Generalized Heyting Algebras and Duality

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Let $\mathcal{A} = (A, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. A tuple $(\mathcal{A}, \nabla, \to)$ is called a ∇-algebra if $\nabla c \wedge a \leq b$ is equivalent to $c \leq a \to b$, for any $a, b, c \in A$.

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Example

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- For *Heyting algebras*, set $\nabla c = c$ and $a \to b = a \supset b$, for any $a, b, c \in A$, where $\supset$ is the Heyting implication.

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Hence, ∇-algebras generalize both bounded lattices and Heyting algebras.

Intuitionistic Kripke Frames

Example

Let $(W, \leq)$ be a poset. By an *intuitionistic Kripke frame*, we mean a tuple $\mathcal{K} = (W, \leq, R)$, where R is a binary relation over W, compatible with the partial order, i.e., if $k' \leq k \, R \, l \leq l'$, then $k' \, R \, l'$, for any $k, k', l, l' \in W$. To any intuitionistic Kripke frame, we can assign a canonical ∇-algebra, encoding its structure via topology. Set $\mathcal{X}$ as the locale of all upsets of $(W, \leq)$ and define $\nabla : \mathcal{X} \to \mathcal{X}$ as $\nabla_{\mathcal{K}} U = \{x \in W \mid \exists y \in U \; R(y, x)\}$ and $U \to_{\mathcal{K}} V = \{x \in W \mid \forall y \in W[R(x, y) \wedge y \in U \Rightarrow y \in V]\}$. It is easy to see that $(\mathcal{X}, \nabla_{\mathcal{K}}, \to_{\mathcal{K}})$ is a ∇-algebra.

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Motivation I: Intuitionistic Temporal Logics

A Heyting ∇-algebra is the algebraic model for basic intuitionistic temporal logic.

Example

Let X be a topological space and $f : X \to X$ be a continuous function. Define $\to_f$ over $\mathcal{O}(X)$ by $U \to_f V = f_*(int[U^c \cup V])$, where $f_* : \mathcal{O}(X) \to \mathcal{O}(X)$ is the right adjoint of f^{-1}. Then, the structure $(\mathcal{O}(X), f^{-1}, \to_f)$ is a ∇-algebra. This ∇-algebra is the point-free version of the dynamic system (X, f), using the adjunction $f^{-1} \dashv f_*$ to encode the map f.

Dynamic systems

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Motivation II: Point-free Dynamic Systems

A Heyting ∇-algebra in which ∇ commutes with all finite meets is the elementary and point-free version of dynamic systems.

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Motivation III: Implications

∇-algebras represent all possible implications...

Varieties of ∇-algebras

A ∇-algebra is called:

- **(D)**: distributive, if $\mathcal{A}$ is distributive.
- **(N)**: normal, if ∇ commutes with all finite meets.
- **(Fa)**: faithful, if ∇ is surjective.
- **(Fu)**: full, if $\Box$ is surjective, where $\Box a = 1 \to a$.

For any $C \subseteq \{D, N, Fa, Fu\}$, by $\mathcal{V}(C)$ we mean the class of all ∇-algebras with the properties described in the set C.

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The last three has a topological root. More precisely, over locales and in the presence of enough separation axioms on the space:

- **(N)**: ∇ is the inverse image of a continuous function.
- **(N)+(Fa)**: The continuous function is a topological embedding.
- **(N)+(Fu)**: The continuous function is surjective.

Some Closure Properties

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Let $C \subseteq \{N, Fa, Fu\}$. Then, the variety $\mathcal{V}(C)$ is closed under Dedekind-MacNeille completion.

Theorem (Amalgamation)

The varieties $\mathcal{V}(D, N)$ and $\mathcal{V}(D, N, Fa)$ have the amalgamation property.

Subdirectly Irreducible Normal Distributive ∇-algebras

To investigate the structure of the varieties, we have to study the building blocks of the varieties:

Theorem

A non-trivial normal distributive ∇-algebra $\mathcal{A}$ is subdirectly irreducible iff there exists $x \in A - \{1\}$ such that for any $y \in A - \{1\}$, there exist $m_i, n_i \in \mathbb{N}$ such that $\bigwedge_i \nabla^{m_i} \Box^{n_i} y \leq x$.

