Modal Logics of Cayley Graphs

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Lattices
Universal Algebra
Boolean Algebras
Set Theory
Topology

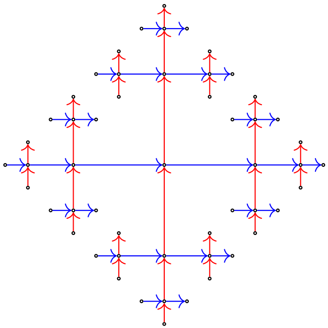
Cayley graphs

Definition

Suppose that G is a group and S is a generating set of G. The *Cayley graph* $\Gamma = \Gamma(G, S)$ is a colored directed graph constructed as follows:

1. Each element g of G corresponds to a vertex: the vertex set $V(\Gamma) = G$.
2. Each generator s of S is assigned a color c_s.
3. For any $g \in G$, $s \in S$ the vertices corresponding to the elements g and gs are joined by a directed edge of color c_s.

Example



$$F = \langle x, y \mid \varnothing \rangle$$

Cayley graphs as frames

Cayley graph $\Gamma(G, S)$, $|S| = n$

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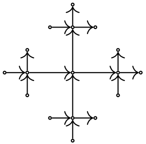
Simple Cayley Frame: $CayleySimple_n$

$F = (G, \mathcal{R})$

One relation for all colors



One modality $\Box$

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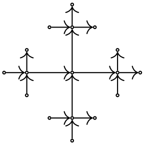
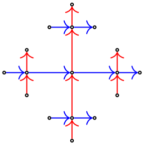


One modality $\Box$

Multimodal Cayley Frame: *Cayley*$_S$

$$F = (G, \{\mathcal{R}_s\}_{s \in S})$$

Each color defines a relation



Modalities $\Box_s$ for each $s \in S$

Four questions

	Simple	**Multimodal**
General	Given $n \in \mathbb{N}$, $\mathrm{Log}(CayleySimple_n) = ?$	Given S, $\mathrm{Log}(Cayley_S) = ?$
Restricted	Given $G = \langle S \mid R \rangle$, $F = (G, \mathcal{R})$, $\mathrm{Log}(F) = ?$	Given $G = \langle S \mid R \rangle$, $F = (G, \{\mathcal{R}_s\}_{s \in S})$, $\mathrm{Log}(F) = ?$

Logic of all multimodal Cayley frames

Definition

The modal logic SL_S is $K_S + \{\Box_s p \leftrightarrow \Diamond_s p\}_{s \in S}$.

Proposition

$Cayley_S \models SL_S$.

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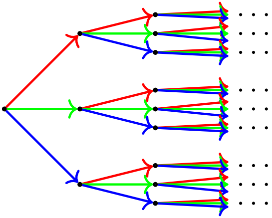
Each element $g \in G$ has exactly one $\mathcal{R}_s$-arrow (g, gs) for any $s \in S$.



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The logic SL_S is canonical. Consider its canonical frame. The unraveling of any of its point-generated subframes embeds into the Cayley graph of a free group as a subframe.

$$\text{Cayley graph of } \langle S \mid \varnothing \rangle \xrightarrow{\uparrow e} F{\uparrow}e \twoheadleftarrow\!\!\twoheadrightarrow F^{\#}_{SL_S} \longrightarrow\!\!\!\!\!\twoheadrightarrow F_{SL_S}$$

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Corollary

$\mathrm{Log}(Cayleys_S) = \mathrm{Log}(\text{Multimodal Cayley Frame of } \langle S \mid \varnothing \rangle) = SL_S$.

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Seems trivial...

$$xy = yx \qquad \Diamond_x \Diamond_y p \leftrightarrow \Diamond_y \Diamond_x p$$

$$xyz = e \qquad \Diamond_x \Diamond_y \Diamond_z p \leftrightarrow p$$

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$$xy^{-1}xy^{-1} = e \qquad ???$$

Positive relations

Definition

1. A word $w = s_1^{\varepsilon_1} s_2^{\varepsilon_2} \dots s_n^{\varepsilon_n}$ in S is positive if $\forall i \in \{1, \dots, n\}$ $\varepsilon_i = 1$, i.e. no inverses of elements of S occur in w.
2. For any words w_1, w_2 in S, we call the equation $w_1 = w_2$ positive, if both w_1 and w_2 are positive.
3. Let $\langle S \mid R \rangle$ be a group presentation. We call a relation $r \in R$ positive, if r is equivalent to a positive equation.

Examples

- $xy = yx$, $xyz = e$, $xy^{-1}x = e$ are positive;
- $xy^{-1}xy^{-1} = e$ is not positive.

