Quantale-theoretic tools for the working logician

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Quantales and modules

Definition

A (unital) quantale $(Q, \bigvee, \cdot, 1)$ is a complete residuated lattice. Quantale homomorphisms preserve arbitrary joins and the monoid structure.
A (left) Q-module $(M, \bigvee)$ is a sup-lattice with a scalar multiplication $\cdot : Q \times M \to M$ satisfying the usual module properties.

Nuclei

A Q-module nucleus over M is a closure operator $\gamma : M \to M$ such that $a \cdot \gamma(x) \leq \gamma(a \cdot x)$ for all $a \in Q$ and $x \in M$. Nuclei are in bijective correspondence with congruences.
The image M_γ of γ is a Q-module with ${}^\gamma\bigvee X := \gamma({}^M\bigvee X)$ and $a \cdot_\gamma x := \gamma(a \cdot x)$, for all $a \in Q$, $X \cup \{x\} \in M_\gamma$.

Propositional languages

Definition

A propositional language is a pair $\mathcal{L} = (L, \nu)$, where L is a set (of connectives) and $\nu : L \to \omega$ is the arity function.
The set $Fm_{\mathcal{L}}$ of $\mathcal{L}$-formulas is defined in the usual manner from a denumerable set of variables $\mathrm{Var} = \{x_n \mid n < \omega\}$; it is the term algebra over $\mathcal{L}$. We denote by $\Sigma_{\mathcal{L}}$ the monoid of substitutions of $\mathcal{L}$, i.e., of the endomorphisms of the $\mathcal{L}$-algebra $Fm_{\mathcal{L}}$.
The set of $\mathcal{L}$-equations is defined as $Eq_{\mathcal{L}} = \{\varphi \approx \psi \mid \varphi, \psi \in Fm_{\mathcal{L}}\}$ and shall be identified with $Fm_{\mathcal{L}}^2$. For any $T \subseteq \omega^2$, the set

$$Seq_T = \{\varphi_1, \ldots, \varphi_m \Rightarrow \psi_1, \ldots, \psi_n \mid \varphi_i, \psi_j \in Fm_{\mathcal{L}}, (m, n) \in T\}$$

of $\mathcal{L}$-sequents closed under the types of T will be identified with $\bigcup_{(m,n)\in T} Fm_{\mathcal{L}}^m \times Fm_{\mathcal{L}}^n$.

Propositional logics

Definition

By a propositional logic we mean a pair $(D, \vdash)$, where D – called the domain – is the set of formulas, the one of equations or a set of sequents closed under type over a propositional language $\mathcal{L} = (L, \nu)$, and $\vdash$ is a binary relation on $\mathcal{P}D$ satisfying, for all $\Phi, \Psi, \Xi \in \mathcal{P}D$, the following conditions:

- if $\Psi \subseteq \Phi$, then $\Phi \vdash \Psi$;
- if $\Phi \vdash \Psi$ and $\Psi \vdash \Xi$, then $\Phi \vdash \Xi$;
- $\Phi \vdash \bigcup_{\Phi \vdash \Psi} \Psi$;
- $\Phi \vdash \Psi$ implies $\sigma[\Phi] \vdash \sigma[\Psi]$ for each substitution $\sigma \in \Sigma_{\mathcal{L}}$.

(Not just) closure operators

The consequence operator

It is well-known that the mapping $\gamma_\vdash : \Phi \in \mathcal{P}D \mapsto \bigcup_{\Phi \vdash \Psi} \Psi \in \mathcal{P}D$ is a closure operator on $(\mathcal{P}D, \subseteq)$, but this representation of deductive systems is somewhat unsatisfactory. Not every closure operator is induced by a consequence relation.

Refining the representation [Galatos and Tsinakis 2009]

$(\mathcal{P}D, \bigcup)$ is a left module over the quantale $(\mathcal{P}\Sigma_{\mathcal{L}}, \bigcup, \cdot, \{id\})$ with the action $(\Sigma, \Phi) \mapsto \Sigma \cdot \Phi = \{\sigma(\varphi) \mid \sigma \in \Sigma, \varphi \in \Phi\}$ and $\gamma_\vdash$ is a $\mathcal{P}\Sigma_{\mathcal{L}}$-module nucleus on $\mathcal{P}D$, i.e. a closure operator such that $\Sigma \cdot \gamma_\vdash(\Phi) \subseteq \gamma_\vdash(\Sigma \cdot \Phi)$, whose image is the module of theories $Th_\vdash$ of $\vdash$.
Moreover, the correspondence between nuclei and substitution invariant consequence relations on $\mathcal{P}D$ is bijective.

