A Grothendieck problem for finitely presented MV-algebras

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Definition of the profinite completion of algebras

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4. Then, $\big((\mathrm{Con}_{\mathfrak{f}}(A), \supseteq), \{A/\theta\}, \{\varphi^{A}_{\theta\alpha}\}\big)$ in an inverse system in **A**.
5. The inverse (projective) limit of this system is known as the profinite completion of the algebra A, and commonly denoted by $\widehat{A}$.

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5. The profinite completion induces a functor $\widehat{-} : \mathbb{A} \to \mathbb{SA}$, where $\mathbb{SA}$ denotes the category of Stone $\mathbb{A}$-algebras with continuous homomorphisms

Definition of an MV-algebra

An MV-algebra can be defined as an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying:

MV-1: $(A, \oplus, 0)$ is an Abelian monoid;

MV-2: $\neg : A \to A$ is an involution;

MV-3: $1 := \neg 0$ is absorbant;

MV-4: $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Note: The rule $x \leq y$ iff $\neg x \oplus y = 1$, defines a bounded distributive lattice order on A.

Examples of MV-algebras

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3. The following subalgebras of $[0, 1]$. For every $n \geq 2$, $Ł_n = [0, 1] \cap \mathbb{Z}\frac{1}{n-1} = \left\{ \frac{k}{n-1} : 0 \leq k \leq n-1 \right\}$.

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4. Let X be any topological space and $A := \mathcal{C}(X, [0,1])$ be the set of continuous functions from $X \to [0,1]$.
   Given $f, g \in A$ and $x \in X$,
   - $(\neg f)(x) = 1 - f(x)$
   - $(f \oplus g)(x) = Min(1, f(x) + g(x))$

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3. Each ideal I of an MV-algebra A correspond to a unique congruence θ_I (and conversely) and the quotient MV-algebra formed is denoted by A/I.
4. **Ex.** Let X be any topological space and $A := \mathcal{C}(X, [0, 1])$ be the MV-algebra above. For every $p \in X$, let

   $$O_p := \{f \in A : f \text{ vanishes on some neighborhood } U \text{ of } p\}$$

   Then O_p is an ideal of A.

Motivation of the study

- A. Grothendieck posed the following problem in 1970 [2]: Given Γ_1 and Γ_2 two finitely presented, residually finite groups, and $u : \Gamma_1 \to \Gamma_2$ a homomorphism such that the induced map of profinite completions $\widehat{u} : \widehat{\Gamma_1} \to \widehat{\Gamma_2}$ is an isomorphism. Does it follow that u is an isomorphism?

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- Recall [4] that an MV-algebra A is called residually finite if for every $x \neq y$ in A, there exists a finite MV-algebra F and a homomorphism $p : A \to F$ such that $p(x) \neq p(y)$.

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- Residually finite MV-algebras are exactly the subalgebras of products of finite Łukasiewicz chains.

A refinement of the problem in the case of MV-algebras

- **Proposition.** Finitely presented MV-algebras are subdirect products of finite Łukasiewicz chains. In particular, finitely presented MV-algebras are residually finite.

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- **Proposition.** Finitely presented MV-algebras are subdirect products of finite Łukasiewicz chains. In particular, finitely presented MV-algebras are residually finite.
- **Proof (sketch)**: Given A, a finitely presented MV-algebra, there exists a rational polyhedron $P \subseteq [0,1]^n$ such that $A = \mathcal{M}(P)$, the MV-algebra of all $\mathbb{Z}$-maps from $P \to [0,1]$. Since P is a rational polyhedron, the subset $R(P)$ of points of P having rational coordinates is dense in P. For each such point $x \in R(P)$, the evaluation of members of A at x yields a function $A \to [0,1]$ which can be easily proved to be a homomorphism of MV-algebras whose image is a finite subalgebra of $[0,1]$, which we denote by $L_A(x)$. Therefore, there is an induced homomorphism $\mu_A : A \to \prod_{x \in R(P)} L_A(x)$ which is indeed a subdirect embedding.

The profinite completion of finitetely presented MV-algebras

Let $A := \mathcal{M}(P)$, where P is a rational polyhedron and $R(P)$ is the subset of points of P having rational coordinates. With the notations above

- $\sigma_A : x \mapsto \mathfrak{h}_x$ is a homeomorphism from P to Max$A$, where $\mathfrak{h}_x := \{f \in A : f(x) = 0\}$.

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- $\widehat{A} \cong \prod_{x \in R(P)} L_A(x)$.

The homomorphism of finitetely presented MV-algebras

- Let $\varphi : \mathcal{M}(P) \to \mathcal{M}(Q)$ be a homomorphism, where $P \subseteq [0,1]^n$ and $Q \subseteq [0,1]^m$ are rational polyhedra.

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- One has a map $\sigma_\varphi : Q \to P$ defined by $\sigma_\varphi(y) = (f_1(y), f_2(y), \ldots, f_n(y))$ which is indeed a $\mathbb{Z}$-map [5, Lem. 3.8(ii)].

