Generalized Heyting Algebras and Duality

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Let $\mathcal{A} = (A, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. A tuple $(\mathcal{A}, \nabla, \rightarrow)$ is called a $\nabla$-algebra if $\nabla c \wedge a \leq b$ is equivalent to $c \leq a \rightarrow b$, for any $a, b, c \in A$. 

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**Example**

- For *bounded lattices*, set $\nabla c = 0$ and $a \rightarrow b = 1$, for any $a, b, c \in A$. 
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**Definition**

Let \( \mathcal{A} = (A, \leq, \wedge, \vee, 0, 1) \) be a bounded lattice. A tuple \( (\mathcal{A}, \triangledown, \rightarrow) \) is called a \( \triangledown \)-algebra if \( \triangledown c \wedge a \leq b \) is equivalent to \( c \leq a \rightarrow b \), for any \( a, b, c \in A \).

**Example**

- For **bounded lattices**, set \( \triangledown c = 0 \) and \( a \rightarrow b = 1 \), for any \( a, b, c \in A \).
- For **Heyting algebras**, set \( \triangledown c = c \) and \( a \rightarrow b = a \supset b \), for any \( a, b, c \in A \), where \( \supset \) is the Heyting implication.
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**Definition**

Let $\mathcal{A} = (A, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. A tuple $(\mathcal{A}, \nabla, \rightarrow)$ is called a $\nabla$-algebra if $\nabla c \wedge a \leq b$ is equivalent to $c \leq a \rightarrow b$, for any $a, b, c \in A$.

**Example**

- For *bounded lattices*, set $\nabla c = 0$ and $a \rightarrow b = 1$, for any $a, b, c \in A$.
- For *Heyting algebras*, set $\nabla c = c$ and $a \rightarrow b = a \supset b$, for any $a, b, c \in A$, where $\supset$ is the Heyting implication.

Hence, $\nabla$-algebras generalize both bounded lattices and Heyting algebras.
Intuitionistic Kripke Frames

Example

Let \((W, \leq)\) be a poset. By an *intuitionistic Kripke frame*, we mean a tuple \(\mathcal{K} = (W, \leq, R)\), where \(R\) is a binary relation over \(W\), compatible with the partial order, i.e., if \(k' \leq k R l \leq l'\), then \(k' R l'\), for any \(k, k', l, l' \in W\).

To any intuitionistic Kripke frame, we can assign a canonical \(\nabla\)-algebra, encoding its structure via topology. Set \(X\) as the locale of all upsets of \((W, \leq)\) and define \(\nabla : X \to X\) as \(\nabla_{\mathcal{K}} U = \{x \in W \mid \exists y \in U \ R(y, x)\}\) and \(U \to_{\mathcal{K}} V = \{x \in W \mid \forall y \in W[R(x, y) \land y \in U \Rightarrow y \in V]\}\). It is easy to see that \((X, \nabla_{\mathcal{K}}, \to_{\mathcal{K}})\) is a \(\nabla\)-algebra.
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Motivation I: Intuitionistic Temporal Logics

A Heyting \(\nabla\)-algebra is the algebraic model for basic intuitionistic temporal logic.
Example

Let $X$ be a topological space and $f : X \to X$ be a continuous function. Define $\to_f$ over $\mathcal{O}(X)$ by $U \to_f V = f_*(\text{int}[U^c \cup V])$, where $f_* : \mathcal{O}(X) \to \mathcal{O}(X)$ is the right adjoint of $f^{-1}$. Then, the structure $(\mathcal{O}(X), f^{-1}, \to_f)$ is a $\nabla$-algebra. This $\nabla$-algebra is the point-free version of the dynamic system $(X, f)$, using the adjunction $f^{-1} \dashv f_*$ to encode the map $f$. 
Example

Let $X$ be a topological space and $f : X \rightarrow X$ be a continuous function. Define $\rightarrow_f$ over $\mathcal{O}(X)$ by $U \rightarrow_f V = f_*(\text{int}[U^c \cup V])$, where $f_* : \mathcal{O}(X) \rightarrow \mathcal{O}(X)$ is the right adjoint of $f^{-1}$. Then, the structure $(\mathcal{O}(X), f^{-1}, \rightarrow_f)$ is a $\nabla$-algebra. This $\nabla$-algebra is the point-free version of the dynamic system $(X, f)$, using the adjunction $f^{-1} \dashv f_*$ to encode the map $f$.

