The topology of closure systems in algebraic lattices

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Let A be a ring, always commutative and unitary. The set $\mathcal{I}(A)$ of ideals

- is a **closure system** in $\mathfrak{P}(A)$, i.e., is closed in $\mathfrak{P}(A)$ under arbitrary infima (= intersections),
- and the closure system is **algebraic**, i.e., is closed in $\mathfrak{P}(A)$ under up-directed suprema (= unions).
- In particular, $\mathcal{I}(A)$ is a complete lattice.

Now let P be a complete lattice. Then:

- $\mathbb{K}(P)$ is the set of compact elements: $a \in P$ is **compact** if $a^{\uparrow} \cap D \neq \emptyset$ for each up-directed set $D \subseteq P$ with $a \leq \bigvee D$.
- $\mathbb{P}(P)$ is the set of prime elements: $a \in P$ is **prime** if $x \wedge y \leq a$ implies $x \leq a$ or $y \leq a$.

The bottom element $\perp$ is always compact. There need not be any other compact elements. The top element $\top$ is always prime, and there need not be any other prime elements.

- $\mathbb{K}(P)$ is a join semilattice. But in general $x, y \in \mathbb{K}(P)$ does not imply $x \wedge y \in \mathbb{K}(P)$.
- P is **arithmetic** if $\mathbb{K}(P)$ is a sublattice of P, i.e., $x, y \in \mathbb{K}(P) \Rightarrow x \wedge y \in \mathbb{K}(P)$.
- P is **coherent** if $\mathbb{K}(P)$ is a bounded sublattice of P.

Definition.
P is an **algebraic lattice** if $a = \bigvee a^{\downarrow} \cap \mathbb{K}(P)$ for each $a \in P$.

Recall that every subset $X \subseteq P$ generates a closure system, whose elements are the infima of subsets of X. This closure system is denoted by $\langle X \rangle_P$.

Fact.
An algebraic lattice P is a frame if and only if it is distributive, if and only if $P = \langle \mathbb{P}(P) \rangle_P$.

Returning to the lattice $\mathcal{I}(A)$ one checks easily:

- $\mathbb{K}(\mathcal{I}(A))$ is the set of finitely generated ideals. Hence:
- $\mathcal{I}(A)$ is an algebraic lattice.

An ideal $\mathfrak{a} \lhd A$ is **strongly irreducible** if $\mathfrak{b} \cap \mathfrak{c} \subseteq \mathfrak{a}$ always implies $\mathfrak{b} \subseteq \mathfrak{a}$ or $\mathfrak{c} \subseteq \mathfrak{a}$, where $\mathfrak{b}, \mathfrak{c} \in \mathcal{I}(A)$. Thus: $\mathbb{P}(\mathcal{I}(A))$ is the set of strongly irreducible ideals.

- $\mathrm{Spec}(A) \subseteq \mathbb{P}(\mathcal{I}(A)$. Frequently the inclusion is proper: Let A be a nontrivial valuation ring. Then $\mathcal{I}(A)$ is totally ordered, hence every ideal is strongly irreducible. But most ideals are not prime.
- $\mathcal{I}(A)$ is always a modular lattice. The ring is called **arithmetic** if $\mathcal{I}(A)$ is distributive, i.e., if every ideal is an intersection of strongly irreducible ideals.
- A domain is arithmetic if and only if it is a Prüfer domain. Thus the polynomial ring $\mathbb{Q}[S]$ is arithmetic, whereas $\mathbb{Q}[S,T]$ is not arithmetic.

Continuing with a ring A, consider a subset $X \subseteq \mathrm{Spec}(A)$. For $a \in A$ we define

- the **X-zero set** $\mathrm{Z}_X(a) = \{\mathfrak{p} \in X \mid a \in \mathfrak{p}\}$ and
- the **X-cozero set** $\mathrm{Coz}_X(a) = X \setminus \mathrm{Z}_X(a)$.

If $X = \mathrm{Spec}(A)$ we follow tradition and write $V(a) = \mathrm{Z}_{\mathrm{Spec}(A)}(a)$ and $D(a) = \mathrm{Spec}(A) \setminus V(a)$.

