Directed games

Modifying Ehrenfeucht-Fraïssé games for stratified sentences

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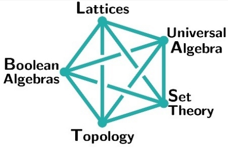

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Summary

In this talk, we are going to

- define a modified version of Ehrenfeucht-Fraïssé games we call directed games.
- show how these games can be used to establish certain decidability results.
- describe how these games can be used to attack more complex decidability problems.

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Introduction

Many-sorted languages

Directed games are more naturally defined for many-sorted structures. Let us briefly review some basic facts about many-sorted languages.

In many-sorted languages, each symbol is assigned a unique **type**[1] or tuple of types from a given non-empty set I:

- For each $i \in I$, there are countably many variables $x_1^i, x_2^i, \ldots$ of type i.
- Each constant symbol c^i is of a certain type $i \in I$.
- Each n-place relation symbol R is of a certain tuple of types $(i_1, \ldots, i_n) \in I^n$.
- Each n-place function symbol f a of a certain tuple of types $(i_1, \ldots, i_{n+1}) \in I^{n+1}$.

[1] Also called sort.

Many-sorted languages

Each **term** t of a many-sorted language $\mathcal{L}$ is defined and assigned a type recursively in the usual way, e.g.,

> *if f is an n-place function symbol of type $(i_1, \ldots, i_{n+1})$ and $t_1, \ldots, t_n$ are terms of types $i_1, \ldots, i_n$, then $f(t_1, \ldots, t_n)$ is a term of type i_{n+1}.*

The **atomic formulas** of a many-sorted language $\mathcal{L}$ are all $R(t_1, \ldots, t_n)$, where R is a an n-place relation symbol of type $(i_1, \ldots, i_n)$ and $t_1, \ldots, t_n$ are terms with types $i_1, \ldots, i_n$.

Formulas are built up from atomic formulas using connectives and $\forall x^i \phi$ in the usual way.

Many-sorted languages

As we know,

Theorem[2]

Many-sorted logic can be translated into one-sorted logic.

Still, many-sorted languages are quite useful for expressing theories where there are more than one kinds of objects (e.g., Simple Type Theory).

It is important to remember that

Note

All the usual model-theoretic notions and results can be expressed in terms of many-sorted languages (e.g. structures, isomorphism between structures, Compactness, Löwenheim-Skolem theorem, saturated models, etc).

[2]For example, see H. Enderton. *A mathematical introduction to logic*. Second edition. Harcourt/Academic Press, Burlington, MA, 2001.

Let $\mathcal{L}$ be some many-sorted language with a non-empty set of types I. Let $\mathscr{A}$ and $\mathscr{B}$ be two $\mathcal{L}$-structures. Let $n > 0$.

Description of the game $\mathrm{G}_n(\mathscr{A}, \mathscr{B})$

The game has n rounds. Suppose that we are in the i-th round of the game. Player[3] I plays first and chooses an element a_i from $\mathscr{A}$ or an element b_i from $\mathscr{B}$, in which case Player II must respond by choosing some element b_i of the same type from $\mathscr{B}$ or some element a_i from $\mathscr{A}$ respectively.

Winning condition in $\mathrm{G}_n(\mathscr{A}, \mathscr{B})$

Player II wins the game if the mapping corresponding to the pairs of elements $\{(a_i, b_i) : 1 \leq i \leq n\}$ is a partial isomorphism from $\mathscr{A}$ to $\mathscr{B}$, otherwise he loses.

[3] The two players are often called Spoiler and Duplicator (or Eloise and Abelard).

Theorem

If Player II has a winning strategy in $\mathrm{G}_n(\mathscr{A}, \mathscr{B})$, then any sentence with n quantifiers is true in $\mathscr{A}$ if and only if it is true in $\mathscr{B}$.

Ehrenfeucht–Fraïssé games are used to establish decidability.

Definition

An $\mathcal{L}$-sentence σ is **decidable** by an $\mathcal{L}$-theory T if $T \vdash \sigma$ or $T \vdash \neg\sigma$.

Corollary

If for all models $\mathscr{A}, \mathscr{B}$ of an $\mathcal{L}$-theory T Player II has a winning strategy in $\mathrm{G}_n(\mathscr{A}, \mathscr{B})$, then T decides all sentences with n-quantifiers.