Translation

A translation for positive equation is defined as follows:

1. $e^{\#}$ is an empty word;
2. $s^{\#} := \Diamond_s$ for any $s \in S$;
3. $(As)^{\#} := A^{\#}s^{\#}$ for any positive word A in S and any $s \in S$;
4. $(A = B)^{\#} := (A^{\#}p \leftrightarrow B^{\#}p)$ for any positive words A, B in S.

Proposition

If G is a group and a positive equation r is true in G, then $r^{\#}$ is valid in the multimodal Cayley frame of G.

Corollary

If $G = \langle S \mid R \rangle$ and $S = S^{-1}$, then $\{r^{\#} \mid r \in R\}$ is valid in the multimodal Cayley frame of G.

Proposition

If $G = \langle S \mid R \rangle$, R contains only positive relations, and F the multimodal Cayley frame of G. Then

$$\mathrm{Log}(F) = SL_S + \{r^{\#} \mid r \in R\}.$$

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1. $f(e) = w_0$;
2. If $f(g)$ is already set, let $f(gs)$ be the unique element in $\mathcal{R}_s(f(g))$.

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If some $g \in G$ is discovered twice, then

$$g = s_1 s_2 \dots s_m = s'_1 s'_2 \dots s'_k.$$

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$$g = s_1 s_2 \dots s_m = s_1' s_2' \dots s_k'.$$

But at the same time

$$F_c \models \Diamond_{s_1} \Diamond_{s_2} \dots r_{s_m} p \leftrightarrow \Diamond_{s_1'} \Diamond_{s_2'} \dots \Diamond_{s_k'} p.$$

Non-positive relations

If r is not positive, it does not affect $Log(F)$.

Example

$$G = \left\langle x, y \mid xy^{-1}xy^{-1} = e \right\rangle; \quad F = (G, \{\mathcal{R}_x, \mathcal{R}_y\}).$$

$$\mathrm{Log}(F) \subseteq \mathrm{Log}(F{\uparrow}e)$$

Consider the rooted frame $F{\uparrow}e$. Its points are

$$g = xx\ldots xyy\ldots yxx\ldots xyy\ldots y\ldots$$

No pattern $xy^{-1}xy^{-1}$ occurs.

Non-positive relations

Definition

Let S be a generating set of a group, and let R be a set of relations in S. We denote $Pos(R)$ the minimal set R' of positive relations that satisfies the following conditions:

1. Any positive equality that is true in $\langle S \mid R \rangle$ is also true in $\langle S \mid R' \rangle$.
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Proposition

Let $G = \langle S \mid R \rangle$, where S is finite. Let F be the multimodal Cayley frame of $\Gamma(G, S)$.
Consider the group $G' = \langle S \mid Pos(R) \rangle$, and the multimodal Cayley frame F' of $\Gamma(G', S)$.
The point-generated subframes $F{\uparrow}e$ and $F'{\uparrow}e$ are isomorphic.

Main result for multimodal frames

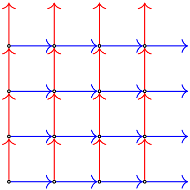
Theorem

Let $G = \langle S \mid R \rangle$, where S is finite. Let $F = (G, \{\mathcal{R}_s\}_{s \in S})$ be the multimodal Cayley frame for G. Then

$$\mathrm{Log}\ F = KC_S + \{r^{\#} \mid r \in Pos(R)\}.$$

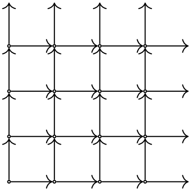
Simple Cayley frames. The difference

$$G = \langle x, y \mid xy = yx \rangle$$



Multimodal Cayley frame

$$F_M = (G, \{\mathcal{R}_x, \mathcal{R}_y\})$$



Simple Cayley frame

$$F_S = (G, \mathcal{R})$$

Frame conditions for simple Cayley frames

$$G = \langle S \mid R \rangle \,, \; |S| = n, \; F = (G, \mathcal{R})$$

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- Serial: $AD = \Diamond\top$
- Bounded branching:

$$Alt_n = \Box\left(\bigvee_{i=1}^{n+1} p_i\right) \to \Diamond\left(\bigvee_{i=1}^{n+1}\bigvee_{j=i+1}^{n+1}(p_i \wedge p_j)\right)$$

Logic of all simple Cayley frames

Theorem

$$\mathrm{Log}(CayleySimple_n) = \mathrm{Log}(\text{Simple Cayley Frame for } \langle S_n \mid \varnothing \rangle) =$$
$$= K + AD + Alt_n.$$

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$\supseteq$: Frame conditions

