A Point-Free Version of the Alexandroff-Hausdorff Theorem

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Overview

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- Repeat ...
- *Ad Infinitum*

In a More Pleasant Notation

- Of course the above construction can be summarized by declaring the Cantor set $\mathcal{C}$ as the subset of the closed unit interval
$$[0,1] - \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

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- Equivalently $\mathcal{C}$ consists of real numbers whose ternary expansion contains only zeroes or twos, in other words

$$\mathcal{C} = \left\{ x \in [0,1] \mid x = \sum_{n=0}^{\infty} \frac{b_n}{3^{n+1}}, b_n = 0, 2 \right\}$$

Some Properties of $\mathcal{C}$

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Classical Results

Theorem

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The Cantor set is homeomorphic to a coutable product of copies of itself.

"The" Classical Result

The Alexandroff-Hausdorff Theorem

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The Alexandroff-Hausdorff Theorem

Every compact metric space X is a continuous image of the Cantor set.

Equivalently...

For every compact metric space X there is a continuous onto function $f : \mathcal{C} \to X$

A Particularization

Another Model of $\mathcal{C}$

The set $\mathbb{Z}_2$ of the 2-adic integers is homeomorphic to the Cantor set, so there is continuous map from $\mathbb{Z}_2$ onto $[0, 1]$

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A Map

In fact, the map $\phi : \mathbb{Z}_2 \to [0,1]$ where

$$\phi\left(\sum_{i\geq 0} b_i 2^i\right) = \sum_{i\geq 0} \frac{b_i}{2^{i+1}}.$$

is continuous and onto.

Some Properties of ϕ

Proposition

Let u be an integer coprime to 2, g a nonnegative integer where $0 < u < 2^g$, then $\phi^{-1}(u/2^g)$ has exactly two elements.

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Lemma

For $\phi\colon \mathbb{Z}_2 \to [0,1] \subset \mathbb{R}$ as above, we have

$$\phi^{-1}(-\infty, u/2^g) = \bigcup \left\{ B\left(a, \frac{1}{2^{g+k}}\right) \mid \phi(a) < \frac{u}{2^g} - \frac{1}{2^{g+k}},\ k \geq 0 \right\}$$

$$\phi^{-1}(u/2^g, +\infty) = \bigcup \left\{ B\left(a, \frac{1}{2^{g+k}}\right) \mid \phi(a) > \frac{u}{2^g} + \frac{1}{2^{g+k}},\ k \geq 0 \right\}$$

for every integer u and nonnegative integer g.

A Couple of Free Frames

The Frame of Reals

The frame of the reals $\mathcal{L}(\mathbb{R})$ is generated by $(p, -)$ and $(-, q)$ with $p, q \in \mathbb{D}$, for $\mathbb{D}$ a countably dense subset of $\mathbb{R}$, subject to the following relations.

(1) $(p, -) \wedge (-, q) = 0$ whenever $p \geq q$.

(2) $(p, -) \vee (-, q) = 1$ whenever $p < q$.

(3) $(p, -) = \bigvee\{(r, -) \mid r > p\}$.

(4) $(-, q) = \bigvee\{(-, s) \mid s < q\}$.

(5) $\bigvee\{(p, -) \mid p \in \mathbb{D}\} = 1$

(6) $\bigvee\{(-, q) \mid q \in \mathbb{D}\} = 1$

A Couple of Free Frames (cont.)

The Frame of p-adic Integers

Let $\mathcal{L}(\mathbb{Z}_p)$ be the frame generated by the elements $B_r(a)$, where $a \in \mathbb{Z}$ and $r \in |\mathbb{Z}| := \{p^{-n+1} \mid n \in \mathbb{N}\}$, subject to the following relations:

(1) $B_r(a) \wedge B_s(b) = 0$ whenever $|a-b|_p \geq r$ and $s \leq r$.

(2) $1 = \bigvee \left\{ B_r(a) : a \in \mathbb{Z},\ r \in |\mathbb{Z}| \right\}$.

(3) $B_r(a) = \bigvee \left\{ B_s(b) : |a-b|_p < r,\ s < r,\ b \in \mathbb{Z},\ s \in |\mathbb{Z}| \right\}$.

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Notation

For every generator B of $\mathcal{L}(\mathbb{Z}_2)$, there is a unique natural number n with $B = B_{1/2^n}(a)$ for some 2-adic integer a. For simplicity, we write $B(a, n)$ instead of $B_{1/2^n}(a)$.

A Frame Map

Defining on Generators

Following properties of $\phi : \mathbb{Z}_2 \to [0,1]$, we define $\widetilde{\phi}\colon \mathcal{L}[0,1] \to \mathcal{L}(\mathbb{Z}_2)$ on generators of $\mathcal{L}[0,1]$ where

$$\widetilde{\phi}(-, u/2^g) = \bigvee \left\{ B(a, g+k) \mid \phi(a) < \frac{u}{2^g} - \frac{1}{2^{g+k}} \text{ and } k \geq 0 \right\}$$

$$\widetilde{\phi}(u/2^g, -) = \bigvee \left\{ B(a, g+k) \mid \phi(a) > \frac{u}{2^g} + \frac{1}{2^{g+k}} \text{ and } k \geq 0 \right\}$$

On $\widetilde{\phi}$

Lemma

Let q be a dyadic rational, then $\widetilde{\phi}(-, q) = 1$ if and only if $q > 1$.