Let $\mathcal{L}$ be some many-sorted language with a non-empty set of types I. Let $\mathscr{A}$ and $\mathscr{B}$ be two $\mathcal{L}$-structures. Let $\bar{s} = (s_1, \ldots, s_n) \in I^n$ for some $n > 0$.

Description of the directed game $\mathrm{G}_n^{\bar{s}}(\mathscr{A}, \mathscr{B})$

The game has n rounds. Suppose that we are in the i-th round of the game. Player I plays first, and

- either chooses an element a_i of type s_i from $\mathscr{A}$ in which case player II must respond by choosing b_i of type s_i from $\mathscr{B}$,
- or an element b_i of type s_i from $\mathscr{B}$ in which case player II must respond by choosing a_i of type s_i from $\mathscr{A}$.

Winning condition in $\mathrm{G}_n^{\bar{s}}(\mathscr{A}, \mathscr{B})$

Player II wins the game if the mapping corresponding to the pairs of elements $\{(a_i, b_i) : 1 \leq i \leq n\}$ is a partial isomorphism from $\mathscr{A}$ to $\mathscr{B}$, otherwise he loses.

Note

The games $\mathrm{G}_n(\mathscr{A}, \mathscr{B})$ and $\mathrm{G}_n^{\bar{s}}(\mathscr{A}, \mathscr{B})$ have the same winning conditions. The only difference between the two games is that in $\mathrm{G}_n^{\bar{s}}(\mathscr{A}, \mathscr{B})$, Player I can only choose elements of a certain prespecified type in each round.

This difference has the following effect on decidability.

Theorem

If Player II has a winning strategy in $\mathrm{G}_n^{\bar{s}}(\mathscr{A}, \mathscr{B})$, then for all quantifiers $\mathrm{Q}_1, \ldots, \mathrm{Q}_n$ and quantifier-free $\mathcal{L}$-formulas ϕ, the sentence

$$\mathrm{Q}_1 x_1^{s_1} \ldots \mathrm{Q}_n x_n^{s_n} \phi(x_1^{s_1}, \ldots, x_n^{s_n})$$

is true in $\mathscr{A}$ if and only if it is true in $\mathscr{B}$.

Corollary

If for all models $\mathscr{A}, \mathscr{B}$ of an $\mathcal{L}$-theory T Player II has winning strategy in $\mathrm{G}_n^{\bar{s}}(\mathscr{A}, \mathscr{B})$, then T decides all sentences $\mathrm{Q}_1 x_1^{s_1} \ldots \mathrm{Q}_n x_n^{s_n} \phi(x_1^{s_1}, \ldots, x_n^{s_n})$, where $\mathrm{Q}_1, \ldots, \mathrm{Q}_n$ are quantifiers and ϕ is quantifier-free.

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Using Directed games

The Theory of Simple Types or **Simple Type Theory (TST) is a simplification**[4] **of Russell's Type Theory**.

The **language $\mathcal{L}_{\mathrm{TST}}$ of Simple Type Theory** is the many-sorted language of set theory with two binary relation symbols ε_i and $=_i$ for each type $i \in \omega$ (where ε_i is of type $(i, i+1)$) and $=_i$ is of type (i, i)). The $\mathcal{L}_{\mathrm{TST}}$-formulas are built inductively from the atomic formulas $x^i \varepsilon_i y^{i+1}$ and $x^i =_i y^i$ in the usual way.

We often expand $\mathcal{L}_{\mathrm{TST}}$ to include the binary relation symbols $\subseteq_i$ (for each $i > 0$) interpreted as the usual subset relation, as well as the unary relation symbols $F_{i,n}$ (for each $i, n > 0$) interpreted in such a way that $F_{i,n}(x^i)$ is equivalent to the statement "x^i has at least n elements". The language $\mathcal{L}_{\mathrm{TST}} \cup \{\subseteq_1, \subseteq_2, \ldots, F_{i,1}, F_{i,2}, \ldots\}$ is denoted as $\mathcal{L}_{\mathrm{TST}}^{\subseteq, F}$.

[4] Proposed by Frank Ramsey in *The Foundations of Mathematics*, Proceedings of the London Mathematical Society, 1926

Simple Type Theory is axiomatized by the following two axioms.

