Lovász-type theorems and polyadic spaces

Luca Reggio

University of Oxford

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A “square root” problem

- Chang, Jónsson, Tarski, *Refinement properties for relational structures*, Fund. Math. **55** (1964):

  ‘When two direct factorisations of a structure have a common refinement?’

  ‘Does $A^2 \cong B^2$ entail $A \cong B$?’

- Lovász, *Operations with structures*, Acta Math. Acad. Sci. Hungar. **18** (1967):

  ‘If A and B are finite then, for all $n \in \mathbb{N}$, $A^n \cong B^n$ entails $A \cong B$.’

Homomorphism counting

Here is what Lovász proved:

Theorem

Let A and B be finite (relational) structures. Then $A \cong B$ iff

$$\text{for all finite structures } C, \quad |\hom(C, A)| = |\hom(C, B)|.$$

Back to the square root problem:

$$\begin{aligned} A^2 \cong B^2 &\Leftrightarrow |\hom(C, A^2)| = |\hom(C, B^2)| \;\; \forall C \\ &\Leftrightarrow |\hom(C, A)|^2 = |\hom(C, B)|^2 \;\; \forall C \\ &\Leftrightarrow |\hom(C, A)| = |\hom(C, B)| \;\; \forall C \\ &\Leftrightarrow A \cong B. \end{aligned}$$

Homomorphism counting in FMT

Lovász' result has had a considerable impact in graph theory but also in finite model theory, where the isomorphism relation is replaced by equivalence in a logic fragment:

Theorem (Dvořák, 2009 link)

For all finite structures A and B,

$$A \equiv_{\mathrm{FO}^k(\#)} B \iff |\hom(C,A)| = |\hom(C,B)| \quad \forall \textit{ finite structures } C \textit{ with } \mathrm{tw}(C) < k.$$

Theorem (Grohe, 2020 link)

For all finite structures A and B,

$$A \equiv_{\mathrm{FO}_k(\#)} B \iff |\hom(C,A)| = |\hom(C,B)| \quad \forall \textit{ finite structures } C \textit{ with } \mathrm{td}(C) \leq k.$$

In this talk

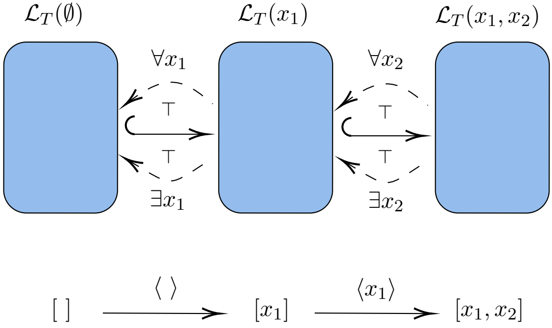
This talk is about a connection between homomorphism counting results and Joyal's polyadic spaces, which are the Stone (pointwise) duals of Boolean hyperdoctrines.

- Most of what I know about polyadic spaces I have learnt from André Joyal (e-mail communication).
- Jérémie Marquès has been doing quite a lot of interesting work on polyadic spaces.
- The results pertaining to FMT that I will mention have been obtained jointly with Anuj Dawar and Tomáš Jakl (*Lovász-type theorems and game comonads*, LiCS'21).

Polyadic spaces

Boolean hyperdoctrines

A basic tool of categorical logic is given by hyperdoctrines (Lawvere, 1969). Fix a first-order theory T. For all finite contexts $\overline{x}$, let $\mathcal{L}_T(\overline{x})$ be the Lindenbaum-Tarski algebra of formulas with free variables in $\overline{x}$, modulo T-equiprovability.



Boolean hyperdoctrines (on **Fin**)

If the signature contains no function symbols nor constants, then the category of contexts can be identified with the category **Fin** of finite sets and functions.

A Boolean hyperdoctrine on **Fin** is a functor $F\colon \mathbf{Fin} \to \mathbf{BA}$ s.t.

1. For all arrows h in **Fin**, Fh has a left adjoint;
   (Existential quantifiers)
2. F takes pushout squares to *Beck-Chevalley* squares.
   ($\exists$ commutes with substitutions)

Recall that a commutative square $\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle g} \\ c & \xrightarrow{k} & d \end{array}$ in **BA** is a BC square

if $\exists_g \circ k = f \circ \exists_j$.

Polyadic spaces (on **Fin**)

A polyadic space on **Fin** is a *contravariant* functor $E\colon \mathbf{Fin} \to \mathbf{Stone}$ satisfying the following conditions:

1. For all arrows h in **Fin**, Eh is an open map;
   (Existential quantifiers)
2. F takes pushout squares to *quasi-pullbacks*.
   ($\exists$ commutes with substitutions)

A commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ W & \xrightarrow{k} & Z \end{array}$$

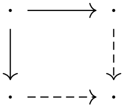
in **Stone** is a quasi-pullback if the unique mediating morphism $X \to Y \times_Z W$ is a surjection.

