Reducts of representable relation algebras: finite representability and axiomatisability

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BLAST 2021, that is:

Boolean Algebras

Lattices, **A**lgebraic and Quantum logic

Universal Algebra

Set Theory, Set-theoretic and Point-free **T**opology

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Relation algebras and their representations

Relation algebras

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Relation algebra

A relation algebra is an algebra $\mathcal{R} = \langle R, 0, +, -, ;, ^\smile, \mathbf{1}' \rangle$ such that

- $\langle R, 0, +, - \rangle$ is a Boolean algebra,
- $\langle R, ;, \mathbf{1}' \rangle$ is a monoid
- $a^{\smile\smile} = a$,
- $(a + b)^\smile = a^\smile + b^\smile$,
- $(a; b)^\smile = b^\smile; a^\smile$,
- $a^\smile; (-(a; b)) \leqslant -b$.

$$(a + b); c = a; c + b; c$$

RA is the class of all relation algebras.

Representable Relation Algebras

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Representable relation algebras (RRA)

A proper relation algebra (**PRA**) is an algebra $\mathcal{R} = \langle R, \varnothing, 1, \cup, -, ;, ^{\smile}, \mathbf{1'} \rangle$ such that

1. $R \subseteq \mathcal{P}(X \times X)$, where X is a non-empty base set
2. $0 = \varnothing$, 1 is the top element of R
3. R is closed under Boolean operations, composition, taking converses and it contains the diagonal relation (denoted as **1'**).

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3. R is closed under Boolean operations, composition, taking converses and it contains the diagonal relation (denoted as **1'**).

A relation algebra $\mathcal{R} \in$ **RA** is *representable* if it is isomorphic to the subalgebra of some $\mathcal{R}' \in$ **PRA**.

The connection between RRA and RA

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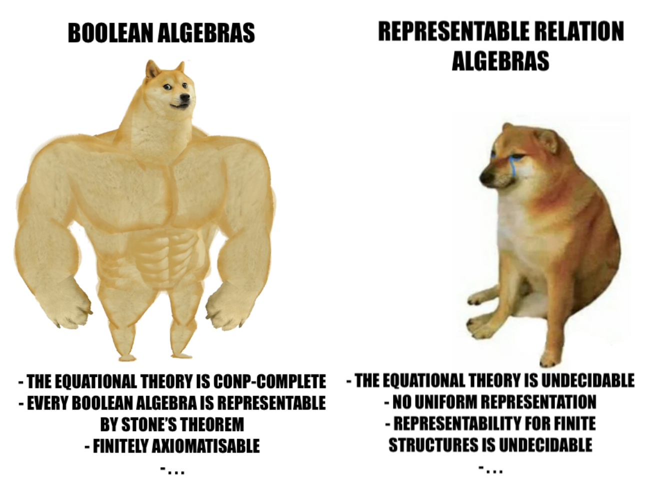
In constrast to Boolean algebras, relation algebras are quite badly behaved due to the following results.

- There exist non-representable relation algebras [Lyndon 1950],
- **RRA** is a variety, but it has no finite axiomatisation [Monk 1964], the equational theory is undecidable [Tarski 1941],
- Representability is undecidable for finite relation algebras [Hirsch, Hodkinson 2001]
- ...

The connection between RRA and RA

One may compare Boolean and representable relation algebras as follows. We consider Boolean algebras as algebras of unary relations.



Reducts of relation algebras

Representable reducts of relation algebras

We need a bit of definitions to deal with reducts of representable relation algebras.

Subsignatures

Let τ be a subset of operations and predicates definable in **RA**. $\mathbf{R}(\tau)$ is the class of subalgebras of τ-subreducts of algebras belonging to **RRA**. We assume that $\mathbf{R}(\tau)$ is closed under isomorphic copies.