Axiom of Extensionality (Ext)

For each $i \in \omega$,

$$\forall x^{i+1} \forall y^{i+1} (x^{i+1} =_{i+1} y^{i+1} \leftrightarrow \forall z^{i} (z^{i} \, \varepsilon_i \, x^{i+1} \leftrightarrow z^{i} \, \varepsilon_i \, y^{i+1})).$$

Axiom of Comprehension (Co)

For each $i \in \omega$ and $\mathcal{L}_{\mathrm{TST}}$-formula ϕ such that y^{i+1} is not free in ϕ,

$$\forall \bar{u} \exists y^{i+1} \forall x^{i} (x^{i} \, \varepsilon_i \, y^{i+1} \leftrightarrow \phi(x^{i}, \bar{u})),$$

Two important weak versions of Comprehension are the following:

- Co(O) is the axiom we get if we restrict ϕ to quantifier-free $\mathcal{L}_{\mathrm{TST}}$-formulas.
- Co$_n$ is the axiom we get if restrict Co to sentences that contain only variables of n consecutive types.

We usually want our $\mathcal{L}_{\mathrm{TST}}$-theories to also satisfy the following weak axiom of infinity.

Scheme of Infinity (Inf)

There are infinitely many (with respect to the metatheory) elements of type 0, i.e.

$$\{\exists x_1^0 \dots \exists x_n^0 (\bigwedge_{i \neq j} x_i^0 \neq_i x_j^0) : n > 0\}.$$

We let

$$\begin{aligned} \mathrm{TST} &= \mathsf{Ext} + \mathsf{Co}, & \mathrm{TST}^{\infty} &= \mathsf{Ext} + \mathsf{Co} + \mathsf{Inf}, \\ \mathrm{TSTO} &= \mathsf{Ext} + \mathsf{Co}(O), & \mathrm{TSTO}^{\infty} &= \mathsf{Ext} + \mathsf{Co}(O) + \mathsf{Inf}, \\ \mathrm{TST}_{(n)} &= \mathsf{Ext} + \mathsf{Co}_n, & \mathrm{TST}_{(n)}^{\infty} &= \mathsf{Ext} + \mathsf{Co}_n + \mathsf{Inf}. \end{aligned}$$

Note

We also denote by TST (similarly for the other theories) the extension by definitions that includes the definitions of $\subseteq_i$ and $F_{i,n}$ for each $i, n > 0$.

A simple example: Decidability of increasing sentences

Definition

An $\mathcal{L}_{\mathrm{TST}}$-formula is **increasing** if it is of the form

$$\mathrm{Q}_1 x_1^{s_1} \ldots \mathrm{Q}_n x_n^{s_n} \phi(x_1^{s_1} \ldots x_n^{s_n}, \bar{y}),$$

where

- ϕ is quantifier-free,
- $\mathrm{Q}_1, \ldots, \mathrm{Q}_n$ are quantifiers,
- $s_1 \leq \ldots \leq s_n$, and
- the types of all variables $\bar{y}$ are less or equal to s_1.

In particular, all quantifier-free $\mathcal{L}_{\mathrm{TST}}$-formulas are considered increasing.

We will show that

Theorem

$\mathrm{TST}^{\infty}_{(2)}$ decides all increasing $\mathcal{L}_{\mathrm{TST}}$-sentences.

Proof. Let $\mathscr{A} = (A_0, A_1, \ldots, \varepsilon_0^{\mathscr{A}}, \varepsilon_1^{\mathscr{A}}, \ldots)$ and $\mathscr{B} = (B_0, B_1, \ldots, \varepsilon_0^{\mathscr{B}}, \varepsilon_1^{\mathscr{B}}, \ldots)$ be models of $\mathrm{TST}^{\infty}_{(2)}$. Let $\bar{s} = (s_1, \ldots, s_n) \in \omega^n$ such that $s_1 \leq \ldots \leq s_n$. It suffices to show that Player II has a winning strategy in $G_n^{\bar{s}}(\mathscr{A}, \mathscr{B})$.