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- Then, $\sigma_\varphi(R(Q)) \subseteq R(P)$ and for every $y \in R(Q)$, and every $f \in \mathcal{M}(P)$, $f(\sigma_\varphi(y)) = \varphi(f)(y)$.
- In particular, $L_A(\sigma_\varphi(y)) \subseteq L_B(y)$ for all $y \in R(Q)$.

The Main Theorem

THEOREM. Let A, B be two finitely presented MV-algebras and $\varphi : A \to B$ be a homomorphism. Then φ is an isomorphism if and only if $\widehat{\varphi}$ is an isomorphism.

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- We write $A := \mathcal{M}(P)$ and $B := \mathcal{M}(Q)$, where $P \subseteq [0,1]^n$ and $Q \subseteq [0,1]^m$ are rational polyhedra. It follows from earlier facts that $\widehat{A} = \prod_{x \in R(P)} L_A(x)$ and $\widehat{B} = \prod_{y \in R(Q)} L_B(y)$.

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- In addition, $\widehat{\varphi} : \prod_{x \in R(P)} L_A(x) \to \prod_{y \in R(Q)} L_B(y)$ is defined by $\widehat{\varphi}(\alpha)(y) = \alpha(\sigma_\varphi(y))$ for all $\alpha \in \prod_{x \in R(P)} L_A(x)$ and $y \in R(Q)$.

Sketch of the proof (sufficiency)

Assume that $\widehat{\varphi}$ is an isomorphism.

- First, we claim that $\sigma_\varphi(R(Q)) = R(P)$. We already had the inclusion $\sigma_\varphi(R(Q)) \subseteq R(P)$. We argue by contradiction and assume that there exists $x_0 \in R(P)$ with $\sigma_\varphi(y) \neq x_0$ for all $y \in R(Q)$. Define $\alpha \in \widehat{A}$ by $\alpha(x_0) = 1$ and $\alpha(x) = 0$ for all $x \neq x_0$. Clearly, $\alpha \neq 0$ and $\widehat{\varphi}(\alpha) = 0$, which contradicts the fact that $\widehat{\varphi}$ is one-to-one.

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- Moreover, assume that $\sigma_\varphi(y) = \sigma_\varphi(y')$ for some $y, y' \in R(Q)$. Let $\Theta : \widehat{B} \to \widehat{A}$ be the inverse of $\widehat{\varphi}$. Then $(\widehat{\varphi} \circ \Theta)(\beta)(y) = \beta(y)$ for all $\beta \in \widehat{B}$, that is $\Theta(\beta)(\sigma_\varphi(y)) = \beta(y)$ for all $\beta \in \widehat{B}$. Since $\sigma_\varphi(y) = \sigma_\varphi(y')$, it follows that $\beta(y) = \beta(y')$ for all $\beta \in \widehat{B}$ and $y = y'$.

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- Hence $\sigma_\varphi : R(Q) \to R(P)$ is a bijection.

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- Hence $\sigma_\varphi : R(Q) \to R(P)$ is a bijection.
- One can now use all the ingredients assembled thus far to prove that φ is an isomorphism.

Comparing the profinite completion and the MacNeille completion

- Besides the profinite completion, another popular completion that is well investigated in the MacNeille completion. We have gathered in the process of proving the main result above enough ingredients to characterize all finitely presented MV-algebras for which the two completions coincide.

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- **Sketech of the proof:** Assume that $P \subseteq [0,1]^n$ is a rational polyhedron such that $\widehat{\mathcal{M}(P)} \cong \overline{\mathcal{M}(P)}$. One proves that $\mathcal{M}(P)$ has only finitely many atoms and that there is a bijection between the set of atoms of $\mathcal{M}(P)$ and $R(P)$. Thus, $R(P)$ is finite and since $\mathcal{M}(P) \subseteq \widehat{\mathcal{M}(P)} \cong \prod_{x \in R(P)} L_{\mathcal{M}(P)}(x)$, we conclude that $\mathcal{M}(P)$ is finite.

THANK YOU FOR YOUR ATTENTION !!!!!!!!

MERCI POUR VOTRE AIMABLE ATTENTION!!!!!!!!!!!

- M. R. Bridson, F. J. Grunewald, Grothendieck's problems concerning profinite completions and representations of groups, *Ann. of Math.*, **160** (2004)359–373
- A. Grothendieck, Représentations linéaires et compactification profinie des groupes discrets, *Manuscripta Math.*, **2** (1970)375–396.
- V. Marra, L. Spada, The dual adjunction between MV-algebras and Tychonoff spaces, *Studia Logica*, **100** no. 1-2 (2012)253–278.
- D. Mundici, Hopfian ℓ-groups, MV-algebras and AF C*-algebras, *Forum Math.*, **28**(6)(2016)1111–1130.
- D. Mundici, Advanced Łukasiewicz calculus and MV-algebras. Trends Log. Stud. Log. Libr. **35**, Springer, New York (2011).
- J. B. Nganou, Profinite completions of MV-algebras, *Houston J. Math.* **44** (3) (2018)753–767.