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Examples

- $\mathbf{R}(;, \mathbf{1}', \leqslant)$ — the class of representable ordered monoids.
- $\mathbf{R}(+, \cdot, ;, 0, 1)$ — the class of representable bounded distributive lattice ordered semigroups.
- ..., etc

Finite axiomatisability

Aspects of finite axiomatisability

- In constrast to the class of representable relation algebras **RRA**, some classes of reducts are finitely axiomatisable.
- Finite axiomatisability of some class might be refuted by providing a sequence of non-representable algebras whose non-principal ultraproduct is representable.
- There are plenty of open questions related to finite axiomatisability of varieties generated by representable reducts, etc.

Finite representability

The finite representation property

- Let τ be a subsignature, the class $\mathbf{R}(\tau)$ has the finite representation property if any finite member of $\mathbf{R}(\tau)$ is representable over a finite base.
- The example of a finite relation algebra that has no finite representation is the point algebra, the algebra of relations $=$, $>$, and $<$ on the rational line [Maddux 1991], [Hirsch 1995].
- One may investigate the aspects of finite representability for other algebras of relations, such as cylindric or polyadic ones [Andréka, Hodkinson, Németi 1999].

Finite representability

A bit of background on the FRP

- Let $\mathbf{R}(\tau)$ be a class of representable reducts, then the presence of a recursively enumerable axiomatisation and finite representability of $\mathbf{R}(\tau)$ implies that the representability problem is decidable for finite τ-algebras. Recall that this does not hold for all representable algebras due to [Hirsch, Hodkinson 2001],
- Decidability of determining whether an arbitrary finite relation algebra has the representation over a finite base is an open question [Maddux, Hirsch, Hodkinson 2002],
- There are plenty of signatures that have (or have no) the FRP, see [Hirsch 2004].

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Does $\mathbf{R}(;,+)$ have the finite representation property?

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The problems we are interested in are the following:

Hirsch, Hodkinson. "Relation algebras by Games", Problem 19.17

Does $\mathbf{R}(;, \backslash, /, \leqslant)$ have the finite representation property? **Yes, we have found a positive solution**

Hirsch, 2004

Does $\mathbf{R}(;, +)$ have the finite representation property? **Perhaps, we guess that it does have the FRP. We only show that $\mathbf{R}(;, +)$ has a universal recursively enumerable axiomatisation**

Residuated semigroups

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The notion of a residuated semigroup

A residuated semigroup is a structure $\mathcal{A} = \langle A, ;\,, \leqslant, \backslash, / \rangle$ such that:

1. $\langle A, ;\,, \leqslant \rangle$ is a partially ordered semigroup,
2. $\backslash, /$ are binary operations satisfying the residuation property:

$$b \leqslant a \backslash c \Leftrightarrow a; b \leqslant c \Leftrightarrow a \leqslant c / b$$

Relation algebras and residuals

Residuals are definable in relation algebras as follows:

Residuals in RA

1. $a \backslash b = -(a^{\smile}; -b)$
2. $a / b = -(-a; b^{\smile})$

Residuals in RRA

1. $a \backslash b = \{\langle x, y \rangle \mid \forall z\ (z, x) \in a \Rightarrow (z, y) \in b\}$
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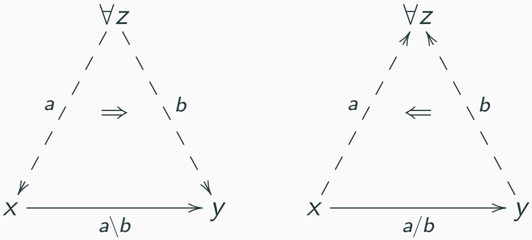
A bit of visualisation:



Quantales

Our argument uses quantales and some related results.

Quantale

A quantale is a structure $\mathcal{Q} = \langle Q, ;, \Sigma \rangle$ such that $\langle Q, \Sigma \rangle$ is a complete lattice, $\langle Q, ; \rangle$ is a semigroup such that the operation ; preserves suprema in each argument.

