}) = \mathbb{AOL}$, but even the lattice reducts of antiortholattices are directly irreducible
- $\mathbb{SAOL} := \mathbb{SDM} \cap \mathcal{V}(\mathbb{AOL}) = \mathcal{V}(\{L \in \mathbb{AOL} \mid L \vDash \mathrm{SDM}\}) = \mathcal{V}(\mathbb{SDM} \cap \mathbb{AOL})$
- $\mathbb{SDM} \cap \mathbb{AOL} = \{L \in \mathbb{AOL} \mid 0^L \in \mathrm{Mi}(L)\}) \supset$ the class of the PBZ*–chains, i.e. bounded involution chains with the trivial $\cdot^{\sim}$

Meet–subsemilattice of the lattice of subvarieties of $\mathbb{PBZL}^*$



for all $n \in \mathbb{N}^*$, $\mathcal{C}_n :=$ the n–element chain

$\supset$ the class of all antiortholattice chains

in any variety $\mathbb{V}$: $\mathbb{T} := \{L \in \mathbb{V} \mid |L| = 1\}$

A: bounded lattice
$B \in \mathbb{BI}/\mathbb{PKA}$

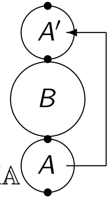
$A \oplus B \oplus A' \in \mathbb{BI}/\mathbb{PKA}$, respectively

$\Uparrow$

$\cdot^{\prime A\oplus B\oplus A'}|_A = f$

$\cdot^{\prime A\oplus B\oplus A'}|_B = \cdot^{\prime B}$

$\cdot^{\prime A\oplus B\oplus A'}|_{A'} = f^{-1}$



f: dual lattice isomorphism

$|A| > 1$ and $B \in \mathbb{PKA}$

$\Downarrow$

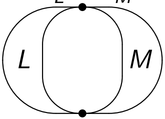
$A \oplus B \oplus A' \in \mathbb{AOL}$

$L, M \in \mathbb{PBZL}^*, \quad |L|, |M| > 1$

$\cdot^{\prime L\boxplus M}|_L = \cdot^{\prime L}, \quad \cdot^{\prime L\boxplus M}|_M = \cdot^{\prime M}$

$\cdot^{\sim L\boxplus M}|_L = \cdot^{\sim L}, \quad \cdot^{\sim L\boxplus M}|_M = \cdot^{\sim M}$

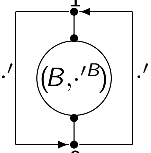
$\Downarrow$



$L \boxplus M \in \mathbb{PBZL}^*$
iff
$L \in \mathbb{OML}$
or
$M \in \mathbb{OML}$

(arbitrary $\boxplus \in \mathbb{PBZL}^*$ iff all but one $\in \mathbb{OML}$)

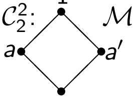
In particular: $A \oplus A' = A \oplus \mathcal{C}_1 \oplus A'$, and:



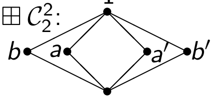
$\mathcal{C}_2 \oplus B \oplus \mathcal{C}_2 \in \mathbb{AOL}$

$\mathcal{MO}_0 := \mathcal{C}_2$
for any $\kappa > 0$, $\mathcal{MO}_\kappa := \boxplus_{\iota<\kappa} \mathcal{C}_2^2 \in \mathbb{MOL}$, thus:

$\mathcal{MO}_1 = \mathcal{C}_2^2$:



$\mathcal{MO}_2 = \mathcal{C}_2^2 \boxplus \mathcal{C}_2^2$:



$\mathbb{K} \subseteq \mathbb{BI}/\mathbb{PKA} \Rightarrow \mathcal{C}_2 \oplus \mathbb{K} \oplus \mathcal{C}_2 := \{\mathcal{C}_2 \oplus K \oplus \mathcal{C}_2 \mid K \in \mathbb{K}\} \subset \mathbb{BI}/\mathbb{AOL}$, respectively

