A chain of adjunctions between BA and the variety generated by a semi-primal bounded lattice expansion.

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**Definition**

A finite algebra $L$ is *semi-primal* if every map $f : L^n \to L$ which preserves subalgebras is term-definable.

**Proposition**

Let $L$ be a finite algebra with bounded lattice reduct. Then TFAE:

- $L$ is semi-primal.
- For every $a \in L$ the map $T_a : L \to L$

\[
T_a(x) = \begin{cases} 
1 & \text{if } x = a \\
0 & \text{if } x \neq a 
\end{cases}
\]

is term-definable.

Throughout the presentation $L$ will denote a semi-primal algebra with bounded lattice reduct and (smallest) subalgebra $2$. 
Examples

- Chain-based: the \((n + 1)\)-element Łukasiewicz-chain
  \(\mathcal{L}_n = (\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, \land, \lor, \oplus, \neg, 0, 1)\) where
  \[x \oplus y = \min(x + y, 1)\]
  and \(\neg x = 1 - x\).
Examples

- Chain-based: the \((n + 1)\)-element Łukasiewicz-chain
  \( \mathcal{L}_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, \wedge, \vee, \oplus, \neg, 0, 1 \) where
  \[ x \oplus y = \min(x + y, 1) \text{ and } \neg x = 1 - x. \]

- Not chain-based: \( L = (\mathcal{N}_5, \wedge, \vee, f, 0, 1) \):

(Davey, Schumann, Werner 1991)
For $BA = \mathbb{ISP}(2)$ we have the famous **Stone duality**:

\[ \text{Stone} \leftrightarrow \Pi \rightarrow \Sigma \leftarrow \text{BA} \]

\[ \Sigma(B) = BA(B, 2) \text{ and } \Pi(X) = \text{Stone}(X, 2). \]
For $BA = \mathbb{ISP}(2)$ we have the famous Stone duality:

$$\begin{array}{c}
\text{Stone} & \overset{\Pi}{\longrightarrow} & BA \\
\downarrow & & \downarrow \\
\Sigma & & \Sigma(B) = BA(B, 2) \text{ and } \Pi(X) = \text{Stone}(X, 2). \\
\end{array}$$

For $\mathcal{A} := \mathbb{ISP}(L)$ we have the similar natural duality

$$\begin{array}{c}
\text{Stone}_L & \overset{P}{\longrightarrow} & \mathcal{A} \\
\downarrow & & \downarrow \\
S & & S(\mathcal{A}) = \mathcal{A}(\mathcal{A}, L) \text{ and } P(X) = \text{Stone}_L(X, L). \\
\end{array}$$

"$S(\mathcal{A}) = \mathcal{A}(\mathcal{A}, L) \text{ and } P(X) = \text{Stone}_L(X, L)$"
The category $\text{Stone}_L$ has objects $(X, \nu)$ where

- $X \in \text{Stone}$,
- $\nu : X \to \mathbb{S}(L)$,
- $\nu^{-1}(S \downarrow)$ is closed for every subalgebra $S \leq L$.

A morphism $f : (X, \nu) \to (Y, \nu')$ is a continuous map with

$$\nu'(f(x)) \subseteq \nu(x)$$

for every $x \in X$. 
Semi-primal natural duality (2)

\[(L, v_L) \in \text{Stone}_L \text{ with } v_L(a) = \langle a \rangle. \text{ Then}\]

\[P(X, \nu) = \text{Stone}_L((X, \nu), (L, v_L))\]

\[f \in P(X, \nu) \iff f \in \text{Stone}(X, L) \text{ and } f(x) \in \nu(x)\]

and

\[S(A) = (A(A, L), \text{im})\]

\[\text{im}(h) = h(A)\]
The adjoint functors on the dual side

Let $U$ be the forgetful functor.

$V^\top (X) = (X, v^\top)$ where $v^\top (x) = \mathcal{L}$ for all $x \in X$.

$V_\bot (X) = (X, v_\bot)$ where $v_\bot (x) = 2$ for all $x \in X$.

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The adjoint functors on the dual side

Let $\mathcal{U}$ be the forgetful functor.

\[
\begin{array}{ccc}
\text{Stone}_L & \xleftarrow{P} & \mathcal{A} \\
\downarrow \mathcal{U} & & \downarrow S \\
\text{Stone} & \xleftarrow{\Pi} & \text{BA} \\
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow \mathcal{U} & & \downarrow \Sigma \\
\text{Stone} & \xleftarrow{\Sigma} & \text{BA} \\
\end{array}
\]
Let $\mathcal{U}$ be the forgetful functor.

$\mathcal{V}^\top(X) = (X, v^\top)$ where $v^\top(x) = L$ for all $x \in X$.

