A chain of adjuntions between BA and the variety generated by a semi-primal bounded lattice expansion.

Wolfgang Poiger

Joint work with Alexander Kurz and Bruno Teheux

University of Luxembourg

BLAST 2021

Semi-primal bounded lattice expansion

Definition

A finite algebra $\boldsymbol{L}$ is *semi-primal* if every map $f : L^n \to L$ which preserves subalgebras is term-definable.

Proposition

Let $\boldsymbol{L}$ be a finite algebra with bounded lattice reduct. Then TFAE:

- $\boldsymbol{L}$ is semi-primal.
- For every $a \in L$ the map $T_a : L \to L$

$$T_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

  is term-definable.

Throughout the presentation $\boldsymbol{L}$ will denote a semi-primal algebra with bounded lattice reduct and (smallest) subalgebra $\mathbf{2}$.

Examples

- Chain-based: the $(n+1)$-element Łukasiewicz-chain $\mathbf{Ł}_n = (\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, \wedge, \vee, \oplus, \neg, 0, 1)$ where

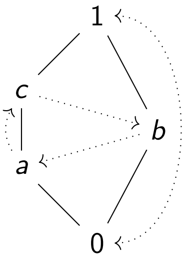
$$x \oplus y = min(x+y, 1) \text{ and } \neg x = 1 - x.$$

Examples

- Chain-based: the $(n+1)$-element Łukasiewicz-chain $\text{Ł}_n = (\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, \wedge, \vee, \oplus, \neg, 0, 1)$ where

$$x \oplus y = min(x+y, 1) \text{ and } \neg x = 1 - x.$$

- Not chain-based: $\boldsymbol{L} = (N_5, \wedge, \vee, f, 0, 1)$:



(Davey, Schumann, Werner 1991)

Stone duality and semi-primal natural duality

For BA $= \mathbb{ISP}(\mathbf{2})$ we have the famous Stone duality:

$$\mathsf{Stone} \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} \mathsf{BA}$$

$\Sigma(B) = \mathsf{BA}(B, \mathbf{2})$ and $\Pi(X) = \mathsf{Stone}(X, 2)$.

Stone duality and semi-primal natural duality

For $\mathsf{BA} = \mathbb{ISP}(\mathbf{2})$ we have the famous Stone duality:

$$\mathsf{Stone} \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} \mathsf{BA}$$

$\Sigma(B) = \mathsf{BA}(B, \mathbf{2})$ and $\Pi(X) = \mathsf{Stone}(X, 2)$.

For $\mathcal{A} := \mathbb{ISP}(\boldsymbol{L})$ we have the similar natural duality

$$\mathsf{Stone}_{\boldsymbol{L}} \underset{S}{\overset{P}{\rightleftarrows}} \mathcal{A}$$

"$S(\boldsymbol{A}) = \mathcal{A}(\boldsymbol{A}, \boldsymbol{L})$ and $P(X) = \mathsf{Stone}_{\boldsymbol{L}}(X, L)$"

The category $\text{Stone}_{\boldsymbol{L}}$

The category $\text{Stone}_{\boldsymbol{L}}$ has objects (X, v) where

- $X \in \text{Stone}$,
- $v : X \to \mathbb{S}(\boldsymbol{L})$,
- $v^{-1}(\boldsymbol{S}{\downarrow})$ is closed for every subalgebra $\boldsymbol{S} \leq \boldsymbol{L}$.

A morphism $f : (X, v) \to (Y, v')$ is a continuous map with

$$v'(f(x)) \subseteq v(x)$$

for every $x \in X$.

Semi-primal natural duality (2)

$$\mathsf{Stone}_{\boldsymbol{L}} \underset{S}{\overset{P}{\rightleftarrows}} \mathcal{A}$$