Lemma

- $\mathbb{C} \subseteq \mathbb{BI} \implies \mathcal{V}_{\mathbb{BI}}(\mathcal{C}_2 \oplus \mathbb{C} \oplus \mathcal{C}_2) = \mathcal{V}_{\mathbb{BI}}(\mathcal{C}_2 \oplus \mathcal{V}_{\mathbb{BI}}(\mathbb{C}) \oplus \mathcal{C}_2)$
- $\mathbb{D} \subseteq \mathbb{PKA} \implies \mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{D} \oplus \mathcal{C}_2) = \mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathcal{V}_{\mathbb{BI}}(\mathbb{D}) \oplus \mathcal{C}_2)$

Proposition

For any $\mathbb{C} \subseteq \mathbb{BI}$:

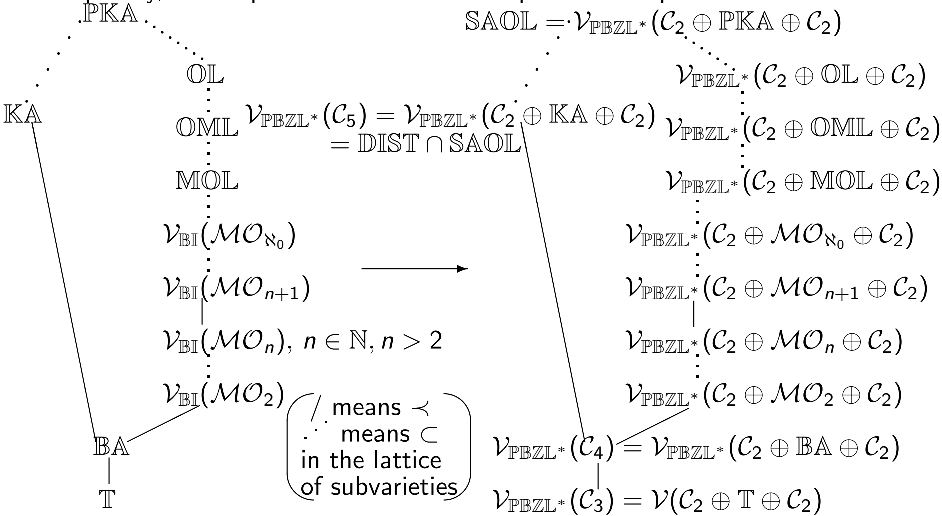
- $\mathcal{C}_3 \in \mathcal{V}_{\mathbb{BI}}(\mathbb{C}) \iff \mathbb{KA} \subseteq \mathcal{V}_{\mathbb{BI}}(\mathbb{C}) \iff \mathcal{V}_{\mathbb{BI}}(\mathbb{C}) = \mathcal{V}_{\mathbb{BI}}(\mathcal{C}_2 \oplus \mathbb{C} \oplus \mathcal{C}_2)$
- $\mathcal{C}_3 \notin \mathcal{V}_{\mathbb{BI}}(\mathbb{C}) \iff \mathcal{V}_{\mathbb{BI}}(\mathbb{C}) \subseteq \mathbb{OL} \iff \mathcal{V}_{\mathbb{BI}}(\mathbb{C}) \subset \mathcal{V}_{\mathbb{BI}}(\mathcal{C}_2 \oplus \mathbb{C} \oplus \mathcal{C}_2)$

Theorem

The operator: *the lattice of subvarieties of* $\mathbb{PKA}$ $\longrightarrow$ *the interval* $[\mathcal{V}(\mathcal{C}_3), \mathbb{SAOL}]$ *of the lattice of subvarieties of* $\mathbb{PBZL}^*$, $\mathbb{V} \mapsto \mathcal{V}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2)$ *is a bounded poset embedding that also reflects order, so its corestriction to its image is a poset isomorphism.*

So this map preserves covers and takes the splitting pair $(\mathbb{OL}, \mathbb{KA} = \mathcal{V}_{\mathbb{BI}}(\mathcal{C}_3))$ in the lattice of subvarieties of $\mathbb{PKA}$ into the splitting pair $(\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{OL} \oplus \mathcal{C}_2), \mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{KA} \oplus \mathcal{C}_2) = \mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_5))$ in its image.

