The Archimedean Property: New Perspectives

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The Archimedean Property

The Archimedean property is one of the most distinctive and useful features of the field of real numbers; it grounds both the theory of magnitudes and classical, as opposed to nonstandard, analysis.

There have been several attempts to define the concept of an Archimedean algebra for individual classes of algebras of logic, but there is not a general definition that subsumes the existing special cases.

In this talk I propose such a definition and single out a large class in which the Archimedean property implies commutativity.

The Conrad Program

In the 1960s, Paul Conrad launched a general program, aimed at capturing relevant information about ℓ-groups by inquiring into the structure of their lattices of convex ℓ-subgroups. He demonstrated that many significant properties of ℓ-groups are purely lattice-theoretic.

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A natural continuation of Conrad's program consists in extending it from ℓ-groups to residuated lattices (to be defined below). This extended Conrad program has led to promising results in the study of semilinear and Hamiltonian varieties, in the investigation of normal-valued residuated lattices, in the description of projectable objects, the construction of the lateral completion of a residuated lattice, etc.

Substructural Logics and Residuated Lattices

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Substructural logics are non-classical logics that are weaker than classical logic, in the sense that they may lack one or more of the structural rules of contraction, weakening and exchange in their Genzen-style axiomatization. They include many non-classical logics related to computer science (linear logic), linguistics (Lambek Calculus), philosophy (relevant logics), and many-valued reasoning.

Residuated Lattices

A residuated lattice (RL) is an algebra
$\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e} \rangle$ such that:

(i) $\langle A, \wedge, \vee \rangle$ is a lattice;
(ii) $\langle A, \cdot, \mathrm{e} \rangle$ is a monoid; and
(iii) for all $x, y, z \in A$,

$$xy \leq z \iff x \leq z/y \iff y \leq x\backslash z.$$

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An algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e}, \mathrm{f} \rangle$ is said to be a pointed residuated lattice (PRL) provided: (i) $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e} \rangle$ is a residuated lattice; and (ii) f is a distinguished element of A.

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The classes $\mathcal{RL}$ (residuated lattices) and $\mathcal{PRL}$ (pointed residuated lattices) are finitely based varieties.

Lattice-Ordered Groups

A lattice-ordered group (ℓ-group) is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, ^{-1}, \mathrm{e} \rangle$ such that

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(iii) multiplication is isotone.

Defining $z/y := zy^{-1}$ and $x\backslash z = x^{-1}z$, condition (iii) is equivalent to

(iiia) For all $x, y, z \in A$,
$xy \leq z \iff x \leq z/y \iff y \leq x\backslash z$.

Thus any ℓ-group may be viewed as an RL. In fact, it is an RL that satisfies the equation $x(x\backslash \mathrm{e}) \approx \mathrm{e}$.

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- Heyting algebras: $xy \approx x \wedge y$ and $x \wedge \mathrm{f} \approx \mathrm{f}$
- Boolean algebras: $xy \approx x \wedge y$, $(x \to y) \to y \approx x \vee y$ (Relative double negation) and $x \wedge \mathrm{f} \approx \mathrm{f}$
- MV-algebras: $xy \approx yx$, $(x \to y) \to y \approx x \vee y$ and $x \wedge \mathrm{f} \approx \mathrm{f}$
- Pseudo-MV-algebras: $y/(x\backslash y) \approx x \vee y \approx (y/x)\backslash y$ (Non-commutative relative double negation) and $x \wedge \mathrm{f} \approx \mathrm{f}$

Negative Cones

The negative cone $A^- = \{a \in A : a \leq e\}$ of an RL $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e} \rangle$ is the universe of an integral RL $\mathbf{A}^- = \langle A^-, \wedge, \vee, \cdot, \backslash_-, /_-, \mathrm{e} \rangle$ such that for all $a, b \in A^-$, $a \backslash_- b = (a \backslash b) \wedge \mathrm{e}$ and $b /_- a = (b/a) \wedge \mathrm{e}$.

Pertinent Equations

- Commutativity: $xy \approx yx$
- Integrality: $x \backslash e \approx e \approx e / x$
- Divisibility: $x(x \backslash y) \approx x \wedge y \approx (y / x)x$
- Cancellativity: $xy / y \approx x \approx y \backslash yx$
- *e*-Cyclicity: $x \backslash e \approx e / x$
- (Left) Prelinearity: $[(x \backslash y) \wedge e)] \vee [(y \backslash x) \wedge e] \approx e$

A Primer of Residuated Lattices (1)

We write $\mathcal{C}(\mathbf{A})$ for the algebraic closure system of all convex subuniverses of an RL $\mathbf{A}$.