Motivation II: Point-free Dynamic Systems

A Heyting $\nabla$-algebra in which $\nabla$ commutes with all finite meets is the elementary and point-free version of dynamic systems.
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Motivation III: Implications

$\nabla$-algebras represent all possible implications...
Varieties of $\triangledown$-algebras

A $\triangledown$-algebra is called:

- (D): distributive, if $\mathcal{A}$ is distributive.
- (N): normal, if $\triangledown$ commutes with all finite meets.
- (Fa): faithful, if $\triangledown$ is surjective.
- (Fu): full, if $\square$ is surjective, where $\square a = 1 \rightarrow a$.

For any $C \subseteq \{D, N, Fa, Fu\}$, by $\mathcal{V}(C)$ we mean the class of all $\triangledown$-algebras with the properties described in the set $C$. 
Varieties of $\sqcap$-algebras

A $\sqcap$-algebra is called:

- **(D)**: distributive, if $\mathcal{A}$ is distributive.
- **(N)**: normal, if $\sqcap$ commutes with all finite meets.
- **(Fa)**: faithful, if $\sqcap$ is surjective.
- **(Fu)**: full, if $\blacksquare$ is surjective, where $\blacksquare a = 1 \rightarrow a$.

For any $C \subseteq \{D, N, Fa, Fu\}$, by $\mathcal{V}(C)$ we mean the class of all $\sqcap$-algebras with the properties described in the set $C$.

The last three has a topological root. More precisely, over locales and in the presence of enough separation axioms on the space:

- **(N)**: $\sqcap$ is the inverse image of a continuous function.
- **(N)+(Fa)**: The continuous function is a topological embedding.
- **(N)+(Fu)**: The continuous function is surjective.
Some Closure Properties

It is possible to rewrite the theory of Heyting algebras for different families of $\sqcap$-algebras. First of all:

Theorem
For any subset $C \subseteq \{D, N, Fa, Fu\}$ the class $V(C)$ is a variety.

Some of the varieties are closed under Dedekind-MacNeille completion:

Theorem
Let $C \subseteq \{N, Fa, Fu\}$. Then, the variety $V(C)$ is closed under Dedekind-MacNeille completion.

Theorem (Amalgamation)
The varieties $V(D, N)$ and $V(D, N, Fa)$ have the amalgamation property.
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**Theorem (Amalgamation)**

The varieties $\mathcal{V}(D, N)$ and $\mathcal{V}(D, N, Fa)$ have the amalgamation property.
Subdirectly Irreducible Normal Distributive $\bigtriangleup$-algebras

To investigate the structure of the varieties, we have to study the building blocks of the varieties:

**Theorem**

A non-trivial normal distributive $\bigtriangleup$-algebra $A$ is subdirectly irreducible iff there exists $x \in A - \{1\}$ such that for any $y \in A - \{1\}$, there exist $m_i, n_i \in \mathbb{N}$ such that $\bigwedge_i \bigtriangleup^{m_i} \Box^{n_i} y \leq x$. 

Example

For Heyting algebras, where $\bigtriangleup a = \Box a = a$, the theorem states that $A$ is subdirectly irreducible iff there exists $x \in A - \{1\}$ such that $y \leq x$, for any $y \in A - \{1\}$. This means that $A$ has the second greatest element.
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For Heyting algebras, where $\nabla a = \Box a = a$, the theorem states that $\mathcal{A}$ is subdirectly irreducible iff there exists $x \in \mathcal{A} - \{1\}$ such that $y \leq x$, for any $y \in \mathcal{A} - \{1\}$. This means that $\mathcal{A}$ has the second greatest element.
Theorem

A normal distributive $\nabla$-algebra $A$ is simple iff for any $x \in A - \{1\}$, there exist $m_i, n_i \in \mathbb{N}$ such that $\bigwedge_i \nabla^{m_i} \Box^{n_i} x = 0$. 