Definition 1.
An ideal $\mathfrak{a} \lhd A$ is a

(a) **vanishing ideal for** X if $\bigcap_{a\in\mathfrak{a}} \mathrm{Z}_X(a) \subseteq \mathrm{Z}_X(c)$ with $c \in A$ implies $c \in \mathfrak{a}$;

(b) **large z-ideal for** X if $\bigcap_{a\in F} \mathrm{Z}_X(a) \subseteq \mathrm{Z}_x(c)$ implies $c \in \mathfrak{a}$, where $F \subseteq \mathfrak{a}$ is finite and $c \in A$;

(c) **small z-ideal for** X if $\mathrm{Z}_X(a) \subseteq \mathrm{Z}_X(c)$ implies $c \in \mathfrak{a}$, where $a \in \mathfrak{a}$ and $c \in A$.

The sets of vanishing ideals, large z-ideals and small z-ideals are denoted by $\mathrm{v}\mathcal{I}(A, X)$, $\mathrm{Z}\mathcal{I}(A, X)$ and $\mathrm{z}\mathcal{I}(A, X)$. Elementary facts:

- $\mathrm{v}\mathcal{I}(A, X) \subseteq \mathrm{Z}\mathcal{I}(A, X) \subseteq \mathrm{z}\mathcal{I}(A, X)$, where both inclusions are proper in general.
- $\mathrm{v}\mathcal{I}(A, X), \mathrm{Z}\mathcal{I}(A, X), \mathrm{z}\mathcal{I}(A, X) \subseteq \mathcal{I}(A)$ are closure systems.
- $\mathrm{Z}\mathcal{I}(A, X)$ and $\mathrm{z}\mathcal{I}(A, X)$ are algebraic closure systems.
- If $X \subseteq Y \subseteq \mathrm{Spec}(A)$ then $\mathrm{v}\mathcal{I}(A, X) \subseteq \mathrm{v}\mathcal{I}(A, Y)$, $\mathrm{Z}\mathcal{I}(A, X) \subseteq \mathrm{Z}\mathcal{I}(A, Y)$ and $\mathrm{z}\mathcal{I}(A, X) \subseteq \mathrm{z}\mathcal{I}(A, Y)$.
- If $Y = \mathrm{Spec}(A)$ then $\mathrm{v}\mathcal{I}(A, Y) = \mathrm{Z}\mathcal{I}(A, Y) = \mathrm{z}\mathcal{I}(A, Y) = \mathcal{I}^{\mathrm{rad}}(A)$.

Let X be a topological space.

- $\mathcal{O}(X)$ is the frame of open sets and $\mathcal{A}(X)$ is the lattice of closed sets.
- The b**-topology** on X: The locally closed subsets, i.e., the sets $O \setminus O'$ with $O, O' \in \mathcal{O}(X)$, are a basis of open sets for the b-topology.
  - For a subset $W \subseteq X$ the closure for the b-topology is denoted by $\overline{W}^{\beta}$.
  - If $Y \subseteq X$ is a subspace then the b-topology of X restricts to the b-topology of Y.
- Let $\mathcal{S} \subseteq \mathcal{O}(X)$ be a subset. Then $\mathcal{O}(X, \mathcal{S}) = \{\bigcup \mathcal{S}' \mid \mathcal{S}' \subseteq \mathcal{S}\}$ is the set of $\mathcal{S}$**-open sets** and $\mathcal{A}(X, \mathcal{S})$, the set of complements, is the set of $\mathcal{S}$**-closed sets**. Every $W \subseteq X$ is contained in a smallest $\mathcal{S}$-closed set $\overline{W}^{\mathcal{S}}$, its $\mathcal{S}$**-closure**.
- $\mathbb{K}(\mathcal{O}(X))$ is the set of the quasi-compact open sets.
- Let X be a **spectral space**,
  - i.e., X is a sober T_0-space, $\mathbb{K}(\mathcal{O}(X)) \subseteq \mathcal{O}(X)$ is a bounded sublattice and a basis of open sets.
  - Let $\mathcal{K}(X)$ be the Boolean algebra generated by the compact elements. Its elements are finite unions of **basic constructible sets** $U \setminus V$ with $U, V \in \mathbb{K}(\mathcal{O}(X))$. The **patch topology** or **constructible topology** is generated by $\mathcal{K}(X)$. The patch topology is Boolean; the patch closure of $W \subseteq X$ is denoted by $\overline{W}^{\text{con}}$.