$(L, v_L) \in \mathsf{Stone}_{\boldsymbol{L}}$ with $v_L(a) = \langle a \rangle$. Then

$$P(X, v) = \mathsf{Stone}_{\boldsymbol{L}}((X, v), (L, v_L))$$

$$f \in P(X, v) \Leftrightarrow f \in \mathsf{Stone}(X, L) \text{ and } f(x) \in v(x)$$

and

$$S(\boldsymbol{A}) = (\mathcal{A}(\boldsymbol{A}, \boldsymbol{L}), im)$$

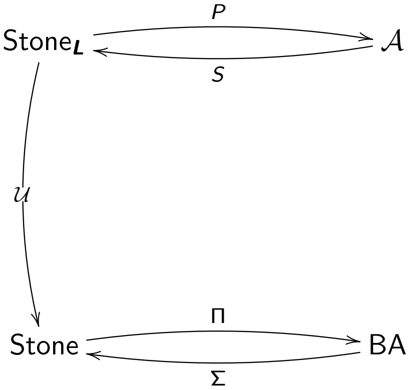
$$im(h) = h(A)$$

The adjoint functors on the dual side

$$\mathsf{Stone}_{\boldsymbol{L}} \underset{S}{\overset{P}{\rightleftarrows}} \mathcal{A}$$

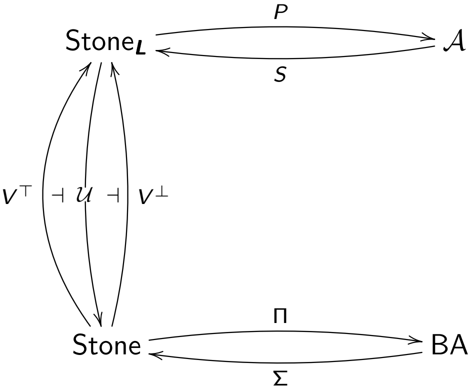
$$\mathsf{Stone} \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} \mathsf{BA}$$

The adjoint functors on the dual side



Let $\mathcal{U}$ be the forgetful functor.

The adjoint functors on the dual side



Let $\mathcal{U}$ be the forgetful functor.

$V^{\top}(X) = (X, v^{\top})$ where $v^{\top}(x) = L$ for all $x \in X$.
$V^{\perp}(X) = (X, v^{\perp})$ where $v^{\perp}(x) = 2$ for all $x \in X$.

The adjoint functors on the dual side (2)

$V^\top \dashv \mathcal{U} \dashv V^\perp$:

$$f \in \mathsf{Stone}_{\boldsymbol{L}}((X, v^\top), (Y, v')) \Leftrightarrow v'(f(x)) \subseteq v^\top(x) \Leftrightarrow f \in \mathsf{Stone}(X, Y)$$

$$f \in \mathsf{Stone}_{\boldsymbol{L}}((X, v), (Y, v^\perp)) \Leftrightarrow v^\perp(f(x)) \subseteq v(x) \Leftrightarrow f \in \mathsf{Stone}(X, Y)$$

The adjoint functors on the dual side (2)

$V^{\top} \dashv \mathcal{U} \dashv V^{\perp}$:

$$f \in \mathsf{Stone}_{\boldsymbol{L}}((X, v^{\top}), (Y, v')) \Leftrightarrow v'(f(x)) \subseteq v^{\top}(x) \Leftrightarrow f \in \mathsf{Stone}(X, Y)$$

$$f \in \mathsf{Stone}_{\boldsymbol{L}}((X, v), (Y, v^{\perp})) \Leftrightarrow v^{\perp}(f(x)) \subseteq v(x) \Leftrightarrow f \in \mathsf{Stone}(X, Y)$$

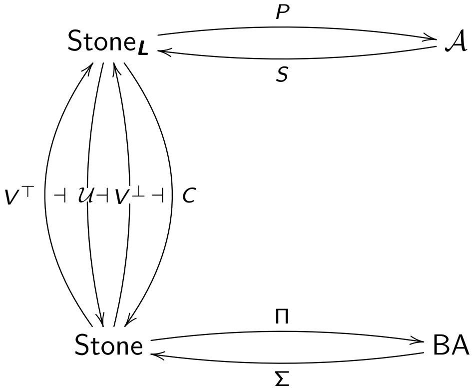
Consider the functor $C : \mathsf{Stone}_{\boldsymbol{L}} \to \mathsf{Stone}$ given by

$$C(X, v) = \{x \in X \mid v(x) = 2\}$$

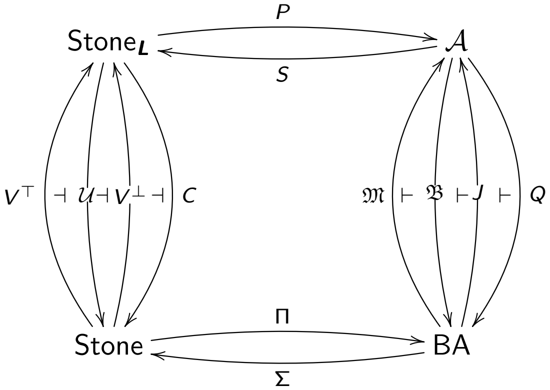
on objects and by $f \mapsto f\restriction_{C(X,v)}$ on morphisms.

$$f \in \mathsf{Stone}_{\boldsymbol{L}}((X, v^{\perp}), (Y, v')) \Leftrightarrow v'(f(x)) \subseteq v^{\perp} \Leftrightarrow f \in \mathsf{Stone}(X, C(Y, v'))$$

The big picture



The big picture



Next goal: Understand the adjunction on the algebraic side.