Example

For Heyting algebras, where $\nabla a = \Box a = a$, the theorem states that $A$ is simple iff for any $x \in A - \{1\}$ we have $x = 0$. This means that $A$ is the boolean algebra $\{0, 1\}$. 

Theorem

There are infinitely many simple finite normal Heyting $\nabla$-algebras.
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Theorem

There are infinitely many simple finite normal Heyting $\nabla$-algebras.
Variants of Kripke Frames

Variants of Kripke frames are defined by:

**Definition**

Let \((\mathcal{W}, \leq, R)\) be an intuitionistic Kripke frame:

- **(N)**: If there exists an order-preserving function \(\pi : \mathcal{W} \to \mathcal{W}\) such that \((x, y) \in R\) iff \(x \leq \pi(y)\).

- **(Fa)**: If for any \(x \in \mathcal{W}\), there exists \(y \in \mathcal{W}\) such that \((y, x) \in R\) and for any \(z \in \mathcal{W}\) such that \((y, z) \in R\) we have \(x \leq z\).

- **(Fu)**: If for any \(x \in \mathcal{W}\), there exists \(y \in \mathcal{W}\) such that \((x, y) \in R\) and for any \(z \in \mathcal{W}\) such that \((z, y) \in R\) we have \(z \leq x\).

For any \(C \subseteq \{N, Fa, Fu\}\), by \(K(C)\), we mean the class of all Kripke frames with the properties described in the set \(C\).
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For any \(C \subseteq \{N, Fa, Fu\}\), by \(K(C)\), we mean the class of all Kripke frames with the properties described in the set \(C\).

- **(N)+(Fa)**: The function \(\pi\) is an order embedding.
- **(N)+(Fu)**: The function \(\pi\) is surjective.
A Representation Theorem

For any $C \subseteq \{N, Fa, Fu\}$, if a Kripke frame is in $K(C)$, then its corresponding $\nabla$-algebra is in $V(C, D)$.

Is it possible to strengthen this representation theorem to a full scale duality theory?
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For any $C \subseteq \{N, Fa, Fu\}$, if a Kripke frame is in $K(C)$, then its corresponding $\nabla$-algebra is in $\mathcal{V}(C, D)$.

The following representation theorems state that any such $\nabla$-algebra can be seen as a subalgebra of such a $\nabla$-algebra:
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The following representation theorems state that any such $\nabla$-algebra can be seen as a subalgebra of such a $\nabla$-algebra:

**Theorem**

For any $C \subseteq \{ N, Fa, Fu \}$ and any $A \in V(C, D)$, there is a Kripke frame $\mathcal{K} \in K(C)$ and a $\nabla$-algebra embedding from $A$ into the $\nabla$-algebra corresponding to $\mathcal{K}$. 
A Representation Theorem

For any $C \subseteq \{N, Fa, Fu\}$, if a Kripke frame is in $K(C)$, then its corresponding $\nabla$-algebra is in $V(C, D)$.

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For any $C \subseteq \{N, Fa, Fu\}$ and any $A \in V(C, D)$, there is a Kripke frame $K \in K(C)$ and a $\nabla$-algebra embedding from $A$ into the $\nabla$-algebra corresponding to $K$.