Note that every quantale is a residuated semigroup, residuals are uniquely defined as follows:

1. $a \backslash b = \Sigma\{c \mid a; c \leqslant b\}$,
2. $a / b = \Sigma\{c \mid b; c \leqslant a\}$.

A quantic nucleus

Let $\langle P, \leqslant \rangle$ be a poset, then a *closure operator* is a monotone map $j : P \to P$ such that $a \leqslant ja = jja$ for each $a \in P$.

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Let $\langle P, \leqslant \rangle$ be a poset, then a *closure operator* is a monotone map $j : P \to P$ such that $a \leqslant ja = jja$ for each $a \in P$. Let $a \in P$, then a lower cone generated by a is a subset having the form $\downarrow a = \{x \in P \mid x \leqslant a\}$.

Quantic nucleus

A *quantic nucleus* on a quantale $\mathcal{Q}$ is a closure operator such that $ja; jb \leqslant j(a; b)$. $a \in \mathcal{Q}$ is j-closed, if $a = ja$. Note that the set of all j-closed elements forms a subquantale of $\mathcal{Q}$ where $\Sigma_j A = j(\Sigma A)$ and $a;_j b = j(a; b)$.

The representation theorem for residuated semigroups

Let $\langle X, \leqslant \rangle$ be a poset and $S \subseteq X$, then lS (uS) is the set of all lower (upper) bounds. Let us put $mS = luS$.

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Let $\langle X, \leqslant \rangle$ be a poset and $S \subseteq X$, then lS (uS) is the set of all lower (upper) bounds. Let us put $mS = luS$.

It is clear that m is the closure operator on the poset $\langle \mathcal{P}(X), \subseteq \rangle$. Let $(\mathcal{P}(X))_m = \{S \subseteq X \mid mS = S\}$, then $\langle (\mathcal{P}(X))_m, \subseteq \rangle$ is a complete lattice, where $\Pi S = \bigcap S$ and $\Sigma S = m(\bigcup S)$.

Then the map $f_m : \langle X, \leqslant \rangle \to \langle (\mathcal{P}(X))_m, \subseteq \rangle$ such that $f_m : x \mapsto \downarrow x$ preserves any existing suprema and infima. Note that this map is well-defined since $\downarrow x$ is m-closed.

See [Davey, Priestley 2002] to have more details.

The representation theorem for residuated semigroups

Now we extend the construction above for an arbitrary residuated semigroup as follows:

Theorem [Goldblatt, 2006]

Let $\mathcal{S}$ be a residuated semigroup, then $\mathcal{S}$ isomorphic to the residuated subsemigroup of some quantale.

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Let us take a look at the proof sketch.

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Proof.

Let us take a look at the proof sketch. Let $\mathcal{S} = \langle S, \leqslant, ;, \backslash, / \rangle$ be a residuated semigroup. The closure operator mX is a quantic nucleus on the free quantale $\langle \mathcal{P}(S), \bullet, \subseteq \rangle$ since $m(X) \bullet m(Y) \subseteq m(X \bullet Y)$. Thus, $\langle (\mathcal{P}(S))_m, \bullet, \subseteq \rangle$ is a quantale.

After that one needs to show that this embedding $f_m : \mathcal{S} \to \langle (\mathcal{P}(S))_m, \bullet, \subseteq \rangle$ preserves products and residuals. □

Representation of quantales

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Relational quantale

Let A be a non-empty set. A relational quantale on A is an algebra $\langle R, \subseteq, ; \rangle$, where

1. $R \subseteq \mathcal{P}(A \times A)$,
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Theorem [Brown, Gurr 1993]

Every quantale $\mathcal{Q} = \langle Q, ; , \Sigma \rangle$ is isomorphic to a relational quantale on Q as the base set.