The Boolean skeleton

Recall: For every $\ell \in L$ the map $T_\ell : L \to L$ given by

$$T_\ell(x) = \begin{cases} 1 & \text{if } x = \ell \\ 0 & \text{if } x \neq \ell \end{cases}$$ is term-definable in $\boldsymbol{L}$.

Definition (Maruyama 2011)

Given any $\boldsymbol{A} \in \mathcal{A}$ we define $\mathfrak{B}(A) = \{a \in A \mid T_1(a) = a\}$. The *Boolean skeleton* of $\boldsymbol{A}$ is the Boolean algebra

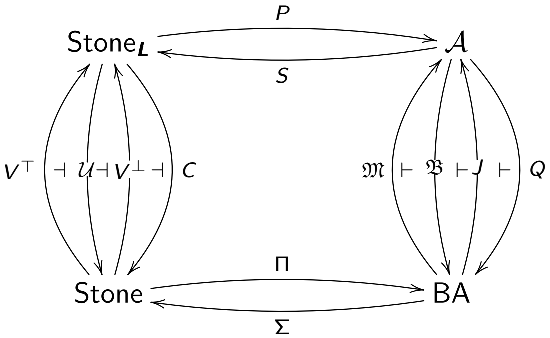
$$\mathfrak{B}(\boldsymbol{A}) = (\mathfrak{B}(A), \wedge, \vee, T_0, 0, 1).$$

The Boolean skeleton (2)

Theorem

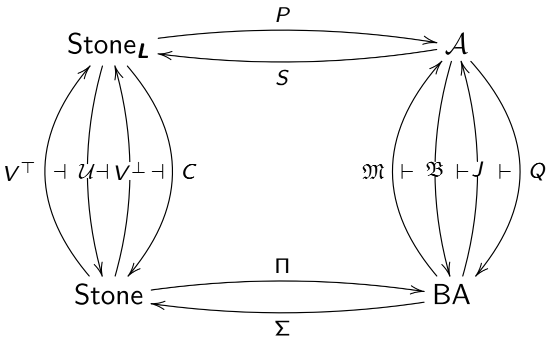
Let $\boldsymbol{A} \in \mathcal{A}$. Then $h \mapsto h\restriction_{\mathfrak{B}(A)}$ is a homeomorphism

$$\mathcal{A}(\boldsymbol{A}, \boldsymbol{L}) \simeq \mathrm{BA}(\mathfrak{B}(\boldsymbol{A}), \mathbf{2}).$$



$$\Pi \mathcal{U} S(\boldsymbol{A}) = \Pi(\mathcal{A}(\boldsymbol{A}, \boldsymbol{L})) \simeq \Pi(\mathrm{BA}(\mathfrak{B}(\boldsymbol{A}), \mathbf{2})) = \Pi\Sigma(\mathfrak{B}(\boldsymbol{A})) \simeq \mathfrak{B}(\boldsymbol{A}).$$

The functor J

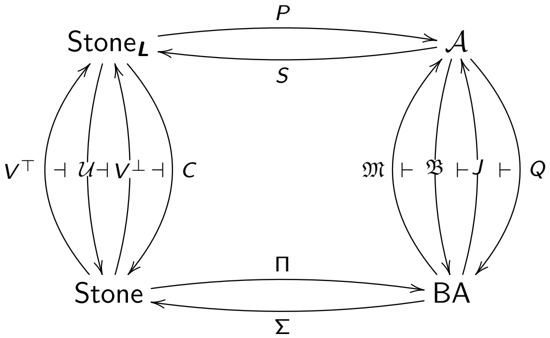


$$J(\boldsymbol{B}) = PV^\perp\Sigma(\boldsymbol{B}) = P(\mathrm{BA}(\boldsymbol{B}, \mathbf{2}), v^\perp)$$

$$J(B) = \mathrm{Stone}(\mathrm{BA}(\boldsymbol{B}, \mathbf{2}), 2) = \Pi\Sigma(B) \simeq B$$

$\Rightarrow$ J can be seen as inclusion.