*71T-E29. ANDRÉ JOYAL, Université de Montréal, Montréal, Québec, Canada. Polyadic spaces and elementary theories. Preliminary report.

Let S_0 be the category of finite sets. A polyadic space E is a contravariant functor from S_0 to the category $\mathcal{C}$ of Stone spaces and continuous mappings satisfying: (1) $(n \xrightarrow{f} m) \in S_0$, $E(n) \xleftarrow{E(f)} E(m)$ is open. (2) E transforms push-out squares in S_0 into quasi pull-back squares in $\mathcal{C}$ (i.e., the object of the initial corner is mapped onto the pull-back of the remaining objects of the square). A morphism $E \xrightarrow{\alpha} E'$ between two polyadic spaces is a natural transformation s.t. $\forall (n \xrightarrow{f} m) \in S_0$, the square constructed by using α_m, α_n, $E(f)$, $E'(f)$ is a quasi pull-back. With every set X, we associate a polyadic space $\widetilde{X}$ as follows: $\widetilde{X}(n) = \beta(X^n)$ (i.e., the Stone-Čech compactification of X^n); $\widetilde{X}(f)$ is defined in the obvious way. A model of E (based on X) is a morphism $\widetilde{X} \to E$. Given an elementary theory, we can associate "canonically" a polyadic space and vice-versa. Under this correspondence, the two concepts of model coincide. Classical theorems of logic are interpreted and proven in this context. Polyadic spaces are naturally found in algebraic geometry. (Received November 4, 1970.)

Amalgamation

A category $\mathcal{A}$ has the amalgamation property if any span of morphisms in $\mathcal{A}$ can be completed to a commutative square:



By extension, we say that a functor $E\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Stone}$ has the amalgamation property if so does its category of elements $\int E$:

The objects of $\int E$ are the pairs (a, x) with $a \in \mathcal{A}$ and $x \in E(a)$, and an arrow $(a, x) \to (a', x')$ in $\int E$ is a morphism $f\colon a \to a'$ in $\mathcal{A}$ such that $Ef(x') = x$.

Polyadic spaces (on arbitrary categories)

Generalising Joyal's definition, let us say that a polyadic space on $\mathcal{A}$ is a contravariant functor $E\colon \mathcal{A} \to \mathbf{Stone}$ s.t.

1. For all arrows f in $\mathcal{A}$, Ef is an open map;
2. E has the amalgamation property.

A (finite) polyadic set is a contravariant functor $E\colon \mathcal{A} \to \mathbf{Set}$ (resp. $\mathcal{A} \to \mathbf{Fin}$) with the amalgamation property.

Example

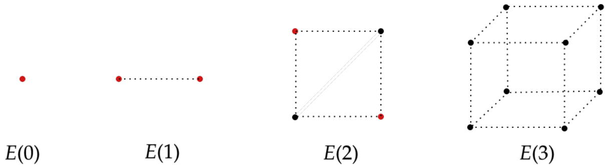
Any representable functor $\hom(-, a)\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ is a polyadic set.

For the homomorphism counting results we only need (finite) polyadic sets, but I think it's worth placing these results in the wider framework of polyadic spaces (this may suggest further developments).

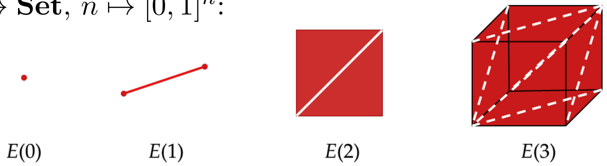
Generic elements

Let $E\colon \mathbf{Fin}^{\mathrm{op}} \to \mathbf{Stone}$ be a polyadic space (or a polyadic set).
An element $x \in E(n)$ is called generic if it is not of the form $Ef(y)$ for any proper surjection $f\colon n \twoheadrightarrow m$ and element $y \in E(m)$.

$E\colon \mathbf{Fin}^{\mathrm{op}} \to \mathbf{Fin}$, $n \mapsto 2^n$ (generic elements in red): link



$E\colon \mathbf{Fin}^{\mathrm{op}} \to \mathbf{Set}$, $n \mapsto [0,1]^n$:



Stirling core

Let $E^\bullet(n)$ denote the subspace of $E(n)$ consisting of the generic elements. Then the polyadic space $E\colon \mathbf{Fin}^{\mathrm{op}} \to \mathbf{Stone}$ induces a polyadic space

$$E^\bullet\colon \mathbf{Fin}_{\mathrm{inj}} \to \mathbf{Stone}$$

on the category of finite sets and injections. We refer to $E^\bullet$ as the Stirling core of E.