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$$\hat{a} = \{\langle g, q \rangle \mid g \in \mathcal{G}(\mathcal{Q}), q \in \mathcal{Q}, g \leqslant a; q\} \quad \widehat{\mathcal{Q}} = \{\hat{a} \mid a \in \mathcal{Q}\}$$

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The map $f : a \mapsto \hat{a}$ satisfies the following conditions:

1. $a \leqslant b$ iff $\hat{a} \subseteq \hat{b}$,
2. $\widehat{\Sigma A} = \Sigma \widehat{A}$, $\hat{a}; \hat{b} = \widehat{a; b}$, and $\langle \widehat{\mathcal{Q}}, \subseteq, \Sigma\rangle$ is a complete lattice,
3. $\langle \widehat{\mathcal{Q}}, \subseteq, ;\rangle$ is a relational quantale,
4. $\mathcal{Q}$ is isomorphic to $\langle \widehat{\mathcal{Q}}, \subseteq, ;\rangle$,
5. f is a quantale isomorphism.

The solution for [Problem 19.17, Hirsch, Hodkinson, 2002]

Representation

Let $\tau = \{;, \backslash, /, \leqslant\}$, $\mathcal{A}$ a τ-structure and X a non-empty set. An interpretation R over a base X maps every $a \in \mathcal{A}$ to a binary relation $a^R \subseteq X \times X$. A representation of $\mathcal{A}$ is an interpretation R satisfying the following conditions:

1. $a \leqslant b$ iff $a^R \subseteq b^R$,
2. $(a; b)^R = \{(x, y) \mid \exists z \in X \; (x, z) \in a^R \;\&\; (z, x) \in b^R\} = a^R; b^R$,
3. $(a \backslash b)^R = \{(x, y) \mid \forall z \in X \; ((z, x) \in a^R \Rightarrow (z, y) \in b^R)\} = a^R \backslash b^R$,
4. $(a \,/\, b)^R = \{(x, y) \mid \forall z \in X \; ((y, z) \in a^R \Rightarrow (x, z) \in b^R)\} = a^R \,/\, b^R$.

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Lemma

Let τ be a signature of residuated semigroups. An interpretation $R : \mathcal{A} \to \widehat{\mathcal{Q}_{\mathcal{A}}}$ such that $R : a \mapsto a^R = \widehat{\downarrow a}$ is a τ-representation.

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Theorem [D.R. 2020]

Every residuated semigroup is isomorphic to the subalgebra of a relational quantale.
Every finite residuated semigroup is representable over a finite base.

The case of join semilattice-ordered semigroups

A bit of definitions

Join semilattice-ordered semigroup

A join semilattice ordered semigroup is an algebra $\mathcal{A} = \langle A, +, ; \rangle$ such that $\langle A, + \rangle$ is a join semilattice (that is, every non-empty finite set has the least upper bound) and $\langle A, ; \rangle$ is a semigroup. That is, ; is a binary associative operation. ; and $+$ are connected with each other as follows, for all $a, b, c \in \mathcal{A}$:

$$a; (b + c) = a; b + a; c$$

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$$a; (b + c) = a; b + a; c$$

$$(a + b); c = a; c + b; c$$

As usual, $a \leqslant b$ iff $a + b = b$. The set $\mathrm{Up}(\mathcal{A})$ is the set of all upward closed subsets of $\mathcal{A}$. Recall that a set X is called upward closed if $x \in X$ and $x \leqslant y$ implies $y \in X$.

Characterising representability using games

Representation

A representation of a join semilattice-ordered semigroup $\mathcal{A}$ over a base $X \neq \emptyset$ is an injective map $f : \mathcal{A} \to 2^{X \times X}$ such that:

- $f(a + b) = f(a) \cup f(b)$
- $f(a; b) = f(a); f(b)$

We denote the class of representable join semilattice ordered semigroups as $R(;,+)$

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- There exists a countable sequence $(\mathcal{A}_i)_{i<\omega}$ of non-representable join semilattice ordered semigroups such that any non-principal ultraproduct $\Pi_U A_i$ is representable. Here, U is any non-principal ultrafilter over ω. For this reason, $R(;, +)$ is not finitely axiomatisable.
- We characterise representability in terms of games.