Consequently, the map above restricts to the poset isomorphism:



well–known infinite ascending chain in the lattice of subvarieties of $\mathbb{MOL}$, thus also that of $\mathbb{PBZL}^* \supset \mathbb{OML} \supset \mathbb{MOL}$, with $\mathcal{V}_{\mathbb{BI}}(\mathcal{MO}_\kappa) = \mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{MO}_\kappa)$ for all κ; chain: $\mathcal{B} := \{\mathbb{T}, \ldots, \mathbb{OML}\}$

infinite ascending chain in the lattice of subvarieties of $\mathbb{SAOL} \subset \mathbb{PBZL}^*$, disjoint from the previous one; chain: $\mathcal{C} := \{\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_3), \ldots, \mathbb{SAOL}\}$

Axiomatizations for subvarieties of $\mathbb{PKA}$ can be turned into axiomatizations for their image of through the map above (which differ for those in $[\mathbb{KA})$ from those in $(\mathbb{OL}]$)

- $k, n, p \in \mathbb{N}$
- $t(x_1, \ldots, x_k, z_1, \ldots, z_p)$, $u(y_1, \ldots, y_n, z_1, \ldots, z_p)$: terms over $\mathbb{BI}$

(terms over $\mathbb{PBZL}^*$)

$$m(t, u)(x_1, \ldots, x_k, y_1, \ldots, y_n, z_1, \ldots, z_p) =$$
$$\bigvee_{i=1}^{k}(x_i \wedge x_i')^{\sim} \vee \bigvee_{j=1}^{n}(y_j \wedge y_j')^{\sim} \vee \bigvee_{h=1}^{p}(z_h \wedge z_h')^{\sim} \vee t(x_1, \ldots, x_k, z_1, \ldots, z_p).$$

$$\Downarrow$$

$$m(u, t)(x_1, \ldots, x_k, y_1, \ldots, y_n, z_1, \ldots, z_p) =$$
$$\bigvee_{i=1}^{k}(x_i \wedge x_i')^{\sim} \vee \bigvee_{j=1}^{n}(y_j \wedge y_j')^{\sim} \vee \bigvee_{h=1}^{p}(z_h \wedge z_h')^{\sim} \vee u(y_1, \ldots, y_n, z_1, \ldots, z_p).$$

Lemma

- $L \in \mathbb{BI}$, $\mathcal{C}_3 \in V_{\mathbb{BI}}(L) \Longrightarrow [L \vDash t \approx u \Leftrightarrow \mathcal{C}_2 \oplus L \oplus \mathcal{C}_2 \vDash t \approx u]$
- $L \in \mathbb{PKA}$, $k+p>0<n+p \Longrightarrow [L \vDash t \approx u \Leftrightarrow \mathcal{C}_2 \oplus L \oplus \mathcal{C}_2 \vDash m(t, u) \approx m(u, t)]$

For any $L \in \mathbb{BI} \supset \mathbb{PKA}$:

(the ortholattice equation written with nonnullary terms)

$L \in \mathbb{OL} \iff L \vDash x \wedge x' \approx y \wedge y' \iff L \vDash x \vee x' \approx y \vee y'$

(the equations $m(t, u) \approx m(u, t)$ for $t \approx u$ the $\mathbb{OL}$ equations above)

- D2OL∧: $(x \wedge x')^{\sim} \vee (y \wedge y')^{\sim} \vee (x \wedge x') \approx (x \wedge x')^{\sim} \vee (y \wedge y')^{\sim} \vee (y \wedge y')$
- D2OL∨: $(x \wedge x')^{\sim} \vee (y \wedge y')^{\sim} \vee x \vee x' \approx (x \wedge x')^{\sim} \vee (y \wedge y')^{\sim} \vee y \vee y'$

$\mathcal{V}_{\mathbb{PBZL}^*}(\mathbb{AOL}) = \{L \in \mathbb{PBZL}^* \mid L \vDash x \approx (x \wedge y^{\sim}) \vee (x \wedge y^{\sim\sim})\}$

Theorem

$\mathbb{V}$ subvariety of $\mathbb{PKA}$; $\{t_i, u_i \mid i \in I\}$ terms over $\mathbb{BI}$.