We describe the winning strategy of Player II. Assume that we are in Round k of the game, and let

$$(a_1, b_1) \in A_{s_1} \times B_{s_1}$$

$$\vdots$$

$$(a_{k-1}, b_{k-1}) \in A_{s_{k-1}} \times B_{s_{k-1}}$$

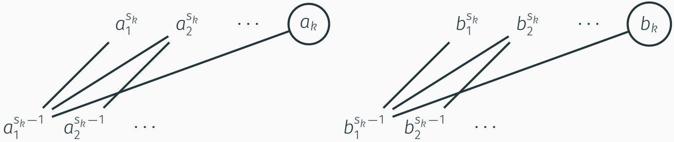
be the pairs of elements chosen by the two player in rounds 1 to $k-1$. Suppose that Player I chooses an element $a_k \in A_{s_k}$ (similarly if he chooses some element $b_k \in B_{s_k}$).

We may assume that $a_k \notin \{a_i : 1 \leq i < k \text{ and } s_i = k\}$. If $s_k = 0$, then let b_k be any element of $B_0 - \{b_1, \ldots, b_{k-1}\}$.

If $s_k > 0$, then let b_k be any element of $B_{s_k} - \{b_i : 1 \leq i < k \text{ and } s_i = s_k\}$ such that for all $1 \leq i < k$ for which $s_i = s_k - 1$,

$$b_i \, \varepsilon^{\mathscr{B}}_{s_k-1} \, b_k \Leftrightarrow a_i \, \varepsilon^{\mathscr{A}}_{s_k-1} \, a_k.$$

Such a b_k exists because B_{s_k} is infinite (since $\mathscr{B} \models \mathsf{Inf}$) and any finite subset of $B_{s_{k-1}}$ is in B_{s_k} (since $\mathscr{B} \models \mathsf{Co}_2$).



Choosing b_k: Let $(a_1^{s_k-1}, b_1^{s_k-1}), (a_2^{s_k-1}, b_2^{s_k-1}), \ldots$ (resp. $(a_1^{s_k}, b_1^{s_k}), (a_2^{s_k}, b_2^{s_k}), \ldots$) be the pairs of elements of type $s_k - 1$ (resp. s_k) played in rounds 1 to $k-1$. The element b_k is chosen in such a way that the two graphs of $\varepsilon^{\mathscr{A}}_{s_k-1}$ and $\varepsilon^{\mathscr{B}}_{s_k-1}$ are isomorphic.

$\square$

Other results obtained using directed games

Theorem

$\mathrm{TST}^{\infty}_{(2)}$ decides all pseudo-increasing $\mathcal{L}^{\subseteq,F}_{\mathrm{TST}}$-sentences, i.e. all $\mathcal{L}^{\subseteq,F}_{\mathrm{TST}}$-sentences

$$\mathrm{Q}_1 x_1^{s_1} \ldots \mathrm{Q}_n x_n^{s_n} \phi,$$

where $\mathrm{Q}_1, \ldots, \mathrm{Q}_n$ are quantifiers, ϕ is quantifier-free, and for all $1 \leq i \leq n$ and $1 \leq j \leq n$,

$$\text{if } x_i^{s_i} \, \varepsilon_{s_i} \, x_j^{s_j} \text{ appears in } \phi\text{, then } s_i \geq \max\{s_k : 1 \leq k < i\}.$$

Note: every increasing sentence is also pseudo-increasing.

Theorem

TSTO^{∞} decides all existential strictly-decreasing $\mathcal{L}_{\mathrm{TST}}$-sentences, i.e. all $\mathcal{L}_{\mathrm{TST}}$-sentences

$$\exists x_1^{r_1} \ldots \exists x_n^{r_n} \mathrm{Q}_1 y_1^{s_1} \ldots \mathrm{Q}_m y_n^{s_m} \phi,$$

where $\mathrm{Q}_1, \ldots, \mathrm{Q}_m$ are quantifiers, ϕ is quantifier-free, and $s_1 > s_2 > \cdots > s_m$.

Breaking down problems

Another easy example: three-quantifier sentences

We already noted that if for all models $\mathscr{A}, \mathscr{B}$ of an $\mathcal{L}$-theory T and all $\bar{s} \in \omega^n$, Player II has a winning strategy in $G_n^{\bar{s}}(\mathscr{A}, \mathscr{B})$, then T decides all $\mathcal{L}$-sentences with n quantifiers.

In most cases, proving that a class of sentences $\mathrm{Q}_1 x_1^{s_1} \ldots \mathrm{Q}_n x_n^{s_n} \phi$ is decidable by some theory T is **easier if we distinguish the cases** for all the different $\bar{s} = (s_1, \ldots, s_n)$.

