Atoms in the lattice of hull operators on $\mathbf{W}$ and in related lattices

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Let $\mathbf{W}$ be the category of Archimedean ℓ-groups with distinguished weak unit and unit-preserving ℓ-group homomorphisms.

A $\mathbf{W}$-morphism $A \xrightarrow{\sigma} B$ is called *essential* just in case σ is one-to-one and for every $B \xrightarrow{\tau} C$ in $\mathbf{W}$, if $\tau\sigma$ is one-to-one, then τ is one-to-one.

Example

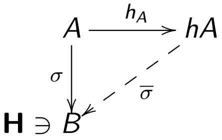
The inclusion $\mathbb{Z} \to \mathbb{R}$ is essential, but the diagonal map $\mathbb{Z} \to \mathbb{R} \times \mathbb{R}$ is not.

Example

Given a $\mathbf{W}$-object A, let E be the Stone space of the complete Boolean algebra of polar subgroups of A, and let $D(E)$ be the $\mathbf{W}$-object consisting of all continuous $f \colon E \to [-\infty, \infty]$ with $f^{-1}(\mathbb{R})$ dense in E. There is an essential $A \xrightarrow{e_A} D(E)$ in $\mathbf{W}$. $eA = D(E)$ is the *maximum essential extension of A*.

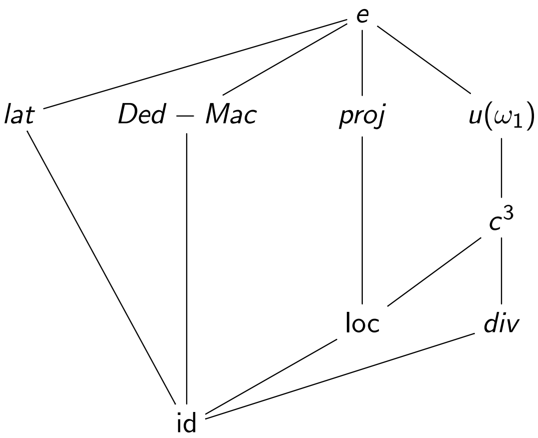
Let $\mathbf{W}^{\#}$ be the category of **W**-objects and essential **W**-morphisms.

Let **H** be a class of **W**-objects closed under the formation of isomorphic copies. We call **H** a *hull class* just in case for every $A \in \mathbf{W}$ there are $hA \in \mathbf{H}$ and $A \xrightarrow{h_A} hA$ in $\mathbf{W}^{\#}$ such that whenever $B \in \mathbf{H}$ and $A \xrightarrow{\sigma} B$ in $\mathbf{W}^{\#}$, there is $hA \xrightarrow{\overline{\sigma}} B$ with $\sigma = \overline{\sigma} h_A$.

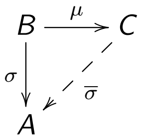


The functor h is called a *hull operator* on **W**. The class of all hull operators on **W** will be denoted **hoW**.

Under the "pointwise" ordering ($h \leq h'$ iff $hG \leq h'G \ \forall G \in \mathbf{W}^{\#}$), **hoW** is a complete lattice with top the maximum essential extension operator e and bottom the identity operator.



We call $B \xrightarrow{\mu} C$ in $\mathbf{W}^{\#}$ a *minimum proper essential extension (mpee)* just in case μ is proper (not an isomorphism) and for every proper $B \xrightarrow{\sigma} A$ in $\mathbf{W}^{\#}$, there is $C \xrightarrow{\overline{\sigma}} A$ in $\mathbf{W}^{\#}$ with $\sigma = \overline{\sigma}\mu$.



An mpee $B \xrightarrow{\mu} C$ is *strong* just in case C is not isomorphic to B. Is every mpee strong? We don't know.

Theorem

If $B \xrightarrow{\mu} C$ is a strong mpee, then $\mathbf{W} \setminus [B]$ is a hull class and the corresponding hull operator $a(\mu)$ is an atom in $\mathbf{hoW}$.

Example

Fix a rational prime p. Let $\mathcal{H}$ be a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$, and choose $r_0 \in \mathcal{H}$ with $r_0 \notin \mathbb{Q}$. Let S be the $\mathbb{Q}$-linear span of $\mathcal{H} \setminus \{r_0\}$ and let

$$G = \left\{ \left(\frac{m}{n}\right) r_0 : \frac{m}{n} \in \mathbb{Q} \text{ and } \gcd(p,n) = 1 \right\}.$$

Finally, let $B_0 = S + G$ and $C_0 = B_0 + \mathbb{Z}\frac{r_0}{p}$. Then the inclusion $B_0 \xrightarrow{\mu_0} C_0$ is a strong mpee.