Topologies on posets. Let P be a complete lattice.

- A topology τ on P is a **lower topology**, resp. an **upper topology**, if every open set is a downset, resp. an upset.
- There is a coarsest lower topology on P, the **coarse lower topology** $\tau^\ell(P)$: the sets $P \setminus a^\uparrow$, $a \in P$, are a subbasis of open sets.

  By default, P will always be considered with the coarse lower topology.
- There is a coarsest upper topology on P, called the **coarse upper topology**.
- The **Scott topology** $\sigma(P)$: A subset $U \subseteq P$ is open if it is an upset and if $D \cap U \neq \emptyset$ for all up-directed $D \subseteq P$ with $\bigvee D \in U$.
- The **lower Scott topology** $\sigma^\ell(P)$: The open sets are downsets, and the condition about up-directed sets is replaced by the corresponding condition about down-directed sets.

Now assume that P is an algebraic lattice.

- The coarse lower topology is a spectral topology. The quasi-compact open sets are the sets $P \setminus F^\uparrow$ with $F \subseteq \mathbb{K}(P)$ finite.
- The inverse topology of $\tau^\ell(P)$ is $\sigma(P)$.
- The sets $F^\uparrow \setminus G^\uparrow$ with $F, G \subseteq \mathbb{K}(P)$ finite are a basis of the patch topology, and the sets $a^\uparrow \setminus b^\uparrow$, $a, b \in \mathbb{K}(P)$ are a subbasis.
- For the coarse lower topology there is a b-topology. The sets $a^\uparrow \setminus F^\uparrow$ with $a \in P$ and $F \subseteq P$ finite are a basis of open sets, and the sets $a^\uparrow \setminus b^\uparrow$, $a, b \in P$ are a subbasis.

Example. Let S be any set. Then $\mathfrak{P}(S)$ with inclusion as partial order is an algebraic lattice, hence the coarse lower topology is spectral. In fact, this is the only lower topology that is spectral.

- The sets $\mathfrak{P}(S) \setminus A^\uparrow = \bigcup_{a \in A} S \setminus a^\uparrow$ with $A \subseteq S$ are a subbasis of open sets, hence the sets $S \setminus a^\uparrow$ with $a \in S$ are a subbasis as well.
- $\mathbb{K}(\mathfrak{P}(S)) = \mathfrak{P}_{\text{fin}}(S)$.
- $\mathbb{P}(\mathfrak{P}(S) = \{a \subseteq S \mid |S \setminus A| \leq 1\}$.

Maps between posets. Let $\varphi : P \to Q$ be map between complete posets.

- It is always assumed that φ is monotonic. Additional properties will always be announced explicitly.
- If $D \subseteq P$ is up-dirceted (or down-directed) then $\varphi(D) \subseteq Q$ is up-directed (or down-directed. We call φ a **dcpo-map** if $\bigvee \varphi(D) = \varphi(\bigvee D)$ for all up-directed sets, or an **fcpo-map** if the corresponding condition holds for down-directed sets.
- φ preserves all infima if and only if there is a **left adjoint map** $\varphi^* : Q \to P$, i.e., $b \le \varphi(a) \Leftrightarrow \varphi^*(b) \le a$ with $a \in P$ and $b \in Q$. Explicitly, $\varphi^*(b)$ is the smallest $a \in P$ with $b \le \varphi(a)$.
- φ preserves all suprema if and only if there is a **right adjoint map** $\varphi_* : Q \to P$, i.e., $\varphi(a) \le b \Leftrightarrow a \le \varphi_*(b)$ with $a \in P$ and $b \in Q$. Explicitly, $\varphi_*(b)$ is the largest $a \in P$ with $\varphi(a) \le b$.

Theorem 2.
Let $\varphi : P \to Q$ be a monotonic map of complete lattices.

(a) φ is an fcpo-map if and only if it is continuous for the lower Scott topology.