- *If $\mathcal{C}_3 \in \mathbb{V}$ (equivalently, if $\mathbb{KA} \subseteq \mathbb{V}$), then:*

$$\mathbb{V} = \{L \in \mathbb{PKA} \mid (\forall i \in I)(L \vDash t_i \approx u_i)\} \iff$$

$$\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2) = \{L \in \mathbb{SAOL} \mid (\forall i \in I)(L \vDash t_i \approx u_i)\} \iff$$

$$\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2) = \{L \in \mathcal{V}_{\mathbb{PBZL}^*}(\mathbb{AOL}) \mid L \vDash \mathrm{SDM}, (\forall i \in I)(L \vDash t_i \approx u_i)\}$$

$$\iff \mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2) = \{L \in \mathbb{PBZL}^* \mid L \vDash x \approx (x \wedge y^{\sim}) \vee (x \wedge y^{\sim\sim}),$$

$$L \vDash \mathrm{SDM}, (\forall i \in I)(L \vDash t_i \approx u_i)\}.$$

- If $\mathcal{C}_3 \notin \mathbb{V}$ (equivalently, if $\mathbb{V} \subseteq \mathbb{OL}$) and, for all $i \in I$, t_i and u_i have nonzero arities, then:

$$\mathbb{V} = \{L \in \mathbb{OL} \mid (\forall\, i \in I)\,(L \vDash t_i \approx u_i)\} \iff$$

$$\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2) = \{L \in \mathcal{V}_{\mathbb{PBZL}^*}(\mathbb{AOL}) \mid L \vDash \text{D2OL}\wedge,$$

$$(\forall\, i \in I)\,(L \vDash m(t_i, u_i) \approx m(u_i, t_i))\} \iff$$

$$\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2) = \{L \in \mathcal{V}_{\mathbb{PBZL}^*}(\mathbb{AOL}) \mid L \vDash \text{D2OL}\vee,$$

$$(\forall\, i \in I)\,(L \vDash m(t_i, u_i) \approx m(u_i, t_i))\} \iff$$

$$\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2) = \{L \in \mathbb{PBZL}^* \mid L \vDash x \approx (x \wedge y^{\sim}) \vee (x \wedge y^{\sim\sim}),$$

$$L \vDash \text{D2OL}\wedge,\ (\forall\, i \in I)\,(L \vDash m(t_i, u_i) \approx m(u_i, t_i))\} \iff$$

$$\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2) = \{L \in \mathbb{PBZL}^* \mid L \vDash x \approx (x \wedge y^{\sim}) \vee (x \wedge y^{\sim\sim}),$$

$$L \vDash \text{D2OL}\vee,\ (\forall\, i \in I)\,(L \vDash m(t_i, u_i) \approx m(u_i, t_i))\}.$$

So we have two different ways of determining an axiomatization for $\mathcal{V}_{\mathbb{PBZL}^*}(\mathcal{C}_2 \oplus \mathbb{V} \oplus \mathcal{C}_2)$ from one for $\mathbb{V}$, depending on whether $\mathbb{V}$ belongs to the filter $[\mathbb{KA})$ or the ideal $(\mathbb{OL}]$ determined by the splitting pair $(\mathbb{OL}, \mathbb{KA})$ in the lattice of subvarieties of $\mathbb{PKA}$.

1. PBZ*–lattices: Definition and Main Subvarieties

2. Embedding the Lattice of Subvarieties of $\mathbb{PKA}$ as a Poset in the Lattice of Subvarieties of $\mathbb{SAOL}$, also Mapping the Axiomatizations

3. An Infinite Ascending Chain of Subvarieties of $\mathbb{DIST}$ and an Infinity of Pairwise Disjoint Ascending Chains of Subvarieties of $\mathbb{PBZL}^*$

Theorem

The following is an infinite ascending chain of subvarieties of $\mathbb{DIST}$:

$$\mathbb{T} = V_{\mathbb{PBZL}^*}(\mathcal{C}_1) = V_{\mathbb{PBZL}^*}(\mathcal{C}_2^0 \oplus \mathcal{C}_2^0) \prec \mathbb{BA} = V_{\mathbb{PBZL}^*}(\mathcal{C}_2) = V_{\mathbb{PBZL}^*}(\mathcal{C}_2^0 \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^0) \prec$$

$$V_{\mathbb{PBZL}^*}(\mathcal{C}_3) = V_{\mathbb{PBZL}^*}(\mathcal{C}_2^1 \oplus \mathcal{C}_2^1) \prec V_{\mathbb{PBZL}^*}(\mathcal{C}_4) = V_{\mathbb{PBZL}^*}(\mathcal{C}_2^1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^1) \prec V_{\mathbb{PBZL}^*}(\mathcal{C}_5)$$

