A new perspective on quantum substructural logics

Gavin St. John

Università degli Studi di Cagliari [1]
GavinC.StJohn@unica.it

Joint work with Wesley Fussner (CNRS)

BLAST 2021
New Mexico State University

9 June 2021

[1] Support from MIUR within the project PRIN 2017:
"Logic and cognition. Theory, experiments, and applications", CUP: 2013YP4N3,
sub-project "Quantum structures and substructural logics: a unifying approach".

Birkhoff/von Neumann Approach to Quantum Logic

quantum events/properties $\Longleftrightarrow$ projection operators on $\mathcal{H}$, a complex separable Hilbert space

Let X a closed subspace of $\mathcal{H}$ and $X^{\perp}$ the subspace orthogonal to X. For all $v \in H$, $v = v_X + v_{X^{\perp}}$ for unique $v_X \in X$ and $v_{X^{\perp}} \in X^{\perp}$.

- $\Pi(\mathcal{H}) := \{P_X : v \mapsto v_X\}$ is the set of projection operators
- $\neg P_X := P_{X^{\perp}}$
- $P_X \wedge P_Y := P_{X \cap Y}$
- $P_X \vee P_Y := P_{(X \cup Y)^{\perp\perp}}$
- $0 := P_{\{\mathbf{0}\}}$
- $1 := P_H$

$(\Pi(\mathcal{H}), \wedge, \vee, \neg, 0, 1)$ is an example of an *orthomodular lattice*

See [Birkhoff and von Neumann, 1936].

OL and OML

Definition

An *involutive lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \neg)$ where:

- $(A, \wedge, \vee)$ is a lattice
- $\neg$ is an antitone involution on $\mathbf{A}$

$\mathbf{A}$ is called *bounded* if it has a *bottom* 0 and *top* 1.

Definition

An *ortholattice* is a bounded involutive lattice $\mathbf{A} = (A, \wedge, \vee, \neg, 0, 1)$ where $\neg$ is an ortho-complementation, i.e., $x \wedge \neg x \approx 0$.

Definition

An *orthomodular lattice* (OML) is an ortholattice satisfying:

(orthomodular law) $\quad x \leq y \implies y \approx x \vee (\neg x \wedge y)$

Ortholattices form a variety OL and OMLs form a variety OML.

Problems and motivations

The problems

- It is unknown whether OML admit any form of completions
  - Not closed under canonical completions [Harding, 1998]
  - or even MacNeille completions [Harding, 1991]
- The decidability of OML remains unknown
- No pair of operations form a (two-sided) residuated pair (see [Chiara et al., 2004])

New approaches: Zooming out

- Sasaki operations form a one-sided residuated pair
- Orthomodular groupoids [Chajda and Länger, 2017]
- Pointed left-residuated ℓ-groupoids [Fazio et al., 2021]
- Residuated ortholattices [Fussner and S., 2021]

Sasaki operations [Sasaki, 1954] are definable in involutive lattices:

$$\begin{array}{rcll} x \cdot y & := & x \wedge (\neg x \vee y) & \text{(Sasaki product)} \\ x \to y & := & \neg x \vee (x \wedge y) & \text{(Sasaki hook)} \end{array}$$

Proposition

Let $\mathbf{A}$ be a an ortholattice. Then the following are equivalent:

1. $\mathbf{A}$ is an OML.
2. $\mathbf{A} \models x \leq y \implies y \approx \neg x \to y$
3. $\mathbf{A} \models x \leq y \implies x \approx y \cdot x$.
4. $(\cdot, \to)$ form a *(right-) residuated pair:* $\mathbf{A} \models x \cdot y \leq z \Leftrightarrow y \leq x \to z$.

Proposition

Let $\mathbf{A}$ be a bounded involutive lattice. Then the operation $\cdot$ is residuated iff the operation $\to$ is co-residuated.
Moreover, if $\mathbf{A}$ is a bounded involutive lattice for which the above equivalent conditions hold, then $\mathbf{A}$ is an ortholattice.