Example

Suppose $E \in \mathbf{Comp}$ is extremally disconnected and $x_0 \in E$ is isolated. If $B_0 \xrightarrow{\mu_0} C_0$ is as in the previous example, then the inclusion map $B \xrightarrow{\mu} C$ is a strong mpee, where $B = \{f \in D(E) : f(x_0) \in B_0\}$ and $C = \{f \in D(E) : f(x_0) \in C_0\}$.

Example

Suppose $E \in \mathbf{Comp}$ is infinite and extremally disconnected. Let $x \neq y$ be non-isolated points in E, and let $E_{xy} \overset{\gamma}{\leftarrow} E$ be the quotient of E which identifies x and y and no other points. If B is maximal among those $\mathbf{W}$-objects G with $YG = E_{xy}$ and

$$C = \mathrm{jm}(B \circ \gamma + F(E, \mathbb{Z})),$$

then the "inclusion" map $B \overset{\mu}{\to} C$ is a strong mpee. Here $B \circ \gamma$ is the image of B in $D(E)$ under the natural map $f \mapsto f \circ \gamma$ and $F(E, \mathbb{Z})$ consists of all $f \in D(E)$ such that $f(E)$ is a finite subset of $\mathbb{Z}$. Note that $F(E, \mathbb{Z})$ is minimum among $\mathbf{W}$-objects G with $YG = E$.

Theorem

A proper $B \xrightarrow{\mu} C$ *in* $\mathbf{W}^{\#}$ *is an mpee if and only if for every* $f \in C \setminus B$*:*

(i) B *is maximal among the sub-*ℓ*-groups of* eB *not containing* f*.*

(ii) C *is the sub-*ℓ*-group of* eB *generated by* $B \cup \{f\}$*.*

Theorem

If every mpee is strong, then:

(i) a *is an atom of* **hoW** *if and only if* $a = a(\mu)$ *for some mpee* $B \xrightarrow{\mu} C$*.*

(ii) *For every* $h > \mathrm{id}$ *in* **hoW***, there is an atom* $a \in$ **hoW** *with* $a \leq h$*.*

(iii) **hoW** *is not atomic.*

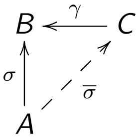
A *cover* $X \xleftarrow{f} Y$ in **Comp** is a continuous surjection which is irreducible (i.e., F proper closed in Y implies $f(F) \neq X$). Let $\mathbf{Comp}^{\#}$ be the category of **Comp**-objects and covers. The notions of *covering class* and *covering operator* are defined in a way that's dual (in the categorical sense) to those of hull class and hull operator.

Example

The absolute or Gleason (projective) cover operator g assigns to X in **Comp** the maximum cover $X \xleftarrow{g_X} gX$, where gX is extremally disconnected.

Let **coComp** be the class of all covering operators with the "pointwise" ordering. Then **coComp** is a complete lattice with top g and bottom the identity operator.

We call $B \overset{\gamma}{\leftarrow} C$ in $\mathbf{Comp}^{\#}$ a *minimum proper cover (mpc)* just in case γ is proper (not a homeomorphism) and for every proper $B \overset{\sigma}{\leftarrow} A$ in $\mathbf{Comp}^{\#}$ there is $C \overset{\overline{\sigma}}{\leftarrow} A$ in **Comp** such that $\sigma = \gamma\overline{\sigma}$:



Theorem

(i) *Every $E_{xy} \overset{\gamma}{\leftarrow} E$ is an mpc.*

(ii) *Every mpc is of the form in (i).*

(iii) $\mathbf{Comp} \setminus [E_{xy}]$ *is a covering class, and so determines a covering operator* $a(\gamma) \in \mathbf{coComp}$.

Theorem

(i) *a is an atom of* **coComp** *if and only if $a = a(\gamma)$ for some mpc $E_{xy} \overset{\gamma}{\leftarrow} E$.*

(ii) *For every $c >$ id in* **coComp**, *there is an atom $a \in$* **coComp** *with $a \leq c$.*

(iii) **coComp** *is not atomic.*

Hager, A.W., Wynne, B.: Atoms in the lattice of covering operators in compact Hausdorff spaces. Topology Appl. **289**, 107402 (2021)