Proposition (Reconstruction Lemma)

Let $E\colon \mathbf{Fin}^{\mathrm{op}} \to \mathbf{Stone}$ be a polyadic space. For all $n \in \mathbf{Fin}$,

$$E(n) \cong \coprod_{n \twoheadrightarrow m} E^\bullet(m)$$

where the coproduct is indexed by the set of equivalence classes of surjections with domain n.
(In fact, E is the left Kan extension of $E^\bullet$ along $\mathbf{Fin}_{\mathrm{inj}} \hookrightarrow \mathbf{Fin}$.)

An example

The number of non-equivalent surjections $n \twoheadrightarrow m$ coincides with the number of ways to partition an n-element set into m non-empty subsets. This is commonly denoted by $S(n, m)$ and known as the Stirling number of the second kind associated with the pair (n, m).

Example

For the finite polyadic set $E\colon \mathbf{Fin}^{\mathrm{op}} \to \mathbf{Fin}$, $n \mapsto 2^n$, we have

$$2^n \cong \coprod_{0<m\leq n} S(n, m)\, E^{\bullet}(m).$$

link

For all $n \geq 3$, $2^n = 2 \cdot S(n, 1) + 2 \cdot S(n, 2) = 2 + 2 \cdot S(n, 2)$.

The Key Lemma

Two parallel functors $E, F\colon \mathcal{A} \to \mathcal{B}$ are pointwise isomorphic if, for all $a \in \mathcal{A}$, there is an isomorphism $\eta_a\colon Ea \to Fa$ in $\mathcal{B}$.
This contrasts with the concept of *natural isomorphism* between E and F, whereby the isomorphisms η_a are required to be natural in a.

Lemma (Key Lemma)

Let $E, F\colon \mathbf{Fin}^{\mathrm{op}} \to \mathbf{Fin}$ *be finite polyadic sets. If* E *and* F *are pointwise isomorphic, then so are their Stirling cores* $E^\bullet$ *and* $F^\bullet$*.*

- The Key Lemma generalises to finite polyadic sets on any category $\mathcal{A}$ equipped with a proper factorisation system $(\mathcal{Q}, \mathcal{M})$ such that, for every $a \in \mathcal{A}$, the poset of quotients of a is well-founded.
- The proof relies on the Reconstruction Lemma and Rota's *Möbius inversion formula*.

Homomorphism counting

Combinatorial categories

A category $\mathcal{A}$ is said to be

- locally finite if, for all $a, b \in \mathcal{A}$, the set $\hom(a, b)$ is finite;
- combinatorial if it is locally finite and, for all $a, b \in \mathcal{A}$,

$$a \cong b \iff |\hom(c, a)| = |\hom(c, b)| \;\; \forall c \in \mathcal{A}.$$

Lovász' theorem states that, if σ is a finite relational signature, the category of finite σ-structures and homomorphisms is combinatorial.

Pultr (1973) showed that any finitely well-powered, locally finite category with (extremal epi, mono) factorisations is combinatorial.

An abstract homomorphism counting result

Theorem

Let $\mathcal{A}$ be a locally finite category admitting a proper factorisation system $(\mathcal{Q}, \mathcal{M})$ such that the poset of quotients of each $a \in \mathcal{A}$ is well-founded. Then $\mathcal{A}$ is combinatorial.

Proof.

Suppose that $|\hom(c, a)| = |\hom(c, b)|$ for all $c \in \mathcal{A}$. Then $E := \hom(-, a)$ and $F := \hom(-, b)$ are pointwise isomorphic finite polyadic sets. By the Key Lemma, $E^\bullet$ is pointwise isomorphic to $F^\bullet$.

It is not difficult to see that, for all $c \in \mathcal{A}$, $E^\bullet(c) = \mathcal{M}(c, a)$ and $F^\bullet(c) = \mathcal{M}(c, b)$. As $\emptyset \neq \mathcal{M}(a, a) \cong \mathcal{M}(a, b)$, there is $i \in \mathcal{M}(a, b)$. Similarly, there is $j \in \mathcal{M}(b, a)$ and so $j \circ i \in \mathcal{M}(a, a)$.