Residuated Ortholattices

Definition

A *(Sasaki)* *residuated ortholattice* (or *ROL*) is an expansion of a bounded involutive lattice $(A, \wedge, \vee, \neg, 0, 1)$ by a binary operation $\backslash$ satisfying

$$x \cdot y \leq z \iff y \leq x \backslash z \tag{R}$$

where $\cdot$ is the Sasaki product: $x \cdot y = x \wedge (\neg x \vee y)$.

Properties

- Residuated ortholattices form a variety ROL.
- OML is a subvariety of ROL (taking $\backslash$ to be the Sasaki hook $\rightarrow$).

Residuated Ortholattices

Definition

A *(Sasaki)* *residuated ortholattice* (or *ROL*) is an expansion of a bounded involutive lattice $(A, \wedge, \vee, \neg, 0, 1)$ by a binary operation $\backslash$ satisfying

$$x \cdot y \leq z \iff y \leq x \backslash z \tag{R}$$

where $\cdot$ is the Sasaki product: $x \cdot y = x \wedge (\neg x \vee y)$.

Properties

- Both $\cdot$ and $\backslash$ or order-preserving in their right-coordinates.
- $\cdot$ distributes over arbitrary joins, when they exist, from the left.
- $\backslash$ distributes over arbitrary meets, when they exist, from the left.

Residuated Ortholattices

Definition

A *(Sasaki) residuated ortholattice* (or *ROL*) is an expansion of a bounded involutive lattice $(A, \wedge, \vee, \neg, 0, 1)$ by a binary operation $\backslash$ satisfying

$$x \cdot y \leq z \iff y \leq x \backslash z \tag{R}$$

where $\cdot$ is the Sasaki product: $x \cdot y = x \wedge (\neg x \vee y)$.

Properties

- Generally, $\cdot$ is not order-preserving in its left-coordinate.
- Generally, $\backslash$ is not order-reversing in its left-coordinate.
- Generally, $\cdot$ is neither commutative nor associative.

Residuated Ortholattices

Definition

A *(Sasaki)* *residuated ortholattice* (or *ROL*) is an expansion of a bounded involutive lattice $(A, \wedge, \vee, \neg, 0, 1)$ by a binary operation $\backslash$ satisfying

$$x \cdot y \leq z \iff y \leq x \backslash z \tag{R}$$

where $\cdot$ is the Sasaki product: $x \cdot y = x \wedge (\neg x \vee y)$.

Proposition

In ROL, finite products consisting only of variables x and y, where x is the *left-most* variable, are equal to $x \cdot y$.

E.g.,

$$\mathsf{ROL} \vDash x \cdot y \approx (x \cdot ((y \cdot x) \cdot y)) \cdot (y \cdot x)$$

Definition

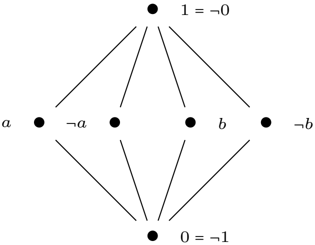
A *(Sasaki)* *residuated ortholattice* (or *ROL*) is an expansion of a bounded involutive lattice $(A, \wedge, \vee, \neg, 0, 1)$ by a binary operation $\backslash$ satisfying

$$x \cdot y \leq z \iff y \leq x \backslash z \tag{R}$$

where $\cdot$ is the Sasaki product: $x \cdot y = x \wedge (\neg x \vee y)$.

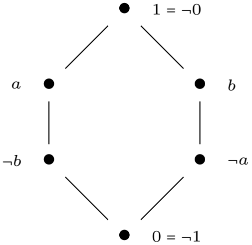
The number of OMLs and ROLs of size n up to isomorphism

n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
OMLs	1	0	1	0	1	0	2	0	2	0	3
ROLs	1	0	1	0	2	0	4	0	7	0	15



$\backslash$	0	$\neg a$	$\neg b$	a	b	1
0	1	1	1	1	1	1
$\neg a$	a	1	a	a	a	1
$\neg b$	b	b	1	b	b	1
a	$\neg a$	$\neg a$	$\neg a$	1	$\neg a$	1
b	$\neg b$	$\neg a$	$\neg a$	$\neg a$	1	1
1	0	$\neg a$	$\neg b$	a	b	1

Figure: The orthomodular lattice $\mathbf{MO}_2$. Expanded by $\backslash = \to$ it is an ROL.