Let us explain how by giving a simple example. We will show that

Theorem

TST^∞ decides all $\mathcal{L}_{\mathrm{TST}}$-sentences with three quantifiers.

Another easy example: three-quantifier sentences

Proof. Let σ be an $\mathcal{L}_{\mathrm{TST}}$-sentence

$$\mathrm{Q}_1 x^i \mathrm{Q}_2 y^j \mathrm{Q}_3 z^k \phi,$$

where $\mathrm{Q}_1, \mathrm{Q}_2, \mathrm{Q}_3$ are quantifiers and ϕ is quantifier-free.

First of all notice that

> *any two-quantifier sentence is equivalent to an increasing or strictly decreasing sentence,*

which means that it is decidable by TST^∞.

We may therefore assume that $\{i, j, k\}$ is a set of consecutive numbers, otherwise σ is equivalent to a boolean combination of a two-quantifier sentence and an one-quantifier sentence, which means that σ is decidable by TST^∞.

Another easy example: three-quantifier sentences

There are 13 possible cases for (i, j, k):

(i) $(i, i, i+1)$
(ii) (i, i, i)
(iii) $(i, i, i-1)$
(iv) $(i, i+1, i+2)$
(v) $(i, i+1, i+1)$
(vi) $(i, i+1, i)$
(vii) $(i, i+1, i-1)$
(viii) $(i, i-1, i+1)$
(ix) $(i, i-1, i)$
(x) $(i, i-1, i-1)$
(xi) $(i, i-1, i-2)$
(xii) $(i, i+2, i+1)$
(xiii) $(i, i-2, i-1)$

In cases (i), (ii), (iv), (v) σ is increasing, whereas in (iii), (vi), (vii), (xi), (xii), σ is equivalent to an existential strictly-decreasing sentence or a negation of such a sentence.

Another easy example: three-quantifier sentences

It remains to examine the following cases for (i, j, k):

(viii) $(i, i-1, i+1)$

(ix) $(i, i-1, i)$

(x) $(i, i-1, i-1)$

(xiii) $(i, i-2, i-1)$

For case (viii) (similarly for (ix) and (x)), notice that by replacing in σ,

$$z^{i-1} \, \varepsilon_{i-1} \, x^i \text{ with } F_{i,1}(z^i) \wedge \neg F_{i,2}(z^i) \wedge z^i \subseteq_i x^i$$

(similarly for $z^{i-1} \, \varepsilon \, y^i$) and

$$z^{i-1} =_{i-1} z^{i-1} \text{ with } z^i =_i z^i$$

we get TST^∞-equivalent increasing $\mathcal{L}_{\mathrm{TST}}^{\subseteq,F}$-sentence. Case (xiii) is also easy to treat by examining all the possible formulas or by finding a winning strategy for Player II in the corresponding directed game. □

Combining strategies

Combining games

It is often very useful to combine or extend winning strategies for different $\bar{s}$ in a game $\mathrm{G}_n^{\bar{s}}(\mathscr{A}, \mathscr{A}')$. Let us describe yet another simple example.

Lemma

Let $\mathscr{A}$ and $\mathscr{A}'$ be models of $\mathrm{TST}_{(2)}^{\infty}$, and let

$$\bar{s} = (s_1, \ldots, s_n)$$
$$\bar{t} = (s_1, \ldots, s_m, k, s_{m+1}, \ldots, s_n),$$

where

$$n \geq m > 0,$$
$$k \geq \max\{s_1, \ldots, s_m\},$$
$$k > \max\{s_{m+1}, \ldots, s_n\} + 1.$$

If Player II has a winning strategy in $\mathrm{G}_n^{\bar{s}}(\mathscr{A}, \mathscr{A}')$, then he has a winning strategy in $\mathrm{G}_{n+1}^{\bar{t}}(\mathscr{A}, \mathscr{A}')$.

Proof. Let $\mathscr{A} = (A_0, A_1, \ldots, \varepsilon_1^{\mathscr{A}}, \varepsilon_2^{\mathscr{A}}, \ldots)$ and $\mathscr{A}' = (A_0', A_1', \ldots, \varepsilon_1^{\mathscr{A}'}, \varepsilon_2^{\mathscr{A}'}, \ldots)$. We describe the winning strategy of Player II in $\mathrm{G}_{n+1}^{\bar{t}}(\mathscr{A}, \mathscr{A}')$.