Is it possible to strengthen this representation theorem to a full scale duality theory?
A $\triangledown$-space is a tuple $(X, \leq, R)$ of a Priestley space $(X, \leq)$ and a binary relation $R$ on $X$ such that:

- $R$ is compatible with the order, i.e., if $x' \leq x$, $(x, y) \in R$ and $y \leq y'$, then $(x', y') \in R$,
- $R[x] = \{ y \in X \mid (x, y) \in R \}$ is closed, for every $x \in X$,
- $\Diamond_R(U) = \{ x \in X \mid \exists y \in U \ (x, y) \in R \}$ is clopen, for any clopen $U$,
- $\nabla_R(V) = \{ x \in X \mid \exists y \in V \ (y, x) \in R \}$ is a clopen upset, for any clopen upset $V$.
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Example

- For \(R = \emptyset\), a \(\nabla\)-space is just a Priestley space.
∇-spaces as the Unification of Priestley and Esakia Spaces

Definition

A ∇-space is a tuple \((X, \leq, R)\) of a Priestley space \((X, \leq)\) and a binary relation \(R\) on \(X\) such that:

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Example

- For \(R = \emptyset\), a ∇-space is just a Priestley space.
- For \(R = \leq\), a ∇-space is just an Esakia space.
Variants of $\nabla$-spaces

Note that any $\nabla$-space is a Kripke frame, if we forget the topology of the space. A $\nabla$-space satisfies a condition in the set \(\{N, Fa, Fu\}\), if it satisfies the condition as a Kripke frame.

\[\pi\] is a Priestley map and \(\downarrow\pi[U]\) is clopen, for any clopen \(U\).

\(\pi\) is also an order embedding or a regular monic in the category of Priestley spaces.

\(\pi\) is also surjective or an epic map in the category of Priestley spaces.
Variants of $\nabla$-spaces

Note that any $\nabla$-space is a Kripke frame, if we forget the topology of the space. A $\nabla$-space satisfies a condition in the set $\{N, Fa, Fu\}$, if it satisfies the condition as a Kripke frame.

- **(N)**: The function $\pi : X \to X$ is a Priestley map and $\downarrow \pi[U]$ is clopen, for any clopen $U$.

- **(N)+(Fa)**: The function $\pi$ is also an order embedding or a regular monic in the category of Priestley spaces.

- **(N)+(Fu)**: The function $\pi$ is also surjective or an epic map in the category of Priestley spaces.
Definition

By a $\nabla$-space map $f : (X, \leq_X, R_X) \to (Y, \leq_Y, R_Y)$, we mean an order-preserving continuous map such that:

- For any $x, x' \in X$, if $(x, x') \in R_X$ then $(f(x), f(x')) \in R_Y$,
- for any $y' \in Y$ such that $(f(x), y) \in R_Y$, there exists $x' \in X$ such that $(x, x') \in R_X$ and $f(x') = y$,
- for any $y \in Y$ such that $(y, f(x)) \in R_Y$, there exists $x' \in X$ such that $(x', x) \in R_X$ and $f(x') \geq_Y y$. 

Let $C \subseteq \{N, Fa, Fu\}$. The $\nabla$-spaces in $\mathcal{K}(C)$ together with $\nabla$-space maps form a category. Denote this category by $\text{Space}_{\nabla}(C)$. If we also denote the category of all $\nabla$-algebras in $\mathcal{V}(D, C)$ together with corresponding algebraic morphisms by $\text{Alg}_{\nabla}(D, C)$, then:

**Theorem (Priestley-Esakia duality for distributive $\nabla$-algebras)**

Let $C \subseteq \{N, Fa, Fu\}$. Then, $\text{Alg}_{\nabla}(D, C) \simeq \text{Space}_{\nabla}(C)^{op}$. 

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Let $C \subseteq \{N, Fa, Fu\}$. Then, $\textbf{Alg}_{\nabla}(D, C) \simeq \textbf{Space}_{\nabla}(C)$.
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**Theorem (Priestley-Esakia duality for distributive $\nabla$-algebras)**

Let $C \subseteq \{N, Fa, Fu\}$. Then, $\textbf{Alg}_{\nabla}(D, C) \simeq \textbf{Space}^{op}_{\nabla}(C)$.
Thank you for your attention!