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Example

For Heyting algebras, where $\nabla a = \Box a = a$, the theorem states that $\mathcal{A}$ is subdirectly irreducible iff there exists $x \in A - \{1\}$ such that $y \leq x$, for any $y \in A - \{1\}$. This means that $\mathcal{A}$ has the second greatest element.

Simple Normal Distributive ∇-algebras

Theorem

A normal distributive ∇-algebra $\mathcal{A}$ is simple iff for any $x \in A - \{1\}$, there exist $m_i, n_i \in \mathbb{N}$ such that $\bigwedge_i \nabla^{m_i} \Box^{n_i} x = 0$.

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Theorem

There are infinitely many simple finite normal Heyting ∇-algebras.

Variants of Kripke Frames

Variants of Kripke frames are defined by:

Definition

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- **(Fa)**: If for any $x \in W$, there exists $y \in W$ such that $(y, x) \in R$ and for any $z \in W$ such that $(y, z) \in R$ we have $x \leq z$.
- **(Fu)**: If for any $x \in W$, there exists $y \in W$ such that $(x, y) \in R$ and for any $z \in W$ such that $(z, y) \in R$ we have $z \leq x$.

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- **(N)+(Fa)**: The function π is an order embedding.
- **(N)+(Fu)**: The function π is surjective.

A Representation Theorem

For any $C \subseteq \{N, Fa, Fu\}$, if a Kripke frame is in $\mathbf{K}(C)$, then its corresponding ∇-algebra is in $\mathcal{V}(C, D)$.

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For any $C \subseteq \{N, Fa, Fu\}$ and any $\mathcal{A} \in \mathcal{V}(C, D)$, there is a Kripke frame $\mathcal{K} \in \mathbf{K}(C)$ and a ∇-algebra embedding from $\mathcal{A}$ into the ∇-algebra corresponding to $\mathcal{K}$.

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Is it possible to strengthen this representation theorem to a full scale duality theory?

∇-spaces as the Unification of Priestley and Esakia Spaces

Definition

A ∇-space is a tuple $(X, \leq, R)$ of a Priestley space $(X, \leq)$ and a binary relation R on X such that:

- R is compatible with the order, i.e., if $x' \leq x$, $(x, y) \in R$ and $y \leq y'$, then $(x', y') \in R$,
- $R[x] = \{y \in X \mid (x, y) \in R\}$ is closed, for every $x \in X$,
- $\Diamond_R(U) = \{x \in X \mid \exists y \in U \; (x, y) \in R\}$ is clopen, for any clopen U,
- $\nabla_R(V) = \{x \in X \mid \exists y \in V \; (y, x) \in R\}$ is a clopen upset, for any clopen upset V.

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- For $R = \emptyset$, a ∇-space is just a *Priestley* space.

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Variants of ∇-spaces

Note that any ∇-space is a Kripke frame, if we forget the topology of the space. A ∇-space satisfies a condition in the set $\{N, Fa, Fu\}$, if it satisfies the condition as a Kripke frame.

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- **(N)**: The function $\pi : X \to X$ is a Priestley map and $\downarrow \pi[U]$ is clopen, for any clopen U.
- **(N)+(Fa)**: The function π is also an order embedding or a regular monic in the category of Priestley spaces.
- **(N)+(Fu)**: The function π is also surjective or an epic map in the category of Priestley spaces.

Definition

By a ∇-space map $f : (X, \leq_X, R_X) \to (Y, \leq_Y, R_Y)$, we mean an order-preserving continuous map such that:

- For any $x, x' \in X$, if $(x, x') \in R_X$ then $(f(x), f(x')) \in R_Y$,
- for any $y' \in Y$ such that $(f(x), y) \in R_Y$, there exists $x' \in X$ such that $(x, x') \in R_X$ and $f(x') = y$,
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Let $C \subseteq \{N, Fa, Fu\}$. The ∇-spaces in $\mathbf{K}(C)$ together with ∇-space maps form a category. Denote this category by $\mathbf{Space}_\nabla(C)$. If we also denote the category of all ∇-algebras in $\mathcal{V}(D, C)$ together with corresponding algebraic morphisms by $\mathbf{Alg}_\nabla(D, C)$, then:

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Theorem *(Priestley-Esakia duality for distributive ∇-algebras)*

Let $C \subseteq \{N, Fa, Fu\}$. Then, $\mathbf{Alg}_\nabla(D, C) \simeq \mathbf{Space}^{op}_\nabla(C)$.

Thank you for your attention!