Rounds 1 *to* m. For the first m moves, Player II plays according to his winning strategy in $G_n^{\bar{s}}(\mathscr{A}, \mathscr{A}')$. Assume that

$$(a_1, a_1') \in A_{s_1} \times A_{s_1}'$$
$$\vdots$$
$$(a_m, a_m') \in A_{s_m} \times A_{s_m}'$$

are the pairs of elements chosen by the two players in rounds 1 to m respectively.

Rounds $m+1$. Player I chooses an element $b \in A_k$ (similarly if Player I chooses an element $b' \in A'_k$).

We may assume that $b \notin \{a_i : 1 \leq i \leq m \text{ and } s_i = k\}$.

Let b' be any element of $A'_k - \{a'_i : 1 \leq i \leq m \text{ and } s_i = k\}$ such that for all $1 \leq i < m$ for which $s_i = k-1$, we have

$$a'_i \, \varepsilon^{\mathscr{A}'}_{k-1} \, b' \Leftrightarrow a_i \, \varepsilon^{\mathscr{A}'}_{k-1} \, b.$$

Such a b_k exists because A'_k is infinite and any finite subset of A'_k is in A'_k.

<u>*Rounds $m+2$ to n*</u>. Player II continues to play the remaining moves of his winning strategy in $G_n^{\bar{s}}(\mathscr{A}, \mathscr{A}')$.

Assume that

$$(a_{m+1}, a'_{m+1}) \in A_{s_{m+1}} \times A'_{s_{m+1}}$$
$$\vdots$$
$$(a_n, a'_n) \in A_{s_n} \times A'_{s_n}$$

are the pairs of elements chosen by the two players in the remaining rounds $m+1$ to n respectively.

The fact that Player II followed a winning strategy in $G_n^{\bar{s}}(\mathscr{A}, \mathscr{A}')$ combined with the fact that for all $m < i \leq n$, we have $s_i + 1 < k$, $\{(a_1, a'_1), \ldots, (a_n, a'_n), (b, b')\}$ is a partial isomorphism, i.e. Player II wins $G_{n+1}^{\bar{t}}(\mathscr{A}, \mathscr{A}')$. □

Results obtained by combining games

Lemma

TST^∞ decides all four-quantifier $\mathcal{L}_{\mathrm{TST}}$-sentences

$$\mathrm{Q}_1 x^{i+2} \mathrm{Q}_2 y^{i+1} \mathrm{Q}_3 z^{i+3} \mathrm{Q}_4 w^{i} \phi,$$

where $\mathrm{Q}_1, \mathrm{Q}_2, \mathrm{Q}_3, \mathrm{Q}_4$ are quantifiers and ϕ is quantifier-free.

Proof. Let $\mathscr{A}$ and $\mathscr{A}'$ be models of TST^∞. It suffices to show that Player II has a winning strategy in $G_4^{\bar{t}}(\mathscr{A}, \mathscr{A}')$, where $\bar{t} = (i+2, i+1, i+3, i)$.

Since $\bar{s} = (i+2, i+1, i)$ is strictly decreasing, we know that Player II has a winning strategy in $\mathrm{G}_3^{\bar{s}}(\mathscr{A}, \mathscr{A}')$. So, using the previous Lemma, we get that Player II has a winning strategy in $G_4^{\bar{t}}(\mathscr{A}, \mathscr{A}')$. $\square$

Using results like the one above, we get the following.

Theorem

TST^∞ decides all four-quantifier $\mathcal{L}_{\mathrm{TST}}$-sentences

$$\mathrm{Q}_1 x^i \mathrm{Q}_2 y^j \mathrm{Q}_3 z^k \mathrm{Q}_4 w^l \phi,$$

where $\mathrm{Q}_1, \mathrm{Q}_2, \mathrm{Q}_3, \mathrm{Q}_4$ are quantifiers, ϕ is quantifier-free, and i, j, k, l are distinct.

Theorem

TST^∞ decides all existential-universal $\mathcal{L}_{\mathrm{TST}}$-sentences with four quantifiers.

Consequences for Quine's 'New Foundations' (NF)

Let $\mathcal{L}_{\mathrm{NF}}$ be the language of Quine's New Foundations, i.e. the ordinary one-sorted language of set theory $\{\epsilon\}$ with one binary relation symbol.