$\mathcal{V}^\bot(X) = (X, v^\bot)$ where $v^\bot(x) = 2$ for all $x \in X$. 
The adjoint functors on the dual side (2)

\[ \mathcal{V}^\top \dashv \mathcal{U} \dashv \mathcal{V}^\perp : \]

\[ f \in \text{Stone}_L((X, v^\top), (Y, v')) \iff v'(f(x)) \subseteq v^\top(x) \iff f \in \text{Stone}(X, Y) \]

\[ f \in \text{Stone}_L((X, v), (Y, v^\perp)) \iff v^\perp(f(x)) \subseteq v(x) \iff f \in \text{Stone}(X, Y) \]
The adjoint functors on the dual side (2)

\( V^\top \dashv U \dashv V^\perp: \)

\[
f \in \text{Stone}_L((X, v^\top), (Y, v')) \iff v'(f(x)) \subseteq v^\top(x) \iff f \in \text{Stone}(X, Y)
\]

\[
f \in \text{Stone}_L((X, v), (Y, v^\perp)) \iff v^\perp(f(x)) \subseteq v(x) \iff f \in \text{Stone}(X, Y)
\]

Consider the functor \( C : \text{Stone}_L \rightarrow \text{Stone} \) given by

\[
C(X, v) = \{ x \in X \mid v(x) = 2 \}
\]
on objects and by \( f \mapsto f\upharpoonright_{C(X, v)} \) on morphisms.

\[
f \in \text{Stone}_L((X, v^\perp), (Y, v')) \iff v'(f(x)) \subseteq v^\perp \iff f \in \text{Stone}(X, C(Y, v'))
\]
Next goal: Understand the adjunction on the algebraic side.
The big picture

Next goal: Understand the adjunction on the algebraic side.
Recall: For every $\ell \in L$ the map $T_\ell : L \to L$ given by

$$T_\ell(x) = \begin{cases} 1 & \text{if } x = \ell \\ 0 & \text{if } x \neq \ell \end{cases}$$

is term-definable in $L$.

**Definition (Maruyama 2011)**

Given any $A \in \mathcal{A}$ we define $\mathcal{B}(A) = \{a \in A \mid T_1(a) = a\}$. The Boolean skeleton of $A$ is the Boolean algebra

$$\mathcal{B}(A) = (\mathcal{B}(A), \wedge, \vee, T_0, 0, 1).$$
The Boolean skeleton (2)

Theorem
Let $A \in \mathcal{A}$. Then $h \mapsto h|_{\mathcal{B}(A)}$ is a homeomorphism

$$\mathcal{A}(A, L) \simeq BA(\mathcal{B}(A), 2).$$

A chain of adjunctions

$$\Pi \mathcal{L} S(A) = \Pi(\mathcal{A}(A, L)) \simeq \Pi(BA(\mathcal{B}(A), 2)) = \Pi \Sigma(\mathcal{B}(A)) \simeq \mathcal{B}(A).$$
The functor $J$

\[
\begin{align*}
J(B) &= PV^\perp \Sigma(B) = P(BA(B, 2), v^\perp) \\
J(B) &= \text{Stone}(BA(B, 2), 2) = \Pi\Sigma(B) \simeq B
\end{align*}
\]

$\implies J$ can be seen as inclusion.
The functor $Q$

$A \in \mathcal{A}_\omega \Rightarrow A = \prod S_i = \prod \{h(A) \mid h \in \mathcal{A}(A, L)\}$

$Q(A) = 2^{\mathcal{A}(A, L)}$

$\Rightarrow Q$ can be seen as quotient.
The functor $\mathcal{M}$ - Example

$\mathcal{M}(B)$ is the largest algebra in $\mathcal{A}$ which has $B$ as Boolean skeleton. For finite algebras this is $\mathcal{M}(B) = L^{BA(B,2)}$.

For $MV_n = ISP(L_n)$ (Di Nola, Lettieri 2000):

$$\mathcal{M}(B) = \{ (b_1, \ldots, b_n) \in B^n \mid b_1 \geq \cdots \geq b_n \}$$

turn into $MV_n$-algebra

$$\neg (b_1, \ldots, b_n) = (\neg b_n, \ldots, \neg b_1)$$

$$(b \oplus c)_i = \bigvee_{j+k=i} b_j \wedge c_k$$
For an arbitrary subalgebra $S \leq L$:
Another adjunction

For an arbitrary subalgebra $S \leq L$:

\[
\begin{array}{ccc}
\text{Stone}_L & \overset{P}{\longrightarrow} & \text{ISP}(L) \\
\downarrow V^S & & \uparrow S \\
\text{Stone} & \overset{\Pi}{\longrightarrow} & \text{BA} \\
\end{array}
\]

\[
\begin{array}{ccc}
\uparrow C^S & & \downarrow \text{ISP}(S) \\
\uparrow \iota & & \uparrow Q_S \\
\uparrow M & & \downarrow \mathcal{B} \\
\end{array}
\]
Many-valued modal logic

\[ \begin{array}{c}
\mathcal{P}' \xleftarrow{\text{Set}_L} \mathcal{P} \\
\mathcal{L}' \xrightarrow{\mathcal{A}} \mathcal{L} \\
\mathcal{V}^\top \xrightarrow{\cup} \mathcal{U} \xrightarrow{\vee^\bot} \mathcal{C} \\
\mathcal{M} \xrightarrow{\mathcal{B} \vdash \mathcal{J} \vdash \mathcal{Q}} \\
\mathcal{P} \xleftarrow{\Pi} \mathcal{B} \xrightarrow{\Sigma} \mathcal{L} \\
\end{array} \]

\( \mathcal{P} \)-coalgebras: (Kripke) frames  \( \mathcal{P}' \)-coalgebras:  \( L \)-frames

\( \mathcal{L} \)-algebras: Classical modal logic  \( \mathcal{L}' \)-algebras: Modal logic over  \( L \)
Thanks for your attention!