The functor Q



$$\boldsymbol{A} \in \mathcal{A}_\omega \Rightarrow \boldsymbol{A} = \prod \boldsymbol{S}_i = \prod \{h(A) \mid h \in \mathcal{A}(\boldsymbol{A}, \boldsymbol{L})\}$$

$$Q(\boldsymbol{A}) = \mathbf{2}^{\mathcal{A}(\boldsymbol{A},\mathbf{2})}$$

$\Rightarrow$ Q can be seen as quotient.

The functor $\mathfrak{M}$ - Example

$\mathfrak{M}(\boldsymbol{B})$ is the largest algebra in $\mathcal{A}$ which has $\boldsymbol{B}$ as Boolean skeleton. For finite algebras this is $\mathfrak{M}(\boldsymbol{B}) = \boldsymbol{L}^{\mathrm{BA}(\mathrm{B},\mathbf{2})}$.

For $\mathrm{MV}_n = \mathbb{ISP}(\text{Ł}_n)$ (Di Nola, Lettieri 2000):

$$\mathfrak{M}(B) = \{(b_1, \dots, b_n) \in B^n \mid b_1 \geq \cdots \geq b_n\}$$

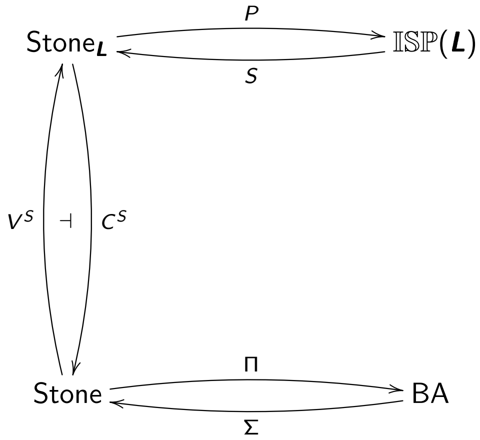
turn into MV_n-algebra

$$\neg(b_1, \dots, b_n) = (\neg b_n, \dots, \neg b_1)$$

$$(b \oplus c)_i = \bigvee_{j+k=i} b_j \wedge c_k$$

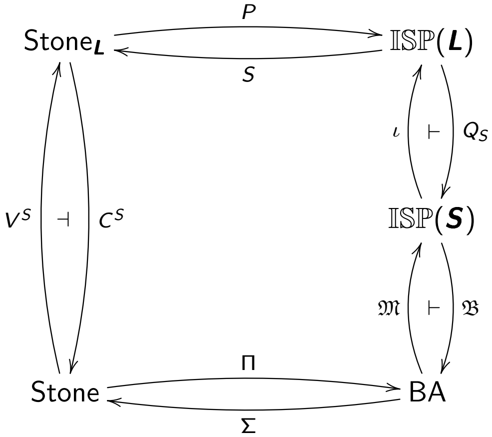
Another adjunction

For an arbitrary subalgebra $\boldsymbol{S} \leq \boldsymbol{L}$:

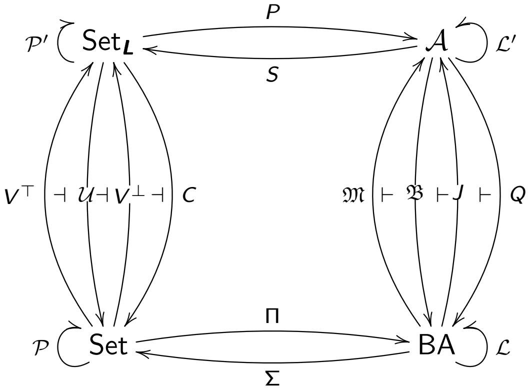


Another adjunction

For an arbitrary subalgebra $\boldsymbol{S} \leq \boldsymbol{L}$:



Many-valued modal logic



$\mathcal{P}$-coalgebras: (Kripke) frames

$\mathcal{L}$-algebras: Classical modal logic

$\mathcal{P}'$-coalgebras: $\boldsymbol{L}$-frames

$\mathcal{L}'$-algebras: Modal logic over $\boldsymbol{L}$

Thanks for your attention!