Characterising representability using games

Networks for building representations

Let $\mathcal{A}$ be a join-semilattice ordered semigroup. A *prenetwork* over $\mathcal{A}$ is a triple (V, E, l), where V is a set of vertices, E is a set of edges such that $\langle V, E \rangle$ is a directed graph, and l is a labelling function $l : E \to \mathrm{Up}(\mathcal{A})$.

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1. **(The saturation condition)** For every $u, v \in V$ and for every $x, y, z \in \mathcal{A}$ $z \in l(u, v)$ and $z \leqslant x ; y$ implies $x \in l(u, w)$ and $y \in l(w, v)$ for some $w \in V$.
2. **(The coherence conidition)** For every $u, v, w \in V$ one has $l(u, v); l(v, w) \subseteq l(u, w)$.
3. **(The join-primeness condition)** For every $u, v \in V$ $l(u, v)$ is join-prime. That is, for each $a, b \in \mathcal{A}$ if $a + b \in l(u, v)$, then either $a \in l(u, v)$ or $b \in l(u, v)$.

Let $n \leqslant \omega$ and $\mathcal{A}$ a join semilattice-ordered semigroup. A play of the game $\mathcal{G}_n(\mathcal{A})$ has n rounds and consists of n finitary networks. We have two players, $\forall$ (Abelard) and $\exists$ (Héloïse). The game consists of the following rounds:

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Round 0

$\forall$ picks $a, b \in \mathcal{A}$ such that $a \not\leqslant b$. $\exists$ has to find a prenetwork $\mathcal{N}_0 = (V_0 = \{x_0, x_1\}, E_0 = \{(x_0, x_1)\}, l_0)$ such that $l_0(x_0, x_1) = A$, where A is an upward closed set such that $a \in A$ and $b \notin A$.

Characterising representability using games

Round $k+1$. Suppose, the prenetwork $\mathcal{N}_k = (V_k, E_k, l_k)$ has been played. $\forall$ has several options for his move. Let us have a look only at one of them. We omit the rest ones.

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Composition move

$\forall$ picks $x, y, z \in V_k$ with $b \in l_k(x, y)$ and $c \in l_k(y, z)$. Then $\exists$ must play with $\mathcal{N}_{k+1} = (V_{k+1}, E_{k+1}, l_{k+1})$ such that $\mathcal{N}_{k+1}$ is the same as $\mathcal{N}_k$, but $l_{k+1}(x, z) = \uparrow(l_k(x, z) \cup \{b \,;\, c\})$.

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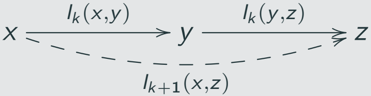

$\mathcal{N}_\omega = \bigcup_{i<\omega} \mathcal{N}_i$ (unless the game termitates at some finite round)

Characterising representability using games. The presence of a winning strategy

$\forall$ wins the play if $b \in I_i(x, y)$ for some $i \in n$ (where $n \leqslant \omega$). This b is the same b as in Round 0. Otherwise, $\exists$ wins the play.

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Lemma

Let $\mathcal{A} = \langle A, ;, + \rangle$ be a join semilattice-ordered semigroup,

1. If $\mathcal{A}$ is representable then $\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathcal{A})$.
2. If $|\mathcal{A}| \leqslant \omega$ and $\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathcal{A})$ then $\mathcal{A}$ is representable.

The axiomatisation of R$(;,+)$

Definition

Let Var be a set of variables. The set of terms T is generated by the following grammar:

$$\text{Term} ::= \text{Var} \mid (T + T) \mid (T ; T)$$

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$$\text{Term} ::= \text{Var} \mid (T + T) \mid (T; T)$$

A *term network* is a triple $\langle V, E, l\rangle$, where $\langle V, E\rangle$ is a finite directed graph and $l : E \to 2^{Term}$ is a labelling function such that every $l(x, y)$ is finite for each $(x, y) \in E$.