$\backslash$	0	$\neg a$	$\neg b$	a	b	1
0	1	1	1	1	1	1
$\neg a$	a	1	a	a	1	1
$\neg b$	b	b	1	1	b	1
a	b	b	b	1	b	1
b	a	a	a	a	1	1
1	0	$\neg a$	$\neg b$	a	b	1

Figure: The ortholattice $\mathbf{B}_6$, also called *Benzene*, witnesses the failure of the orthomodular law. Expanded by $\backslash$ defined above, Benzene forms an ROL.

Proposition

Let $\mathbf{A}$ be an ortholattice. The following are equivalent.

(1) $\mathbf{A}$ is orthomodular.

(2) $\mathbf{B}_6$ is not a subalgebra of $\mathbf{A}$.

Theorem

Let V *be a subvariety of* ROL *such that* $\backslash$ *is definable in* V *by a term in the language* $\{\wedge, \vee, \neg, 0, 1\}$*. Then* V *is a variety of OMLs.*

Proof: If a non-OML subvariety $\mathcal{V}$ satisfies $t(x, y) \approx x\backslash y$, then so does $\mathbf{B}_6$. Hence t is compatible with every OL-congruence on $\mathbf{B}_6$. However, the OL-congruence θ generated by $(a, \neg b)$ would imply

$$b = a\backslash\neg b = t(a, \neg b)\ \theta\ t(\neg b, \neg b) = \neg b\backslash\neg b = 1,$$

a contradiction. □

The logic of ROL and algebraizability

Let K be a 1-pointed class of algebras in the signature $\mathcal{L}$.

- The *assertional logic of* K is the logic $(\mathcal{L}, \vdash_{\mathsf{K}})$, where for all sets of $\mathcal{L}$-formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \vdash_{\mathsf{K}} \varphi \iff \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{K}} \varphi \approx 1$$

Note

The assertional logic of OL is a textbook example of a logic that is weakly algebraizable but not algebraizable

The logic of ROL and algebraizability

Let K be a 1-pointed class of algebras in the signature $\mathcal{L}$.

- The *assertional logic of* K is the logic $(\mathcal{L}, \vdash_{\mathsf{K}})$, where for all sets of $\mathcal{L}$-formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \vdash_{\mathsf{K}} \varphi \iff \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{K}} \varphi \approx 1$$

- $\mathcal{K}$ is *1-regular* if for all $\mathbf{A} \in \mathcal{K}$ and congruences θ, ψ of $\mathbf{A}$:

$$[1]_\theta = [1]_\psi \implies \theta = \psi$$

  where $[a]_\theta$ denotes the θ-congruence class of $a \in A$.

Note

The assertional logic of every 1-pointed, 1-regular variety is algebraizable in the sense of Blok and Pigozzi. [Font, 2016]

Proposition [Bruns and Harding, 2000]

OML is 1-regular, hence its assertional logic is algebraizable and its equivalent algebraic semantics is OML.

The logic of ROL and algebraizability

Let K be a 1-pointed class of algebras in the signature $\mathcal{L}$.

- The *assertional logic of* K is the logic $(\mathcal{L}, \vdash_{\mathsf{K}})$, where for all sets of $\mathcal{L}$-formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \vdash_{\mathsf{K}} \varphi \iff \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{K}} \varphi \approx 1$$

- $\mathcal{K}$ is *1-regular* if for all $\mathbf{A} \in \mathcal{K}$ and congruences θ, ψ of $\mathbf{A}$:

$$[1]_\theta = [1]_\psi \implies \theta = \psi$$

where $[a]_\theta$ denotes the θ-congruence class of $a \in A$.