(b) φ is a dcpo-map if and only if it is continuous for the Scott topology.

(c) φ preserves all infima if and only if it preserves the top element and finite meets and is continuous for the coarse lower topology.

(d) φ preserves all suprema if and only if it preserves the bottom element and finite joins and is continuous for the coarse upper topology.

(e) Let P be an algebraic lattice and assume that φ is surjective and preserves all suprema. Then $\mathbb{K}(Q) \subseteq \varphi(\mathbb{K}(P))$.

Example 3.
Let P be a complete poset, $Q \subseteq P$ a closure system, $\iota : Q \to P$ the inclusion, $\eta : P \to P$ the corresponding **closure operator**, and $\vartheta : P \to Q$ the **closure map**, i.e., the corestriction of η.

- ι preserves all infima, hence has a left adjoint map, which is ϑ.
- ϑ has the right adjoint map ι, hence preserves all suprema.
- ι is continuous for the coarse lower topology.
- ϑ is continuous for the Scott topology.

Moreover, $Q \subseteq P$ is closed for the b-topology. (For, if $a \in P \setminus Q$ then $a^{\uparrow} \setminus \eta(a)^{\uparrow}$ is a b-open neighborhood of a and is disjoint from Q.)

Theorem 4.
Let P be an algebraic lattice, $Q \subseteq P$ a closure system, with inclusion ι, closure operator η and closure map ϑ. The following conditions are equivalent:

(a) Q is algebraic in P.

(b) $\sigma(P)|_Q = \sigma(Q)$. (In general one knows that $\sigma(Q) \subseteq \sigma(P)|_Q$.)

(c) Q is a spectral subspace of P.

(d) $\vartheta(\mathbb{K}(P)) = \mathbb{K}(Q)$. (The inclusion $\mathbb{K}(Q) \subseteq \vartheta(\mathbb{K}(P))$ holds more generally by Theorem 2.)

(e) Q is sober with respect to $\sigma(P)|_Q$.

Remark.
Assume that $Q \subseteq P$ is an algebraic closure system in an algebraic lattice. Then ι is a spectral map, and η and ϑ are continuous for the Scott topology. But they need not be spectral maps.

Theorem 5.
Let P be an algebraic lattice and $Q = \langle X \rangle_P \subseteq P$ a closure system. Then:

(a) $\overline{Q}^{\mathrm{con}}$ is an algebraic closure system and is equal to $\langle \overline{X}^{\mathrm{con}} \rangle_P$.

(b) Q is dense in $\overline{Q}^{\mathrm{con}}$ for the b-topology belonging to the Scott topology.

(c) $\overline{Q}^{\mathrm{con}}$ with the Scott topology is the sobrification of $(Q, \sigma(P)|_Q)$.

(d) $\mathbb{K}(\overline{Q}^{\mathrm{con}}) \subseteq Q$ (but $x \in \mathbb{K}(\overline{Q}^{\mathrm{con}})$ need not be compact in Q).

Proof of (a). The claim follows from the following two facts: $\langle \overline{X}^{\mathrm{con}} \rangle_P$ is patch closed, and $\overline{Q}^{\mathrm{con}}$ is a closure system.

Here is a short explanation of the second fact: The meet operation $\wedge : P \times P \to P$ is a spectral map. Since $\wedge(Q \times Q) \subseteq Q$ it follows that $\wedge(\overline{Q}^{\mathrm{con}} \times \overline{Q}^{\mathrm{con}}) = \wedge(\overline{Q \times Q}^{\mathrm{con}}) \subseteq \overline{Q}^{\mathrm{con}}$. Thus, $\overline{Q}^{\mathrm{con}}$ is closed under finite meets. Being patch closed it is also closed under down-dircted infima. Together this shows that $\overline{Q}^{\mathrm{con}}$ is closed under all infima, i.e., is a closure system. The closure system is algebraic by Theorem 4.

Closure systems generated by prime elements.

Proposition 6.
Let P be a complete lattice.

(a) If $Q \subseteq P$ is a meet sub-semilattice. Then $\mathbb{P}(P) \cap Q \subseteq \mathbb{P}(Q)$.