Interpretations and translations

Theorem [Galatos and Tsinakis 2009]

A propositional deductive system $(D, \vdash)$ over $\mathcal{L}$ is interpretable in (resp.: representable in, equivalent to) another $\mathcal{L}$-system $(D', \vdash')$ if and only if there exists a $\mathcal{P}\Sigma_{\mathcal{L}}$-module homomorphism (resp.: embedding, isomorphism) from $Th_{\vdash}$ to $Th_{\vdash'}$.

Translations

In order to extend the previous result to the case of systems over two different languages $\mathcal{L}$ and $\mathcal{L}'$, it is necessary to introduce the concept of language translation. A translation $\tau : \mathcal{L} \to \mathcal{L}'$ turns out to induce a quantale morphism $t : \mathcal{P}\Sigma_{\mathcal{L}} \to \mathcal{P}\Sigma_{\mathcal{L}'}$ (and we also have a complete characterization of quantale morphisms induced by a translation). [R. 2013]

Restricting and extending the scalars

Restricting the scalars

A quantale morphism $h : Q \to R$ turns every R-module into a Q-module. In fact it defines a functor $(\ )_h : R\text{-}\mathcal{M}od \to Q\text{-}\mathcal{M}od$ which has both a left and a right adjoint. An analogous procedure is known in the theory of ring modules as restricting the scalars along h.

The left adjoint

The left adjoint to the functor $(\ )_h$ uses the tensor product:

$$(\ )_h^l : M \in Q\text{-}\mathcal{M}od \mapsto R \otimes_Q M \in R\text{-}\mathcal{M}od.$$

We shall call it the extension of scalars.
Note: $x \in M \mapsto 1 \otimes x \in R \otimes_Q M$ is a Q-module morphism.

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Tensor product and language expansion

Let $\mathcal{L}_1 \supseteq \mathcal{L}$ and $i : \mathcal{P}\Sigma_{\mathcal{L}} \to \mathcal{P}\Sigma_{\mathcal{L}_1}$ be the quantale embedding associated to the inclusion map.
Let $(D, \vdash)$ be a d.s. over $\mathcal{L}$, γ its associated nucleus, and Th its $\mathcal{P}\Sigma_{\mathcal{L}}$-module of theories.

$\mathcal{P}\Sigma_{\mathcal{L}_1} \otimes_{\mathcal{P}\Sigma_{\mathcal{L}}} \mathcal{P}Fm_{\mathcal{L}} \cong \mathcal{P}Fm_{\mathcal{L}_1}$, hence $\mathcal{P}\Sigma_{\mathcal{L}_1} \otimes_{\mathcal{P}\Sigma_{\mathcal{L}}} \mathcal{P}D$ is isomorphic to the domain $\mathcal{P}D_1$ in the language $\mathcal{L}_1$ of the same type of $\mathcal{P}D$.

Theorem

1. *There exists a consequence relation $\vdash_1$ on $\mathcal{P}D_1$ whose $\mathcal{P}\Sigma_{\mathcal{L}_1}$-module of theories Th_1 is isomorphic to $\mathcal{P}\Sigma_{\mathcal{L}_1} \otimes_{\mathcal{P}\Sigma_{\mathcal{L}}} Th$.*
2. *The mapping $\Phi \in Th \mapsto \{\mathrm{id}\} \otimes \Phi \in \mathcal{P}\Sigma_{\mathcal{L}_1} \otimes_{\mathcal{P}\Sigma_{\mathcal{L}}} Th$ is a $\mathcal{P}\Sigma_{\mathcal{L}}$-module embedding.*

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Notations

For $i = 1, 2$, let $\mathcal{L}_i$ be two languages, $(D_i, \vdash_i)$ be $\mathcal{L}_i$-deductive systems of the same type, and $\mathcal{L}$ be the disjoint union of $\mathcal{L}_1$ and $\mathcal{L}_2$. Then $\mathcal{P}\Sigma_{\mathcal{L}} \cong \mathcal{P}\Sigma_{\mathcal{L}_1} \amalg \mathcal{P}\Sigma_{\mathcal{L}_2}$. Let also E be the $\Sigma_{\mathcal{L}}$-domain of the same type of the two given systems, and let us set the following:

- γ_i nuclei associated to $\vdash_i$,
- $Th_i = (\mathcal{P}D_i)_{\gamma_i}$ the corresponding modules of theories,
- $d_i \colon \mathcal{P}D_i \to \mathcal{P}E$ the inclusion maps,
- $\gamma_i' \colon \Phi \in \mathcal{P}E \mapsto \gamma_i(\Phi \cap D_i) \cup (\Phi \setminus D_i) \in \mathcal{P}E$, and
- δ_i be the nuclei associated to the consequence relations $\vdash_{\delta_i}$ on E defined by means of the axioms and rules of $\vdash_i$.