Note

The assertional logic of every 1-pointed, 1-regular variety is algebraizable in the sense of Blok and Pigozzi. [Font, 2016]

Theorem

ROL *is* 1*-regular, and is therefore the equivalent algebraic semantics to its assertional logic.*

Given a residuated ortholattice $\mathbf{A} = (A, \wedge, \vee, \neg, \backslash, 0, 1)$, we define:

$$\sim x := x \backslash 0 \qquad \overline{x} := \sim\sim x$$

$$x * y := x \wedge (\sim x \vee y) \qquad x \mathrel{-\!\!*} y := \sim x \vee (x \wedge y)$$

$$\overline{A} := \{\overline{a} : a \in A\}$$

Lemma

Let $\mathbf{A} = (A, \wedge, \vee, \neg, \backslash, 0, 1)$ *be an ROL. Then for all* $x, y \in A$*:*

(1) $\sim 0 = 1$ *and* $\sim 1 = 0$.

(2) $x \wedge \sim x = 0$.

(3) $x \leq y \implies \sim y \leq \sim x$.

(4) $\sim\sim\sim x = \sim x$.

(5) $\sim(x \vee y) = \sim x \wedge \sim y$ *and* $\sim(x \wedge y) = \sim x \vee \sim y$.

A new negation and a skeleton

Given a residuated ortholattice $\mathbf{A} = (A, \wedge, \vee, \neg, \backslash, 0, 1)$, we define:

$$\sim x := x \backslash 0 \qquad \overline{x} := \sim\sim x$$

$$x * y := x \wedge (\sim x \vee y) \qquad x \multimap y := \sim x \vee (x \wedge y)$$

$$\overline{A} := \{\overline{a} : a \in A\}$$

Lemma

Let $\mathbf{A} = (A, \wedge, \vee, \neg, \backslash, 0, 1)$ *be an ROL. Then for all* $x, y \in A$*:*

(6) $x \leq \overline{x} = \overline{\overline{x}}$.

(7) $\overline{\neg x} = \sim\overline{x}$.

(8) $\overline{x \vee y} = \overline{x} \vee \overline{y}$ *and* $\overline{x \wedge y} = \overline{x} \wedge \overline{y}$.

(9) $\overline{x \cdot y} = \overline{x} * \overline{y}$.

(10) $\overline{x} \leq \overline{y} \implies \overline{x} = \overline{y} * \overline{x}$.

A new negation and a skeleton

Given a residuated ortholattice $\mathbf{A} = (A, \wedge, \vee, \neg, \backslash, 0, 1)$, we define:

$$\sim x := x \backslash 0 \qquad \overline{x} := \sim\sim x$$

$$x * y := x \wedge (\sim x \vee y) \qquad x \twoheadrightarrow y := \sim x \vee (x \wedge y)$$

$$\overline{A} := \{\overline{a} : a \in A\}$$

Theorem

Let $\mathbf{A} = (A, \wedge, \vee, \neg, \backslash, 0, 1)$ *be a residuated ortholattice.*

(1) $\overline{\mathbf{A}} = (\overline{A}, \wedge, \vee, \sim, 0, 1)$ *is an OML.*

(2) $\overline{x} \backslash \overline{y} = \overline{x} \twoheadrightarrow \overline{y}$ *for all* $x, y \in A$.

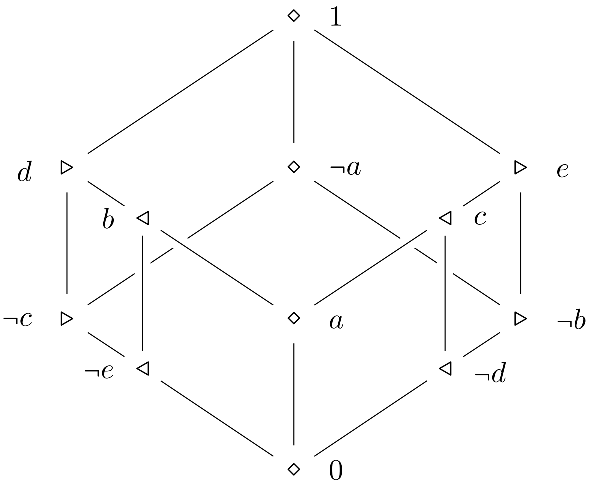
(3) *The map* $x \mapsto \overline{x}$ *is an ortholattice homomorphism of* $\mathbf{A}$ *onto* $\overline{\mathbf{A}}$.