Definition

To every $\mathcal{L}_{\mathrm{TST}}$-formula ϕ we can assign a unique $\mathcal{L}_{\mathrm{NF}}$-formula ϕ^* obtained by deleting all type superscripts from the variables of ϕ.

A formula ϕ of $\mathcal{L}_{\mathrm{NF}}$ is **stratified** if there exists an $\mathcal{L}_{\mathrm{TST}}$-formula ψ such that $\phi = \psi^*$.

For any set of $\mathcal{L}_{\mathrm{TST}}$-sentences Γ, we let $\Gamma^* = \{\sigma^* : \sigma \in \Gamma\}$.

There is a direct correspondence between $\mathcal{L}_{\mathrm{TST}}$-theories and $\mathcal{L}_{\mathrm{NF}}$-theories:

$$\mathrm{NF} = (\mathrm{TST})^*,$$
$$\mathrm{NF}_2 = (\mathrm{TST}_{(2)})^*,$$
$$\mathrm{NFO} = (\mathrm{TSTO})^*.$$

Using the following proposition, we are able to transfer decidability results for an $\mathcal{L}_{\mathrm{TST}}$-theory to the corresponding $\mathcal{L}_{\mathrm{NF}}$-theory.

Proposition

Let T be an $\mathcal{L}_{\mathrm{TST}}$-theory and Γ a set of $\mathcal{L}_{\mathrm{TST}}$-sentences. If T decides Γ, then T^* decides Γ^*.

We denote by $\mathcal{L}_{\mathrm{NF}}^{\subseteq,F}$ the language $\mathcal{L}_{\mathrm{NF}} \cup \{\subseteq, F_1, F_2, \ldots\}$ (where $F_n(x)$ is equivalent to "x has at least n elements").

Corollary

NF_2 decides

- all stratified pseudo-increasing $\mathcal{L}_{\mathrm{NF}}^{\subseteq,F}$-sentences, i.e. all sentences such that there exists an pseudo-increasing $\mathcal{L}_{\mathrm{TST}}^{\subseteq,F}$-sentence τ such that $\sigma = \tau^*$.
- all stratified existential $\mathcal{L}_{\mathrm{NF}}^{\subseteq,F}$-sentences.
- all $\mathcal{L}_{\mathrm{NF}}$-sentences that can be stratified by two types.
- all stratified $\mathcal{L}_{\mathrm{NF}}$-sentences $\forall\bar{x}\exists\bar{y}\phi(\bar{x},\bar{y})$, where ϕ is quantifier-free and all atomic formulas $y_i \mathrel{\varepsilon} y_j$, $y_i \mathrel{\varepsilon} x_j$ do not appear in ϕ.

Corollary

NFO decides all stratified $\mathcal{L}_{\mathrm{NF}}$-sentences $\exists\bar{x}\forall\bar{y}\phi(\bar{x},\bar{y})$, where ϕ is quantifier-free and the types (under some stratification) of all variables $\bar{y}$ are distinct.

Corollary

NF decides

- all stratified $\mathcal{L}_{\mathrm{NF}}$-sentences with three quantifiers.
- all stratified four-quantifier $\mathcal{L}_{\mathrm{NF}}$-sentences with variables of distinct types.
- all stratified existential-universal $\mathcal{L}_{\mathrm{NF}}$-sentences with four quantifiers.

We may now apply a theorem[5] by Richard Kaye relating ambiguity in $\mathcal{L}_{\mathrm{TST}}$-structures with the existence of certain $\mathcal{L}_{\mathrm{NF}}$-structures.

Definition

An $\mathcal{L}_{\mathrm{TST}}$-formula σ is **ambiguous** in some $\mathcal{L}_{\mathrm{TST}}$-theory T, if $T \vdash \sigma \leftrightarrow \sigma^+$, where σ^+ is the formula derived from σ if we raise the type of every variable by one.

Notice that for every $\mathcal{L}_{\mathrm{TST}}$-structure $\mathscr{A}$ and $\mathcal{L}_{\mathrm{TST}}$-sentence σ,

$$\mathscr{A} \models \sigma^+ \Leftrightarrow \mathscr{A}^+ \models \sigma,$$

where $\mathscr{A}^+$ is the structure we get if erase the bottom level of $\mathscr{A}$. So, in all the subtheories that we study, decidability implies ambiguity.