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Cor.

For p, q, r, and s dyadic rationals, $\widetilde{\phi}((p, q) \vee (r, s)) = 1$ if and only if $(p, q) \vee (r, s) = 1$.

On $\widetilde{\phi}$ (cont.)

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$\widetilde{\phi}$ is a frame morphism.

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Lemma

If $\widetilde{\phi}(u/2^g, v/2^h) = 0$ then $(u/2^g, v/2^h) = 0$.

On $\widetilde{\phi}$ (cont.)

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If $\widetilde{\phi}(u/2^g, v/2^h) = 0$ then $(u/2^g, v/2^h) = 0$.

Second Conclusion

$\widetilde{\phi}$ is an injective frame morphism.

Coproducts

An Extension

Let $\varphi\colon L \to M$ a dense morphism between compact regular frames then φ induces an injective frame morphism

$$\varphi^{\flat}\colon {}^{\mathcal{K}}L \to {}^{\mathcal{K}}M$$

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Cor.

There is an injective frame morphism

$$\bar{\phi}\colon {}^{\mathcal{K}}\mathcal{L}[0,1] \to {}^{\mathcal{K}}\mathcal{L}(\mathbb{Z}_2)$$

Cantor and Hilbert

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Definition

The *Hilbert cube frame*, $\mathcal{H}$, is ${}^{\mathbb{N}}\mathcal{L}[0,1]$.

Prop.

There is an injective frame morphism $\eta\colon \mathcal{H} \to \mathcal{L}(\mathbb{Z}_2)$

Urysohn and Retracts

Theorem

The following are equivalent for a frame L:

a) L is regular and has a countable basis.

b) L is a quotient of a $\mathcal{H}$.

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Definition

Let L and M frames, we say that L is *retract* of M if, there is a surjective frame morphism $\rho\colon M \to L$ such that there exists a morphism $\iota\colon L \to M$ such that $\rho\iota = id_L$.

On $\mathcal{L}(\mathbb{Z}_2)$

Lemma

Let $b = B(a, n)$ and x be in $\mathcal{L}(\mathbb{Z}_2)$ where b is a basic element and $x < 1$. If $b \not\leq x$, then there is a basic element $b' = B(a', n+1) < b$ such that $b' \not\leq x$.

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Lemma

Let $\mathcal{B}$ the set of basic elements of $\mathcal{L}(\mathbb{Z}_2)$ and $x < 1$, then there is a function $f\colon \mathcal{B} \to \mathcal{B}$ such that $f(b) \not\leq x$ for all $b \in \mathcal{B}$. Moreover, f can be chosen so that f preserves the "radius" of each ball in $\mathcal{B}$.

Closed Quotient

Prop.

Let x be in $\mathcal{L}(\mathbb{Z}_2)$ where $x < 1$, and let $\mathcal{B}$ the set of basic elements of $\mathcal{L}(\mathbb{Z}_2)$. The function $h\colon \mathcal{B} \to \mathcal{L}(\mathbb{Z}_2)$ given by

$$h(b) = \bigvee f^{-1}(\{b\})$$

where f is as above extends to a frame endomorphism H on $\mathcal{L}(\mathbb{Z}_2)$ with $x = \bigvee H^{-1}(\{0\})$.

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Cor.

Every non-trivial closed quotient of $\mathcal{L}(\mathbb{Z}_2)$ is a retraction of $\mathcal{L}(\mathbb{Z}_2)$.

Our Main Result

Theorem

For every compact metrizable frame L there is an embedding

$$\vartheta\colon L \to \mathcal{L}(\mathbb{Z}_2).$$

In other words every compact metrizable frame can be identified as a subframe of the Cantor frame $\mathcal{L}(\mathbb{Z}_2)$.

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- We have a surjective frame morphism $\rho\colon \mathcal{H} \to L$
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- Diagrammatically

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\theta} & \mathcal{L}(\mathbb{Z}_2) \\ {\scriptstyle\rho}\downarrow & & \\ L & & \end{array}$$

Proof (cont.)

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- We thus have an injective morphism $\uparrow \theta(x) \to \mathcal{L}(\mathbb{Z}_2)$

$\mathcal{L}(\mathbb{Z}_p)$ Generated by Words

Many of the above results were inspired by identifying generators as members a language. In fact, the basic elements of $\mathcal{L}(\mathbb{Z}_p)$ can be finite words over an alphabet with p symbols subject to a prefix relation:

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Example

$0011, 0010 \leq 001$, but $00 \not\leq 01$. In fact $00 \wedge 01 = \perp$ and $00 \vee 01 = 0$

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Example

$0011, 0010 \leq 001$, but $00 \not\leq 01$. In fact $00 \wedge 01 = \bot$ and $00 \vee 01 = 0$

So at least, computations of finite suprema and infima are not terribly complicated.

A Concrete Map

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Thank You!