$$\prec V_{\mathbb{PBZL}^*}(\mathcal{C}_2^2 \oplus \mathcal{C}_2^2) \prec V_{\mathbb{PBZL}^*}(\mathcal{C}_2^2 \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^2) \prec \ldots \prec$$

$$V_{\mathbb{PBZL}^*}(\mathcal{C}_2^n \oplus \mathcal{C}_2^n) \prec V_{\mathbb{PBZL}^*}(\mathcal{C}_2^n \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^n) \prec$$

$$V_{\mathbb{PBZL}^*}(\mathcal{C}_2^{n+1} \oplus \mathcal{C}_2^{n+1}) \prec V_{\mathbb{PBZL}^*}(\mathcal{C}_2^{n+1} \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^{n+1}) \prec \ldots \subset$$

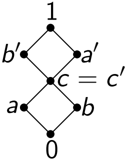
$$V_{\mathbb{PBZL}^*}(\mathcal{C}_2^{\aleph_0} \oplus \mathcal{C}_2^{\aleph_0}) = V_{\mathbb{PBZL}^*}(\mathcal{C}_2^{\aleph_0} \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^{\aleph_0}) \subseteq$$

$$V_{\mathbb{PBZL}^*}(\mathcal{C}_2^{\kappa} \oplus \mathcal{C}_2^{\kappa}) = V_{\mathbb{PBZL}^*}(\mathcal{C}_2^{\kappa} \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^{\kappa}) \subseteq \mathbb{DIST},$$

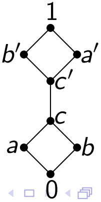
where $n \in \mathbb{N} \setminus \{0, 1, 2\}$ *and* κ *is an infinite cardinality.*

chain: $\mathcal{D} := \{\mathcal{V}(\mathcal{C}_2^2 \oplus \mathcal{C}_2^2), \ldots, \mathbb{DIST}\}$

$\mathcal{C}_2^2 \oplus \mathcal{C}_2^2 \in \mathbb{DIST} \cap \mathbb{AOL}$:



$\mathcal{C}_2^2 \oplus \mathcal{C}_2 \oplus \mathcal{C}_2^2 \in \mathbb{DIST} \cap \mathbb{AOL}$:



Subdirect products of varieties

Lemma (Bjarni JÓNSSON, Congruence–distributive Varieties, Theorem 4.10)

- $\mathbb{U}$: *congruence–distributive variety of similar algebras;*
- $\mathbb{V}$ *and* $\mathbb{W}$: *subvarieties of* $\mathbb{U}$.

Then $Si(\mathbb{V} \vee \mathbb{W}) = Si(\mathbb{V}) \cup Si(\mathbb{W})$, *hence:*

- *the lattice of subvarieties of* $\mathbb{U}$ *is distributive;*
- $\mathbb{V} \vee \mathbb{W} = \mathbb{V} \times_s \mathbb{W}$, *that is:*
  $$\mathbb{V} \vee \mathbb{W} = \bigcup_{A \in \mathbb{V}, B \in \mathbb{W}} \{S \in \mathcal{S}(A \times B) \mid pr_A(S) = A, pr_B(S) = B\}.$$

Consequently:

The lattice of subvarieties of $\mathbb{PBZL}^*$ is distributive.

But also:

Lemma

- $\mathbb{U}$: *variety of similar algebras;*
- $\mathbb{V}$ *and* $\mathbb{W}$: *subvarieties of* $\mathbb{U}$.

$$\begin{cases} [\mathbb{V}\cap\mathbb{W},\mathbb{V}]\times[\mathbb{V}\cap\mathbb{W},\mathbb{W}]\underset{\psi}{\overset{\varphi}{\rightleftarrows}}[\mathbb{V}\cap\mathbb{W},\mathbb{V}\vee\mathbb{W}] \\ (\mathbb{S}_1,\mathbb{S}_2)\overset{\varphi}{\longmapsto}\mathbb{S}_1\vee\mathbb{S}_2 \\ (\mathbb{S}\cap\mathbb{V},\mathbb{S}\cap\mathbb{W})\overset{\psi}{\longleftarrow\!\!\dashv}\mathbb{S} \end{cases}$$