[5]See R. Kaye, A generalization of Specker's theorem on typical ambiguity, J. Symbolic Logic 56 (2) (1991) 458–466.

We therefore have the following.

Corollary

Every existential increasing $\mathcal{L}_{\mathrm{TST}}^{\subseteq,F}$-formula is ambiguous in $\mathrm{TST}_{(2)}^{\infty}$.

Combining the above result with Kaye's Theorem, we get the consistency of $\mathrm{NFINC}(\subseteq, F)$, which is the subtheory of NF that is axiomatized by all σ^*, where σ is a universal-existential-increasing $\mathcal{L}_{\mathrm{TST}}^{\subseteq,F}$-sentence and $\mathrm{TST}^{\infty} \vdash \sigma$.

Proposition

$\mathrm{NF}_2 \subseteq \mathrm{NFINC}(\subseteq, F) \subseteq \mathrm{NF}$.

Directed games on one-sorted structures

Directed games can be defined for one-sorted structures as well. For example, let us adapt the definition of directed game for $\mathcal{L}_{\mathrm{ZF}} = \{\in\}$, the usual one-sorted language of set theroy.

Let $\mathscr{A}$ and $\mathscr{B}$ be two $\mathcal{L}_{\mathrm{ZF}}$-structures. Let $\bar{s} = (s_1, \ldots, s_n) \in I^n$ for some $n > 0$.

Description of the game $\mathrm{G}_n^{\bar{s}}(\mathscr{A}, \mathscr{B})$

The game has n rounds. Suppose that we are in the i-th round of the game. Player I plays first and chooses an element a_i from $\mathscr{A}$ or an element b_i from $\mathscr{B}$, in which case Player II must respond by choosing some element b_i from $\mathscr{B}$ or some element a_i from $\mathscr{A}$ respectively.

Winning condition of $\mathrm{G}_n^{\bar{s}}(\mathscr{A}, \mathscr{B})$

Player II wins the game if for all $1 \leq i \leq n$ and $1 \leq j \leq n$ for which $s_i < s_j$, we have

$$a_i \in^{\mathscr{A}} a_j \Leftrightarrow b_i \in^{\mathscr{B}} b_j.$$

Theorem

Assume that Player II has a winning strategy in $\mathrm{G}_n^{\bar{s}}(\mathscr{A},\mathscr{B})$. Let σ be an $\mathcal{L}_{\mathrm{ZF}}$-sentence

$$\mathrm{Q}_1 x_1 \ldots \mathrm{Q}_n x_n \phi(x_1, \ldots, x_n), \tag{1}$$

where

- $\mathrm{Q}_1, \ldots, \mathrm{Q}_n$ are quantifiers,
- ϕ is a quantifier-free sentence, and
- if $s_i \geq s_j$, then the atomic formula $x_i \in x_j$ does not appear in ϕ.

Then, σ is true in $\mathscr{A}$ if and only if it is true in $\mathscr{B}$

Proposition

If for all models $\mathscr{A},\mathscr{B}$ of an $\mathcal{L}_{\mathrm{ZF}}$-theory T Player II has winning strategy in $\mathrm{G}_n^{\bar{s}}(\mathscr{A},\mathscr{B})$, then T decides all sentences described in (1).

The following result is proved in the same way as the one for increasing sentences in Simple Type Theory.

Theorem

ZF decides all $\mathcal{L}_{\mathrm{ZF}} \cup \{\subseteq, F_1, F_2, \ldots\}$-sentences (where $F_n(x)$ is equivalent to "x has at least n elements")

$$\mathrm{Q}_1 x_1 \ldots \mathrm{Q}_n x_n \phi(x_1, \ldots, x_n),$$

where

- $\mathrm{Q}_1, \ldots, \mathrm{Q}_n$ are quantifiers,
- ϕ is a quantifier-free sentence, and
- if $i \geq j$, then the atomic formula $x_i \in x_j$ does not appear in ϕ.

Uses of directed games

Directed games can be applied in a variety of fields where some kind of stratification can be imposed:

- Set theory (one-sorted like ZF and NF, or many-sorted theories like TST).
- Graph theory.
- Finite model theory.

Directed games can be used to simplify decidability proofs either by braking down problems or by combining strategies.

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