Last, let $\delta = \delta_1 \vee \delta_2$, and $Th = (\mathcal{P}E)_{\delta}$.

The embedding party

Lemma 1

Let $i \neq k \in \{1,2\}$. *Then, for all* $\Phi \cup \{\psi\} \in \mathcal{P}D_k$, $\psi \in \delta_i(\Phi)$ *if and only if* $\psi \in \Phi$.

Proposition 2

There exist $\mathcal{P}\Sigma_{\mathcal{L}i}$*-module embeddings of* Th_i *and* $\mathcal{P}\Sigma_{\mathcal{L}k}$*-module embeddings of* $\mathcal{P}D_k$ *into* Th_{δ_i}, $i \neq k \in \{1,2\}$*:*

$$f_i : \Phi \in Th_i \mapsto \delta_i(\Phi) \in Th_{\delta_i}, \quad \textit{and}$$
$$g_k : \Phi \in \mathcal{P}D_k \mapsto \delta_i(\Phi) \in Th_{\delta_i}.$$

Theorem 3

There exist $\mathcal{P}\Sigma_{\mathcal{L}i}$*-module embeddings* $e_i : Th_i \to Th$, $i = 1, 2$.

The embedding party

Theorem 4

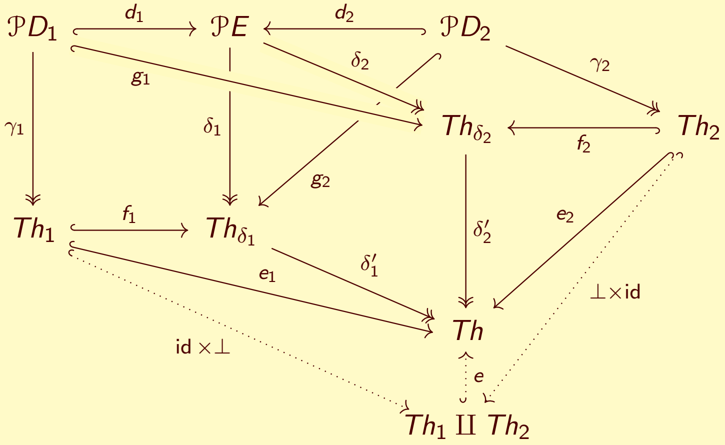
For $i = 1, 2$, the $\mathcal{P}\Sigma_{\mathcal{L}}$-modules $\mathcal{P}\Sigma_{\mathcal{L}} \otimes_{\mathcal{P}\Sigma_{\mathcal{L}i}} Th_i$ and Th_{δ_i} are isomorphic.

Theorem 5

The coproduct of Th_1 and Th_2 embeds as a sup-lattice in Th.

- Each $(D_i, \vdash_i)$ is representable in $(E, \vdash_\delta)$ (Theorem 3 and [R. 2013, Theorem 7.1]).
- Th_{δ_i} is precisely the result of a language expansion (via tensor product) on Th_i (Theorem 4).
- The sup-lattice of theories of δ contains an isomorphic copy of the coproduct of the Th_i's (Theorem 5).

The embedding fireworks



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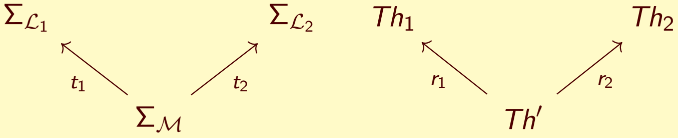
The V-formations

With the same notations, let us add another language $\mathcal{M}$ and a deductive system $(C, \vdash_\beta)$ on $\mathcal{M}$ (again, of the same type of the D_i's), with associated nucleus β and module of theories $Th' = (\mathcal{P}C)_\beta$. Let us also assume the existence of translations $\tau_i : \mathcal{M} \to \mathcal{L}_i$ and structural representations $r_i : Th' \to Th_i$ via τ_i.