(b) If $Q = \langle X \rangle_P$ and $X \subseteq \mathbb{P}(Q)$ then Q is isomorphic to the frame $\mathcal{O}(X^\flat)$, where $X^\flat = X \setminus \{\top\}$.

(c) $\mathbb{P}(P)$ and $\mathbb{P}(P)^\flat$ are closed for the b-topology and are both sober. (But note that the supremum of an up-directed set of prime elements needs not be prime.)

(d) If $X, Y \subseteq \mathbb{P}(P)$ then $\langle X \rangle_P = \langle Y \rangle_P$ if and only if $\overline{X}^\beta = \overline{Y}^\beta$.

Proof of (d). Since $\langle X \rangle_P = \langle \overline{X}^\beta \rangle_P$ and $\langle Y \rangle_P = \langle \overline{Y}^\beta \rangle_P$ we may assume that X and Y are b-closed. Thus the implication $\Leftarrow$ is clear.

For the converse assume that there is some $y \in Y \setminus X$. Then $y = \bigwedge y^\uparrow \cap X$. There is a b-open neighborhood U of y that is disjoint from X. We may assume that $U = y^\uparrow \setminus F^\uparrow$ for some finite $F \subseteq P$. Then $\bigwedge F \not\leq y$ (since y is prime), but $\bigwedge F \leq \bigwedge y^\uparrow \cap X = y$ (since $y^\uparrow \cap X \subseteq F^\uparrow$), a contradiction.

Theorem 7.
Let P be an algebraic closure system, $X \subseteq \mathbb{P}(P)$. Then:

(a) If P is arithmetic then $\mathbb{P}(P)$ is patch closed in P.

(b) If P is coherent then $\mathbb{P}(P)^\flat$ is patch closed in P.

(c) Assume that $X \subseteq P$ is patch closed. Then $\langle X \rangle_P$ is patch closed in P, is an arithmetic algebraic frame isomorphic to $\mathcal{O}(X^\flat)$, and $\mathbb{P}(\langle X \rangle_P) = \mathbb{P}(P) \cap \langle X \rangle_P = X \cup \{\top\}$.

(d) Assume that X is patch closed in P and is contained in $\mathbb{P}(P)^\flat$. Then $\langle X \rangle_P$ is patch closed in P and is a coherent algebraic frame isomorphic to $\mathcal{O}(X)$, and $\mathbb{P}(\langle X \rangle_P)^\flat = \mathbb{P}(P)^\flat \cap \langle X \rangle_P = X$.

Again, consider a ring A, $\mathcal{I}(A)$ the algebraic lattice of ideals, a subset $X \subseteq \operatorname{Spec}(A)$, and the closure systems $\mathrm{v}\mathcal{I}(A,X) \subseteq \mathrm{Z}\mathcal{I}(A,X) \subseteq \mathrm{z}\mathcal{I}(A,X)$ where $X \subseteq \operatorname{Spec}(A)$.
The closure operators belonging to the closure systems are denoted by η_X^v, η_X^Z and η_X^z. Hence $\eta_X^z(\mathfrak{a}) \subseteq \eta_X^Z(\mathfrak{a}) \subseteq \eta_X^v(\mathfrak{a})$ for each $\mathfrak{a} \lhd A$.

Proposition 8.

(a) If $\mathfrak{a} = (a)$ is a principal ideal then $\eta_X^z(\mathfrak{a}) = \eta_X^Z(\mathfrak{a}) = \eta_X^v(\mathfrak{a})$.

(b) If $\mathfrak{a}$ is finitely generated then $\eta_X^Z(\mathfrak{a}) = \eta_X^v(\mathfrak{a})$.

(c) If $\eta_X^z(\mathfrak{c}) = \eta_X^Z(\mathfrak{c})$ for all finitely generated ideals then $\eta_X^z = \eta_X^Z$.

(d) For all $\mathfrak{a} \in \mathcal{I}(A)$ we have $\eta_X^Z(\mathfrak{a}) = \bigcup_{F \in \mathfrak{P}_{\mathrm{fin}}(\mathfrak{a})} \eta_X^v((F))$.

Theorem 9.
Let A be a ring, $X, Y \subseteq \operatorname{Spec}(A)$ subsets.