$\mathrm{C}[S]$: the convex subuniverse of $\mathbf{A}$ generated by $S \subseteq A$, as well as the corresponding algebra.

$\mathrm{C}[a] = \mathrm{C}[\{a\}]$ is the principal convex subuniverse of $\mathbf{A}$ generated by $a \in A$.

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If $\mathbf{A}$ is e-cyclic, then $\mathcal{C}(\mathbf{A})$ is a algebraic frame with the FIP whose compact members are the principal convex subuniverses of $\mathbf{A}$.

If, in addition, $\mathbf{A}$ satisfies the prelinearity equation, then the prime (meet-prime) convex subuniverses of $\mathbf{A}$ form a root-system.

A Primer of Residuated Lattices (2)

A convex subuniverse $B \in \mathcal{C}(\mathbf{A})$ is said to be normal if for all $a, b \in A$,

$$(a \backslash b) \wedge e \in B \text{ iff } (b/a) \wedge e \in B.$$

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The congruence relations of **A** are in bijective correspondence with and are determined by the normal convex subuniverses of **A**.

Normal-Valued RLs

An RL $\mathbf{A}$ is said to be normal-valued provided it is e-cyclic and each completely meet-irreducible convex subuniverse $B \in \mathcal{C}(\mathbf{A})$ is normal in its cover $B^{\sharp}$.

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$$|x|^2|y|^2 \leq |y||x|$$

$$((y/x \wedge \mathrm{e})^{\mathrm{n}} \backslash |\mathrm{x}||\mathrm{y}| \wedge \mathrm{e})^2 \leq |x||y|/(x \backslash y \wedge \mathrm{e})^{4\mathrm{n}},$$

$$(|x||y|/(x \backslash y \wedge \mathrm{e})^{\mathrm{n}} \wedge \mathrm{e})^2 \leq (y/x \wedge \mathrm{e})^{4\mathrm{n}} \backslash |\mathrm{x}||\mathrm{y}|,$$

for all $n \in \mathbb{N}$.

The Archimedean Property

An algebraic frame $\mathbf{L}$ with the FIP is said to satisfy the zero radical compact property if for every compact element $c \in L$, the meet of all maximal elements in $\downarrow c$ is the bottom element $\perp$ of $\mathbf{L}$.

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Jorge Martinez (1973) observed that an Abelian ℓ-group is Archimedean if and only if its lattice of convex subuniverses has the zero radical compact property. In this special case, therefore, the Archimedean property is fully captured in the lattices of convex subuniverses of the ℓ-groups in question.

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However, this is no longer true if non-commutative ℓ-groups are considered, as there exist Archimedean ℓ-groups and non-normal-valued ℓ-groups whose lattices of convex subuniverses are isomorphic.

Our Proposal

We define an RL $\mathbf{A}$ to be Archimedean if
(1) it is normal-valued; and
(2) the lattice $\mathcal{C}(\mathbf{A})$ has the zero radical compact property.

Adequacy

Notation:
$|a| = a \wedge e / a \wedge e$ (the absolute value of a)
$a \ll b : a, b \leq e$ and $a < b^n$ for all $n \in \mathbb{N}$
An RL **A** is said to be unital if there exists $u \in A^-$ (called a strong unit of **A**) such that $\mathrm{C}[u] = A$.

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Let **A** be a (non-trivial) e-cyclic, prelinear RL with a strong order-unit $u \in A^-$. Let $\mathrm{Rad}(\mathbf{A})$ be the intersection of the maximal convex subuniverses of **A** (i.e., the values of u).

Then:
(1) $\mathrm{Rad}(\mathbf{A}) \subseteq \{a \in L : u \ll |a|\}$.
(2) If, moreover, **A** is cancellative and the values of u are normal, then $\mathrm{Rad}(\mathbf{A}) = \{a \in A : u \ll |a|\}$.

Commutativity of Archimedean RLs (1)

A non-trivial RL **A** is called strongly simple provided its only subuniverses are A and $\{e\}$.