Translations and more nuclei

Thanks to the translation morphisms t_i and the inclusion ones of $\mathcal{P}\Sigma_{\mathcal{L}i}$ into $\mathcal{P}\Sigma_{\mathcal{L}}$, all of the modules and embeddings which appeared in the previous section, including e, are now in $\mathcal{P}\Sigma_{\mathcal{M}}$-$\mathcal{M}od$. By previous results from [R. 2013], we also have two $\mathcal{P}\Sigma_{\mathcal{M}}$-module morphisms $s_i : \mathcal{P}C \to \mathcal{P}D_i$ such that $\gamma_i \circ s_i = r_i \circ \beta$.

Let ε be the $\mathcal{P}\Sigma_{\mathcal{L}}$-module nucleus on $\mathcal{P}E$ determined by the union of axioms and rules of $\vdash_1$ and $\vdash_2$, and the set of rules

$$\Theta = \left\{ \frac{e_i r_i(\{\varphi\})}{e_k r_k(\{\varphi\})} \middle| \varphi \in C, i \neq k \in \{1,2\} \right\}.$$

Let ζ be the nucleus determined by Θ, and $\varepsilon_i = \delta_i \vee \zeta$. We have:

$$\varepsilon = \delta \vee \zeta = \delta_1 \vee \delta_2 \vee \zeta = \varepsilon_1 \vee \varepsilon_2.$$

The amalgamation party

Lemma 6

There exist $\mathcal{P}\Sigma_{\mathcal{L}i}$-module embeddings of Th_i into Th_{ε_i}.

Theorem 7

There exist $\mathcal{P}\Sigma_{\mathcal{L}i}$-module embeddings $m_i : Th_i \to Th_\varepsilon$.

Theorem 8

There exist $\mathcal{P}\Sigma_{\mathcal{L}}$-module nuclei ζ_i on $\mathcal{P}\Sigma_{\mathcal{L}} \otimes_{\mathcal{P}\Sigma_{\mathcal{L}i}} Th_i$ such that the $\mathcal{P}\Sigma_{\mathcal{L}}$-modules $(\mathcal{P}\Sigma_{\mathcal{L}} \otimes_{\mathcal{P}\Sigma_{\mathcal{L}i}} Th_i)_{\zeta_i}$ and Th_{ε_i} are isomorphic.

Theorem 9

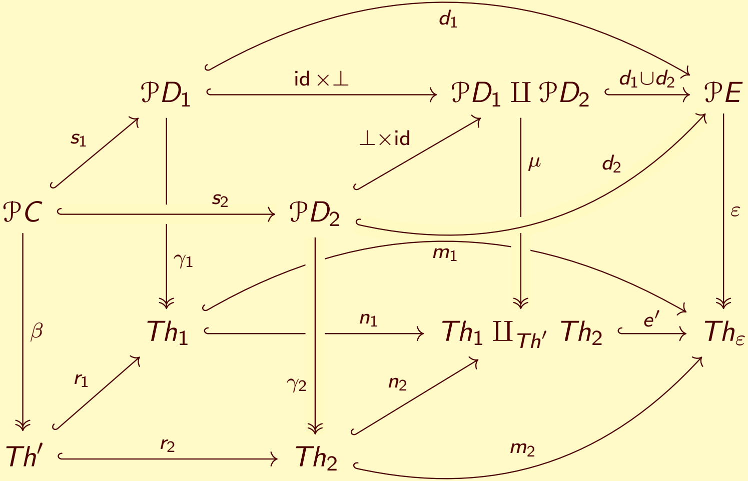
The amalgamated coproduct of the $\mathcal{P}\Sigma_{\mathcal{M}}$-modules Th_1 and Th_2 w.r.t. Th' embeds in Th_ε.

The amalgamation party

Sketch of the proof of Theorem 9

- By the properties of amalgamation, there exists a unique $\mathcal{P}\Sigma_{\mathcal{M}}$-module morphism $e' : Th_1 \amalg_{Th'} Th_2 \to Th_\varepsilon$ which extends the embeddings m_i of Th_i into Th_ε from Theorem 7.
- Let ϑ be the the congruence on $Th_1 \amalg Th_2$ whose quotient structure is $Th_1 \amalg_{Th'} Th_2$, and let $(\Phi, \Psi), (\Phi', \Psi') \in Th_1 \amalg Th_2$ two distinct ϑ-saturated elements.
- Using a previously proved characterization of saturated elements of an "amalgamating congruence", we get an element $\varphi \in r_1[\mathcal{P}C]$ which belongs to both $m_1(\Phi)$ and $m_2(\Psi)$, and neither to $m_1(\Phi')$ nor to $m_2(\Psi')$.
- This shows the injectivity of e'.

The amalgamation fireworks



References

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Thank you!