But $\mathcal{M}(a, a)$ is a finite left-cancellative monoid ($xy = xz \Rightarrow y = z$), hence a group. It follows that $j \circ i$ has an inverse, and so i is an iso. $\square$

Examples

The following categories are combinatorial:

- Finite σ-structures and homomorphism (for σ a relational signature).
- Finite monoids / groups / Abelian groups... and homomorphisms.
- $\mathbb{V}_{\text{fin}}$ for any Birkhoff variety of algebras $\mathbb{V}$.
- Finite Eilenberg-Moore coalgebras for any comonad on **Set**, or on the category of σ-structures.
- Finite trees / forests and forest morphisms.

Grohe's theorem link can be deduced by looking at the finite EM coalgebras for an appropriate *game comonad* on σ-structures. This is a comonad, introduced by Abramsky and Shah, that captures precisely Eherenfeucht-Fraïssé games between structures.

Beyond locally finite categories

- Can we recover Dvořák's result for $\mathrm{FO}^k(\#)$ (in a direct way)?
- What about "small" infinite objects? E.g., is the isomorphism type of *finitely branching* trees determined by homomorphism counts?

We will see that a "local" homomorphism counting result holds for *locally finitely presentable* categories, which are a generalisation of algebraic lattices. A poset P is an algebraic lattice if

1. it is $\bigvee$-complete (hence, a complete lattice), and
2. each $x \in P$ is the directed join of *compact* elements.

An element $x \in P$ is compact if, for all directed subsets $D \subseteq P$,

$$x \leq \bigvee^{\uparrow} D \implies x \leq y \text{ for some } y \in D.$$

Equivalently, $\hom_P(x, \bigvee^{\uparrow} D) = \bigvee^{\uparrow}_{y \in D} \hom_P(x, y)$.

lfp categories

An object x of a category $\mathcal{A}$ is finitely presentable (fp) if the functor $\hom_{\mathcal{A}}(x, -) \colon \mathcal{A} \to \mathbf{Set}$ preserves directed colimits.

$\mathcal{A}$ is locally finitely presentable (lfp) if

1. it is cocomplete,
2. each $x \in \mathcal{A}$ is the directed colimit of fp objects,
3. and there is, up to isomorphism, only a set of fp objects.

Examples of lfp categories include:

- **Set** (but not **Fin**). Here, fp = finite.
- Category of σ-structures. If σ is finite, then fp = finite.
- Any variety of algebras, with the usual fp algebras.

Note: Any lfp category admits a proper factorisation system $(\mathcal{Q}, \mathcal{M})$.

Spaces of homomorphisms

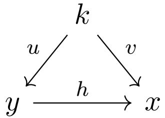
Let $\mathcal{A}$ be a lfp category. An object $x \in \mathcal{A}$ has finite type if $\hom(k, x)$ is finite for every fp object $k \in \mathcal{A}$.

If x has finite type then, for all $y \in \mathcal{A}$, the set $\hom(y, x)$ is naturally equipped with a Stone topology. Just observe that, if $y \cong \operatorname{colim} k_i$ then

$$\hom(y, x) \cong \lim \hom(k_i, x).$$

Explicitly, its Stone topology is generated by the sets of the form

$$\mathcal{O}_{\langle u,v \rangle} := \{h \in \hom(y, x) \mid h \circ u = v\}$$



for k a fp object, $u \in \hom(k, y)$ and $v \in \hom(k, x)$.

A local homomorphism counting result

Lemma

The following hold for every object x of finite type:

1. $\hom(x,x)$ *is a profinite monoid w.r.t. composition.*
2. $\mathcal{M}(x,x)$ *is a closed submonoid of* $\hom(x,x)$.

Theorem

Let $\mathcal{A}$ be a lfp category. For any two objects x, y of finite type,

$$x \cong y \iff |\hom(p,x)| = |\hom(p,y)| \text{ for all fg objects } p.$$

- p is finitely generated (fg) iff $\hom(p,-)$ preserves directed colimits of monomorphisms.
- We can drop the assumption of well-founded posets of quotients.
- The Theorem relies on Numakura's Lemma: A compact Hausdorff topological monoid that satisfies the left-cancellation law is a group.

Examples

The category $\mathcal{T}$ of trees and tree morphisms is lfp. In $\mathcal{T}$, fp = fg = finite trees. On the other hand, the objects of finite type in $\mathcal{T}$ are precisely the finitely branching trees.

Corollary

For any two finitely branching trees x, y,

$$x \cong y \iff |\hom(p, x)| = |\hom(p, y)| \text{ for all finite trees } p.$$

Further, we recover Dvořák's result link by applying the local homomorphism counting result to the (lfp) category of coalgebras for the so-called pebbling comonad on σ-structures (introduced by Abramsky, Dawar and Wang), which captures *pebble games*.

Relevant bibliography

Homomorphism counting:

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Thank you for your attention!