Let $\mathcal{A}$ be a join semilattice-ordered semigroup and $\vartheta : \text{Var} \to \mathcal{A}$ a valuation. Let us define the prenetwork $\mathcal{N}^{\vartheta}$ with the same edges and vertices with labelling $l^{\vartheta}(x, y) = \uparrow \vartheta[l_{\mathcal{N}}(x, y)]$.

The axiomatisation of R$(;, +)$

Lemma

Let $\mathcal{A}$ be a join semilattice-ordered semigroup and $\vartheta : \mathsf{Var} \to \mathcal{A}$ a valuation. For each $n < \omega$ there exists a first-order sentence ρ_n such that $\exists$ has a winning strategy in $\mathcal{G}_n(\mathcal{A})$ iff $\mathcal{A} \models \rho_n$.

As usual, for each $n < \omega$ we construct a formula σ_n such that:

$$\exists \text{ has a winning strategy in } \mathcal{G}_n(\mathcal{N}^{\vartheta}, \mathcal{A}, \vartheta(v)) \text{ if and only if } \mathcal{A} \models \sigma_n(\mathcal{N}, v)$$

where $\mathcal{N}$ is a term network.

The axiomatisation of $\mathsf{R}(;,+)$

Lemma

Let $\mathcal{A}$ be a join semilattice-ordered semigroup and $\vartheta : \mathsf{Var} \to \mathcal{A}$ a valuation. For each $n < \omega$ there exists a first-order sentence ρ_n such that $\exists$ has a winning strategy in $\mathcal{G}_n(\mathcal{A})$ iff $\mathcal{A} \models \rho_n$.

As usual, for each $n < \omega$ we construct a formula σ_n such that:

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where $\mathcal{N}$ is a term network. Let v_0 be any variable and A a finite set of terms that contains v_0, $\mathcal{N}_{v_0}$ denotes the term network having the form $\langle \{\{x_0, x_1\}, \{(x_0, x_1)\}, l\} \rangle$, where $l(x, y) = A$. We define the following sequence of formulas $(\rho_n)_{n<\omega}$:

$$\rho_n = \forall v_0 \forall v_1 (\neg(v_0 \leqslant v_1) \to \sigma(\mathcal{N}_{v_0}, v_0))$$

The axiomatisation of R$(;, +)$

Let us define a two sorted language with sorts **a** (algebra) and **r** (representation). ρ_n might be equivalently transformed into a universal formula. So **R**$(+, ;)$ is a pseudo-universal class defined as $\{\mathcal{M} \restriction_{\mathbf{a}} \mid \mathcal{M} \models (\rho_m)_{n<\omega}\}$. One may obtain the explicit universal axiomatisation of **R**$(;, +)$ using the elementary chain argument.

The axiomatisation of $\mathbf{R}(;,+)$

Let us define a two sorted language with sorts **a** (algebra) and **r** (representation). ρ_n might be equivalently transformed into a universal formula. So $\mathbf{R}(+,;)$ is a pseudo-universal class defined as $\{\mathcal{M} \restriction_{\mathbf{a}} \mid \mathcal{M} \models (\rho_m)_{n<\omega}\}$. One may obtain the explicit universal axiomatisation of $\mathbf{R}(;,+)$ using the elementary chain argument.

Finally, we have

Theorem [D.R. 2021]

$\mathbf{R}(+,;)$ is axiomatised with the axioms of join semilattice-ordered semigroups and $\{\rho_n\}_{n<\omega}$. Moreover, $\mathbf{R}(+,;)$ has a recursively enumerable axiomatisation.

Summary

A couple of open questions

- We have not proved yet that $\mathbf{R}(+, ;)$ has the finite representation property. If so, the representability problem for finite join semilattice-ordered semigroups is decidable.
- (The Failure of) finite axiomatisability of the first-order theory of representable lower semilattice-ordered residuated semigroups. The current conjecture is that this class is not finitely axiomatisable.

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Thank you for your kind attention!

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