$\subseteq$: Canonical frame and truth-preserving operations

$$\text{Cayley graph of } \langle S_n \mid \varnothing \rangle \xrightarrow{\uparrow e} F{\uparrow}e \twoheadrightarrow F^{\#}_{K+AD+Alt_n} \twoheadrightarrow F_{K+AD+Alt_n}$$

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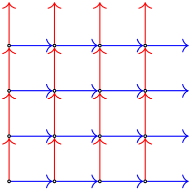
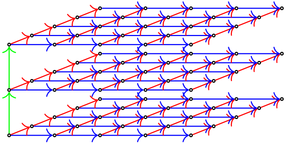
Can we again translate relations R into modal formulas?
No.

$G_1 = \langle x, y \mid xy = yx \rangle$ $\quad$ $G_2 = \langle x, y, z \mid xy = yx \rangle$





$F_{G_1} \models \Diamond\Box p \to \Box\Diamond p$ $\quad$ $F_{G_2} \not\models \Diamond\Box p \to \Box\Diamond p$

Negative result on simple Cayley frames

Theorem

There exist no modal logic L and translation function τ that maps group relations to modal formulas, such that for any group $G = \langle S \mid R \rangle$ and its simple Cayley frame $F = (G, \mathcal{R})$,

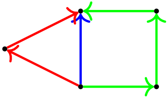
$$Log(F) = L + \bigwedge_{r \in R} \tau(r).$$

The same is true even if we replace R with $Pos(R)$.

A family of validities in a simple Cayley frame

Example

$G = \langle x, y, z \mid x = y^2,\ x = z^3 \rangle$



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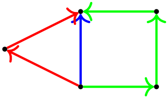

Frame condition: $\forall x \ \left|\mathcal{R}(x) \cap \mathcal{R}^2(x) \cap \mathcal{R}^3(x)\right| = 1$

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3. Let $\sim_l$ be an equivalence relation on G:
$$g_1 \sim_l g_2 \iff L(PW(g_1)) = L(PW(g_2)).$$

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4. For each $C \in G/\sim_l$, denote $L(C) = L(PW(g))$ for any $g \in G$.

A family of validities in a simple Cayley frame

Consider any class of equivalence $C \in G/\sim_I$.

- If C is infinite, we may define

$$\forall x \left| \bigcap_{I \in L(C)} \mathcal{R}^I(x) \right| \geq 1;$$

- If C is finite, we may define

$$\forall x \; 1 \leq \left| \bigcap_{I \in L(C)} \mathcal{R}^I(x) \right| \leq |C|\,;$$

A family of validities in a simple Cayley frame

For an infinite $C \in G/\sim_I$ such that $L(C) = \{l_1, \ldots, l_n\}$ is finite, define

$$\phi^\infty(C) = \bigwedge_{i=1}^{n} \Box^{l_i} p_i \to \Diamond^{l_1} \bigwedge_{i=1}^{n} p_i.$$

For a finite $C \in G/\sim_I$ such that $L(C) = \{l_1, \ldots, l_n\}$ is finite, define

$$\phi(C) = \left(\left(\bigwedge_{i=1}^{n} \Box^{l_i} p_i\right) \wedge \left(\bigwedge_{i=1}^{n} \left(p_i \to \bigvee_{j=1}^{|C|+1} q_j\right)\right)\right) \to$$
$$\to \Diamond^{l_1} \left(\left(\bigwedge_{i=1}^{n} p_i\right) \wedge \left(\bigvee_{j=1}^{|C|+1} \bigvee_{j'=j+1}^{|C|+1} q_j \wedge q_{j'}\right)\right).$$

A family of validities in a simple Cayley frame

Theorem

Let $F = (G, \mathcal{R})$ be the simple Cayley frame for a group $G = \langle S \mid R \rangle$. Then

1. For any finite $C \in G/\overset{l}{\sim}$ such that $L(C)$ is finite,

$$F \models \phi(C)$$

2. For any infinite $C \in G/\overset{l}{\sim}$ such that $L(C)$ is finite,

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If $L(C)$ is infinite, formulas $\phi(C)$ where $L(C)$ is replaced with any finite subset $L' \subseteq L(C)$ are valid in F.

Results

	Simple	Multimodal
General	$K + AD + Alt_n$	SL_S
Restricted	$\{\phi(C) \mid C \in G/\overset{I}{\sim} \text{ (finite)}\}$ $\{\phi^\infty(C) \mid C \in G/\overset{I}{\sim} \text{ (infinite)}\}$ $\dots$	$SL_S + \{r^\# \mid r \in Pos(R)\}$

Thank you for your attention!