Then:

- φ *and* ψ *are order–preserving;*
- *if* $\mathbb{V}\vee\mathbb{W}=\mathbb{V}\times_s\mathbb{W}$, *then* $\varphi\circ\psi=id_{[\mathbb{V}\cap\mathbb{W},\mathbb{V}\vee\mathbb{W}]}$;
- *if* $\mathbb{V}\vee\mathbb{W}=\mathbb{V}\times_s\mathbb{W}$ *and* $\mathbb{S}_1\vee\mathbb{S}_2=\mathbb{S}_1\times_s\mathbb{S}_2$ *for all* $\mathbb{S}_1\in\mathcal{V}\subseteq[\mathbb{V}\cap\mathbb{W},\mathbb{V}]$ *and all* $\mathbb{S}_2\in\mathcal{W}\subseteq[\mathbb{V}\cap\mathbb{W},\mathbb{W}]$, *then:*

$$\mathcal{V}\times\mathcal{W}\underset{\psi}{\overset{\varphi}{\rightleftarrows}}\{\mathbb{S}_1\vee\mathbb{S}_2\mid\mathbb{S}_1\in\mathcal{V},\mathbb{S}_2\in\mathcal{W}\}:$$

mutually inverse order isomorphisms;

- if $\mathbb{S}_1 \vee \mathbb{S}_2 = \mathbb{S}_1 \times_s \mathbb{S}_2$ for all $\mathbb{S}_1 \in [\mathbb{V} \cap \mathbb{W}, \mathbb{V}]$ and all $\mathbb{S}_2 \in [\mathbb{V} \cap \mathbb{W}, \mathbb{W}]$, in particular if $\mathbb{U}$ is congruence–distributive, then:

$$[\mathbb{V} \cap \mathbb{W}, \mathbb{V}] \times [\mathbb{V} \cap \mathbb{W}, \mathbb{W}] \underset{\psi}{\overset{\varphi}{\rightleftarrows}} [\mathbb{V} \cap \mathbb{W}, \mathbb{V} \vee \mathbb{W}] :$$

  mutually inverse lattice isomorphisms.

Therefore:

Theorem

In the lattice of subvarieties of $\mathbb{PBZL}^*$*:*

$$(\mathbb{OML} \vee \mathcal{V}(\mathbb{AOL})] = \{\mathbb{T}\} \cup [\mathbb{BA}, \mathbb{OML} \vee \mathcal{V}(\mathbb{AOL})] \cong$$

$$\cong \mathcal{C}_2 \oplus [\mathbb{BA}, \mathbb{OML} \vee \mathcal{V}(\mathbb{AOL})] \cong \mathcal{C}_2 \oplus ([\mathbb{BA}, \mathbb{OML}] \times [\mathbb{BA}, \mathcal{V}(\mathbb{AOL})]).$$

Consequently, if we denote, for a variety $\mathbb{V}$ and a set $\mathcal{M}$ of varieties similar to $\mathbb{V}$: $\mathcal{M}_{\mathbb{V}} := \{\mathbb{V} \vee \mathbb{W} \mid \mathbb{W} \in \mathcal{M}\}$, then:

For any nontrivial subvarieties $\mathbb{U} \neq \mathbb{Y}$ of $\mathbb{OML}$ and $\mathbb{V} \neq \mathbb{W}$ of $\mathcal{V}(\mathbb{AOL})$: $\mathcal{B}_{\mathbb{V}}$, $\mathcal{B}_{\mathbb{W}}$, $\mathcal{C}_{\mathbb{U}}$, $\mathcal{C}_{\mathbb{Y}}$, $\mathcal{D}_{\mathbb{U}}$ and $\mathcal{D}_{\mathbb{Y}}$: pairwise disjoint infinite ascending chains of subvarieties of $\mathbb{OML} \vee \mathcal{V}(\mathbb{AOL}) \subset \mathbb{PBZL}^*$.

C. Mureșan, A Note on Direct Products, Subreducts and Subvarieties of PBZ*–lattices, to appear in *Mathematica Bohemica*.

THANK YOU FOR YOUR ATTENTION !