Warning: *Generally,* $\overline{x \backslash y} \neq \overline{x} \backslash \overline{y}$

Corollary

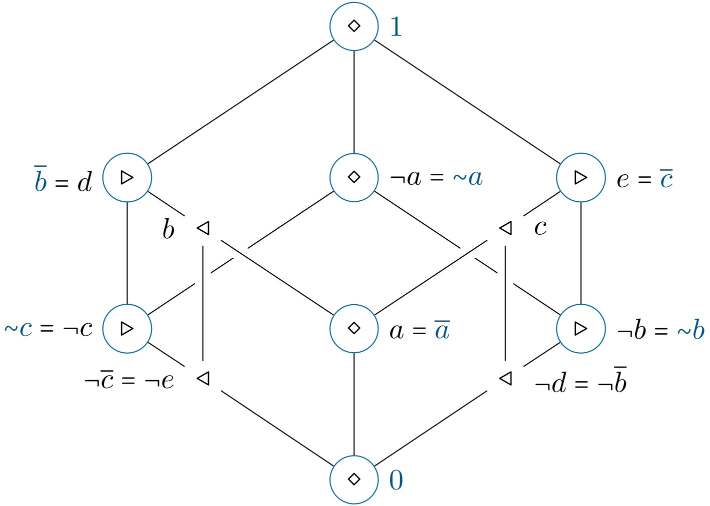
Let $\mathbf{A}$ *be a residuated ortholattice. Then the following are equivalent:*

(1) $\mathbf{A}$ *is an OML.* (2) $\mathbf{A} \vDash \sim x \approx \neg x$. (3) $\mathbf{A} \vDash x \approx \sim\sim x$.



x is:

- $\lhd$: *left-central* if $x \cdot y = x \wedge y$ for all y.
- $\rhd$: *right-central* if $y \cdot x = y \wedge x$ for all y.
- $\diamond$: *central* if it is both left- and right-central.



x is:

- ◁: *left-central* if $x \cdot y = x \wedge y$ for all y.
- ▷: *right-central* if $y \cdot x = y \wedge x$ for all y.
- ◇: *central* if it is both left- and right-central.

A Kolmogorov-style translation

Let t be a residuated ortholattice term. Recall, $\sim t := t\backslash 0$ and $\overline{t} := \sim\sim t$.

We define the term $\mathrm{T}(t)$ inductively on the complexity of t:

- $\mathrm{T}(0) = 0$, $\mathrm{T}(1) = 1$, $\mathrm{T}(x) = \overline{x}$ for all variables x,
- $\mathrm{T}(r \star s) = \mathrm{T}(r) \star \mathrm{T}(s)$ for each $\star \in \{\wedge, \vee, \backslash\}$.
- $\mathrm{T}(\neg s) = \sim\mathrm{T}(s)$

For a set of equations E, $\mathrm{T}[E] := \{\mathrm{T}(u) \approx \mathrm{T}(v) : (u \approx v) \in E\}$.

Definition

For subvarieties W, V of ROL, we say the V *is translatable into* W if

$$E \vDash_{\mathsf{V}} s \approx t \iff \mathrm{T}[E] \vDash_{\mathsf{W}} \mathrm{T}(s) \approx \mathrm{T}(t),$$

for any set of equations $E \cup \{s \approx t\}$ in the language of ROL.

See [Galatos and Ono, 2006].

Definition

For subvarieties W, V of ROL, we say the V is *translatable into* W if

$$E \vDash_{\mathsf{V}} s \approx t \iff \mathrm{T}[E] \vDash_{\mathsf{W}} \mathrm{T}(s) \approx \mathrm{T}(t),$$

for any set of equations $E \cup \{s \approx t\}$ in the language of ROL.

For a variety V and set of equations Σ, define $\mathsf{V}_\Sigma := \mathsf{V} + \Sigma$

Lemma

Let t be a residuated ortholattice term, $\mathbf{A} \in \mathsf{ROL}$, and $\mathbf{a}$ an element of an appropriate power of A. Then $\mathrm{T}(t)^{\mathbf{A}}(\mathbf{a}) = t^{\overline{\mathbf{A}}}(\overline{\mathbf{a}})$.