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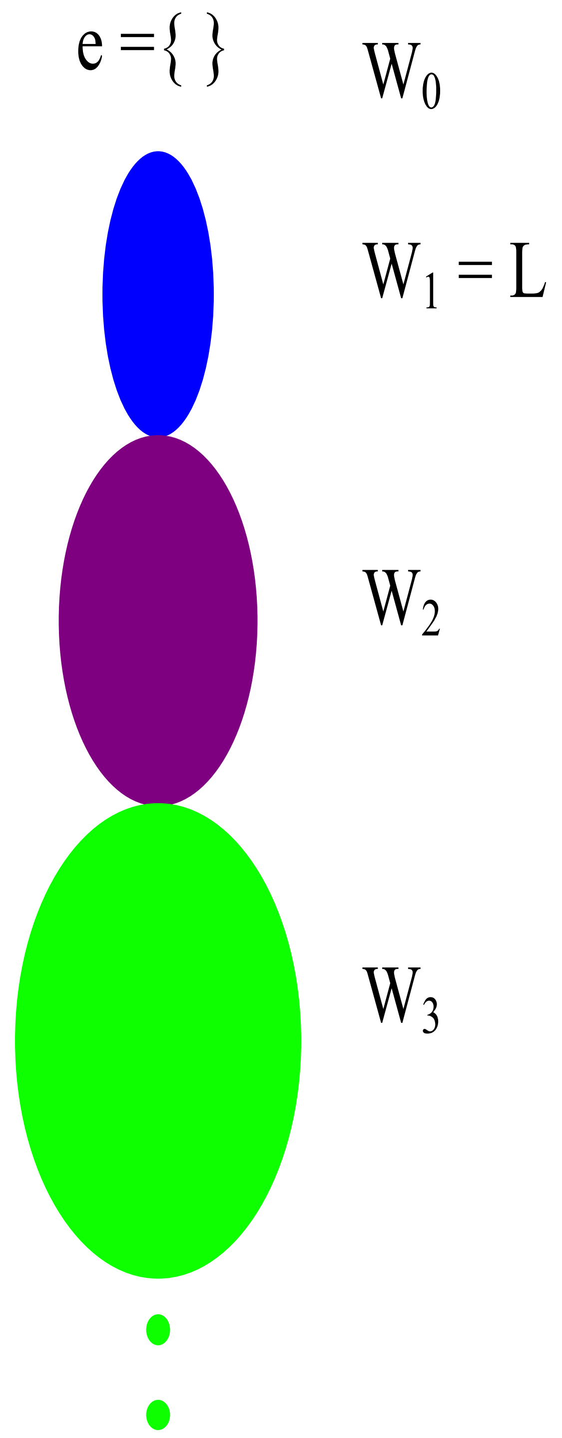
Let **L** be an arbitrary lattice with a top element, and let $\mathbf{L}^*$ be the free monoid over L. We order $\mathbf{L}^*$ as follows:



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Given a variety $\mathcal{V}$ of normal-valued RLs, we seek conditions which will imply that every unital Archimedean member (and hence every Archimedean member) of $\mathcal{V}$ is commutative.

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A simplifying condition is that the variety $\mathcal{V}$ satisfies the prelinearity equation $[(x\backslash y) \wedge e)] \vee [(y\backslash x) \wedge e] \approx e$. Under this condition, the strongly simple RLs in $\mathcal{V}$ are totally ordered.

Commutativity of Archimedean RLs (3)

Further, we assume that $\mathcal{V}$ is a variety of GBL-algebras, that is, it satisfies the following generalizations of the divisibility equations:

- $x[(x\backslash y) \wedge \mathrm{e}] \approx \mathrm{x} \wedge \mathrm{y} \approx [(\mathrm{y}/\mathrm{x}) \wedge \mathrm{e}]\mathrm{x}$

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Now every GBL-algebra **A** is isomorphic to a direct product of an ℓ-group and an integral GBL-algebra. More specifically, **A** is the direct sum of its subalgebras **B** and **C**, where **B** is the ℓ-group of the invertible elements of **A** and **C** is the integral GBL algebra of the integral elements of **A**.

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It follows that a strongly simple RL $\mathbf{A} \in \mathcal{V}$ is either a strongly simple totally ordered ℓ-group or an integral strongly simple totally ordered GBL-algebra.

Commutativity of Archimedean RLs (4)

If the former, then it is (isomorphic to a subalgebra) of $\mathbb{R}$, by Hölder's theorem, and hence is commutative. If the latter, then we use the fact that any totally ordered GBL-algebra is an ordinal sum of pseudo-MV-algebras and negative cones of ℓ-groups. The strong simplicity of **A** implies that it is ordinally indecomposable and hence is a negative cone of an ℓ-group or a pseudo-MV-algebra. Another application of Hölder's theorem implies that **A** is isomorphic to a subalgebra of $\mathbb{R}^-$ or a subalgebra of the MV-algebra $[0, 1]$.

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Theorem: Any Archimedean, prelinear GBL-algebra is commutative. Moreover, every strongly simple totally ordered GBL-algebra is isomorphic to a subalgebra of $\mathbb{R}$, $\mathbb{R}^-$, or the MV-algebra $[0, 1]$.