(a) $\mathrm{v}\mathcal{I}(A,X) = \langle X \rangle_{\mathcal{I}(A)} \simeq \mathcal{O}(X)$.

(b) $\mathrm{v}\mathcal{I}(A,X) = \mathrm{v}\mathcal{I}(A,Y)$ if and only if $\overline{X}^\beta = \overline{Y}^\beta$.

Proof. (a). It is clear that $X \subseteq \mathrm{v}\mathcal{I}(A,X)$, hence $\langle X \rangle_{\mathcal{I}(A)} \subseteq \mathrm{v}\mathcal{I}(A,X)$, and $\mathrm{v}\mathcal{I}(A,X) \subseteq \langle X \rangle_{\mathcal{I}(A)}$ follows from the definition of vanishing ideals. The isomorphism with $\mathcal{O}(X)$: see Proposition 6.

(b) follows from (a) and Proposition 6.

Theorem 10.
Let A be a ring, $X \subseteq \operatorname{Spec}(A)$. Then $\mathrm{Z}\mathcal{I}(A,X) = \langle \overline{X}^{\mathrm{con}} \rangle_{\mathcal{I}(A)} = \overline{\mathrm{v}\mathcal{I}(A,X)}^{\mathrm{con}} \simeq \mathcal{O}(\overline{X}^{\mathrm{con}})$, and $\mathbb{P}(\mathrm{Z}\mathcal{I}(A,X))^{\flat} = \overline{X}^{\mathrm{con}}$. Thus, $\mathrm{Z}\mathcal{I}(A,X)$ is a coherent algebraic frame. The following conditions are equivalent:

(a) $\mathrm{v}\mathcal{I}(A,X) = \mathrm{Z}\mathcal{I}(A,X)$.

(b) $\mathrm{v}\mathcal{I}(A,X)$ is an algebraic closure system.

(c) $\overline{X}^{\beta} = \overline{X}^{\mathrm{con}}$.

Proof, partial. The equality $\langle \overline{X}^{\mathrm{con}} \rangle_{\mathcal{I}(A)} = \overline{\mathrm{v}\mathcal{I}(A,X)}^{\mathrm{con}}$ and the isomorphism with $\mathcal{O}(\overline{X}^{\mathrm{con}})$ follow from Theorems 5 and 9.

The equality $\mathrm{Z}\mathcal{I}(A,X) = \langle \overline{X}^{\mathrm{con}} \rangle_{\mathcal{I}(A)}$ follows from Proposition 8: every large z-ideal is an up-directed union of vanishing ideals, hence belongs to the algebraic closure system $\overline{\mathrm{v}\mathcal{I}(A,X)}^{\mathrm{con}}$.

Example 11. $\mathrm{v}\mathcal{I}(A,X)$ can be a proper subset of $\mathrm{Z}\mathcal{I}(A,X)$.
A general argument: Let A be a ring such that $\operatorname{Spec}(A)$ is not Noetherian. Then there is some $X \in \mathcal{O}(\operatorname{Spec}(A)) \setminus \mathbb{K}(\mathcal{O}(\operatorname{Spec}(A)))$. Being open, X is closed for the b-topology, but is not patch closed.

For a concrete instance, let K be a field, $A = K[T_i \mid i \in \mathbb{N}]$ a polynomial ring with infinitely many variables, let $\mathfrak{m}$ be the ideal generated by the variables and set $X = \operatorname{Spec}(A) \setminus \{\mathfrak{m}\}$.

Finally the comparison of large and small z-ideals.

- The sets $(a)^{\uparrow} \setminus (b)^{\uparrow} \subseteq \mathcal{I}(A)$, with $a, b \in A$, are a subbasis, $\mathcal{T}$, of open sets for the patch topology.
- Let $\mathcal{S}$ be restriction of $\mathcal{T}$ to $\mathrm{Spec}(A)$, i.e., the collection of sets $D(a) \cap V(b)$.
- Every $\mathcal{S}$-closed set in $\mathrm{Spec}(A)$ is patch closed.

Example 12. Patch closed sets that are not $\mathcal{S}$-closed.