Lemma

Let Σ be a set of equations in the language of ROLs. Then

(1) OML_Σ *is a subvariety of* $\mathsf{ROL}_{\mathrm{T}[\Sigma]}$.

(2) *If* $\mathbf{A} \in \mathsf{ROL}_{\mathrm{T}[\Sigma]}$, *then* $\overline{\mathbf{A}} \in \mathsf{OML}_\Sigma$.

Definition

For subvarieties W, V of ROL, we say the V is *translatable into* W if

$$E \vDash_{\mathsf{V}} s \approx t \iff \mathrm{T}[E] \vDash_{\mathsf{W}} \mathrm{T}(s) \approx \mathrm{T}(t),$$

for any set of equations $E \cup \{s \approx t\}$ in the language of ROL.

For a variety V and set of equations Σ, define $\mathsf{V}_\Sigma := \mathsf{V} + \Sigma$

Theorem

Let Σ *be a set of equations in the language of ROLs. Then* OML_Σ *is translatable into any variety* V *where* $\mathsf{OML}_\Sigma \subseteq \mathsf{V} \subseteq \mathsf{ROL}_{\mathrm{T}[\Sigma]}$.
In particular, OML *is translatable into* ROL.

Proposition

For varieties W, V of residuated ortholattices, if V is translatable into W then deciding equations in V is no harder than deciding equations in W. In particular, if the equational theory of W is decidable then the same holds for V.

Definition

For subvarieties W, V of ROL, we say the V *is translatable into* W if

$$E \vDash_{\mathsf{V}} s \approx t \iff \mathrm{T}[E] \vDash_{\mathsf{W}} \mathrm{T}(s) \approx \mathrm{T}(t),$$

for any set of equations $E \cup \{s \approx t\}$ in the language of ROL.

For a variety V and set of equations Σ, define $\mathsf{V}_\Sigma := \mathsf{V} + \Sigma$

Theorem

Let Σ be a set of equations in the language of ROLs. Then OML_Σ *is translatable into any variety* V *where* $\mathsf{OML}_\Sigma \subseteq \mathsf{V} \subseteq \mathsf{ROL}_{\mathrm{T}[\Sigma]}$.
In particular, OML *is translatable into* ROL.

Corollary

OML *has a decidable equational theory if any variety of residuated ortholattices that contains it has a decidable equational theory.*

References I

Birkhoff, G. and von Neumann, J. (1936).
The logic of quantum mechanics.
Ann. of Math. (2), 37:823–843.

Bruns, G. and Harding, J. (2000).
Algebraic aspects of orthomodular lattices.
In Coecke, B., Moore, D., and Wilce, A., editors, *Current Research in Operational Quantum Logic*, pages 37–65. Springer.

Chajda, I. and Länger, H. (2017).
Orthomodular lattices can be converted into left-residuated ℓ-groupoids.
Miskolc Mathematical Notes, 18:685–689.

References II

- Chiara, M. D., Giuntini, R., and Greechie, R. (2004).
  Reasoning in Quantum Theory.
  Kluwer, Dordrecht.

- Fazio, D., Ledda, A., and Paoli, F. (2021).
  Residuated structures and orthomodular lattices.
  Studia Logica.
  DOI 10.1007/s11225-021-09946-1.

- Font, J. (2016).
  Abstract Algebraic Logic: An Introductory Textbook.
  College Publications.

- Galatos, N. and Ono, H. (2006).
  Glivenko theorems for substructural logics over fl.
  J. Symb. Log., 71:1353–1384.

References III

Harding, J. (1991).
Orthomodular lattices whose macneille completions are not orthomodular.
Order, 8:93–103.

Harding, J. (1998).
Canonical completions of lattices and ortholattices.
Tatra Mt. Math. Publ., 15:85–96.

Sasaki, U. (1954).
Orthocomplemented lattices satisfying the exchange axiom.
J. Sci. Hiroshima Univ. Ser. A, 17:293–302.

Thank you!

Preprint can be found at: `arXiv:2106.03656`

W. Fussner and G. St. John (2021)
Negative translations of orthomodular lattices and their logic
To appear in: *Proceedings 18th International Conference on Quantum Physics and Logic*