- Let C be an algebraically closed field, $A = C[S,T]$ the polynomial ring with 2 variables, $\mathfrak{m}$ the maximal ideal (S,T) and $X = \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$.
- X is open in the Noetherian space $\mathrm{Spec}(A)$, hence is patch closed.
- But X is not $\mathcal{S}$-closed: Assume that $\{\mathfrak{m}\}$ is $\mathcal{S}$-open. Then there are $a, b \in A$ with $\mathfrak{m} \in D(a) \cap V(b)$ and $D(a) \cap V(b) \cap X = \emptyset$, i.e., $D(a) \cap V(b) = \{\mathfrak{m}\}$. Thus, in the quotient ring A_a the maximal ideal $\mathfrak{m} \cdot A_a$ is a principal ideal, generated by b. Krull's Principal Ideal Theorem says that every prime ideal $\mathfrak{p}$ that is minimal with $b \in \mathfrak{p}$ has height at most 1. But $\mathfrak{m}$ has height 2 in A_a, a contradiction.

Theorem 13.
Let A be a ring, $X \subseteq \mathrm{Spec}(A)$ and $Y = \overline{X}^{\mathcal{S}}$. Then:

(a) $\mathrm{z}\mathcal{I}(A,X) = \overline{X}^{\mathcal{T}} \subseteq \mathcal{I}(A)$, hence also $\mathrm{z}\mathcal{I}(A,X) = \overline{\mathrm{v}\mathcal{I}(A,X)}^{\mathcal{T}}$.

(b) $\langle Y \rangle_{\mathcal{I}(A)} = \mathrm{z}\mathcal{I}(A,X)$, and $Y = \mathrm{Spec}(A) \cap \mathrm{z}\mathcal{I}(A,X) = \mathbb{P}(\mathrm{z}\mathcal{I}(A,X))^{\flat}$.

(c) $\mathrm{z}\mathcal{I}(A,X)$ is a coherent algebraic frame.

Summary. If X varies in $\mathfrak{P}(\mathrm{Spec}(A)$ then there are bijective correspondences between

- the closure systems $\mathrm{v}\mathcal{I}(A, X)$ and the b-closed subsets of $\mathrm{Spec}(A)$;
- the algebraic closure systems $\mathrm{Z}\mathcal{I}(A, X)$ and the patch closed subsets of $\mathrm{Spec}(A)$;
- the algebraic closure systems $\mathrm{z}\mathcal{I}(A, X)$ and the $\mathcal{S}$-closed subsets of $\mathrm{Spec}(A)$.

$$
\begin{array}{ccccc}
\mathrm{v}\mathcal{I}(A, X) & \xrightarrow{\subseteq} & \mathrm{Z}\mathcal{I}(A, X) & \xrightarrow{\subseteq} & \mathrm{z}\mathcal{I}(A, X) \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\
\mathrm{v}\mathcal{I}(A, \overline{X}^{\beta}) & \xrightarrow{\subseteq} & \mathrm{Z}\mathcal{I}(A, \overline{X}^{\beta}) & \xrightarrow{\subseteq} & \mathrm{z}\mathcal{I}(A, \overline{X}^{\beta}) \\
{\scriptstyle \subseteq}\downarrow & & {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\
\mathrm{v}\mathcal{I}(A, \overline{X}^{\mathrm{con}}) & \xrightarrow{=} & \mathrm{Z}\mathcal{I}(A, \overline{X}^{\mathrm{con}}) & \xrightarrow{\subseteq} & \mathrm{z}\mathcal{I}(A, \overline{X}^{\mathrm{con}}) \\
{\scriptstyle \subseteq}\downarrow & & {\scriptstyle \subseteq}\downarrow & & \downarrow{\scriptstyle =} \\
\mathrm{v}\mathcal{I}(A, \overline{X}^{\mathcal{S}}) & \xrightarrow{=} & \mathrm{Z}\mathcal{I}(A, \overline{X}^{\mathcal{S}}) & \xrightarrow{=} & \mathrm{z}\mathcal{I}(A, \overline{X}^{\mathcal{S}})
\end{array}
$$

The inclusions in the diagram are always proper if the corresponding inclusions between the representation sets are proper.