The automorphism group of the Fraïssé limit of finite Heyting algebras

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Automorphism groups of ultrahomogeneous structures I

A structure L is ultrahomogeneous if every isomorphism between substructures of L extends to an automorphism of L.

Example

- $(\mathbb{Q}, <)$.
- The countable atomless Boolean algebra B_∞.
- The countable random distributive lattice D_∞.

Automorphism groups of ultrahomogeneous structures II

For a countable ultrahomogeneous L, one is interested in the Polish group $\mathrm{Aut}(L)$.

Example

- The topology of $\mathrm{Aut}(\mathbb{Q}, <), \mathrm{Aut}(B_\infty), \mathrm{Aut}(D_\infty)$ can be recovered from their abstract group structure (small index property) (Truss 1989; Anderson 1958; Droste & Macpherson 2000).
- $\mathrm{Aut}(B_\infty)$ is simple (Anderson 1958).
- Each of $\mathrm{Aut}(\mathbb{Q}, <), \mathrm{Aut}(D_\infty)$ has exactly 3 proper nontrivial normal subgroups (e.g., Higman 1954; Droste & Macpherson 2000).

New perspective

Many concrete studies of the topological group $\mathrm{Aut}(L')$ for a countable ultrahomogeneous L' involve a uniformly locally finite and thus ω-categorical L'.
My structure L will not be uniformly locally finite, and it might be similar enough to B_∞. It comes from finite Heyting algebras.

The finite Heyting algebras form a Fraïssé class

Theorem (Maksimova 1977)

HA, *the theory of Heyting algebras, has the Amalgamation Property.*

So does $\mathcal{K}$, the class of finite nontrivial Heyting algebras.
$\mathcal{K}$ obviously has the Joint Embedding Property. Therefore it is a Fraïssé class.
There is a unique countable structure L, the Fraïssé limit of $\mathcal{K}$, whose substructures are exactly those in $\mathcal{K}$.

Theorem (Fraïssé 1954)

A countable structure is ultrahomogeneous if and only if it is the Fraïssé limit of some countable Fraïssé class.

L is locally finite, but not uniformly locally finite: there is no bound on the size of 1-generated substructures.
Therefore, L is not ω-categorical.

Why does one care about L?

Theorem (Pitts; Ghilardi & Zawadowski; van Gool & Reggio)

The theory HA *has a model completion. In other words, IPC admits the uniform interpolation: if $p \vdash q$, there is r s.t.:*

- *variables in r occur in both p and q;*
- $p \vdash r \vdash q$,
- *r does not depend on q.*

L is the prime model of the model completion.
L was used to derive an "axiomatization" of the model completion (Darnière 2018).

Is Aut(L) a new topological group?

It would be futile to study Aut(L) if the topological group turned out to be topologically isomorphic to Aut(L') for an ω-categorical Fraïssé limit L'. One might think that that is impossible as:

> *ω-categorical structures are determined up to biïnterpretability by their automorphism groups.*

Recall that the technical statement of that result is:

Fact

If M and N are ω-categorical, then $\mathrm{Aut}(M) \cong \mathrm{Aut}(N) \iff$ *M and N are biïnterpretable.*

A lemma

Lemma (Y.)

Assume the following:

- *L is a countable structure*
- *L is strongly 2-homogeneous (e.g., ultrahomogeneous & QE)*
- $p \in S_1^L(\varnothing)$
- *M is an ω-categorical structure in a countable language*
- *for every $n_0 < \omega$ there exist $m < \omega$ and a set X of m-types over $\varnothing$ realized in L such that for every $q(x_1, \ldots, x_m) \in X$ and $i < m$ we have $p(x_i) \subseteq q(x_1, \ldots, x_m)$ and $f_M(n_0 m) < |X|$.*

where $f_M(n) := |S_n^M(\varnothing)|$. Then $\mathrm{Aut}(L)$ is not topologically isomorphic to $\mathrm{Aut}(M)$.

This can be proved by adapting the aforementioned Fact.

Aut(L) is probably a new topological group

Corollary (Y.)

The topological group Aut(L) *is not realized as the automorphism group of any of the following structures:*

- *the countable atomless Boolean algebra B,*
- *the Fraïssé limit D of finite distributive lattices, or*
- *Fraïssé limits in finite relational languages.*

Moreover, since Aut(B) *or* Aut(D) *have the small index property,* Aut(L) *is not isomorphic to* Aut(B) *or* Aut(D) *as abstract groups.*

Ideas used in the proof

Proof.

1. Th(L), Th(B), and Th(D) eliminate quantifiers.
2. n-types are essentially isomorphism types of n-generated subalgebras.
3. f_D grows asymptotically faster than f_B.
4. Use free constructions in the variety of Gödel algebras to obtain types.
5. Use a result on a specturm of Gödel algebras (Valota 2019) and in enumerative combinatorics (Sklar 1952) to obtain asymptotic dominance.
6. It can be checked the proof works for any such M by the evaluation of f_M (Cameron 1990).

□

Related problem

Problem

Can one make $|X_2|$ *make infinite?*

If so, $\mathrm{Aut}(L)$ is not Roelcke precompact, whence $\mathrm{Aut}(L) \ncong \mathrm{Aut}(M)$ for any ultrahomogeneous ω-categorical M (Rosendal 2009).

An easy fact on $\mathrm{Aut}(L)$

Proposition (Y.)

$\mathrm{Aut}(L)$ *is not locally compact.*

This is true of uniformally locally finite ultrahomogeneous structures.

Amenable topological groups

Definition

For a topological group G, a G-flow is a continuous action on a compactum.

Definition

A topological group G is amenable if every G-flow has a ρ-invariant Borel probability measure.

Our next goal is to study the amenability of $\mathrm{Aut}(L)$.

A Fraïssé class controls the automorphism group of its limit

There are several facts relating the combinatorics of a Fraïssé class $\mathcal{K}'$ and the topological properties of the automorphism group $\mathrm{Aut}(L')$ of its Fraïssé limit L':

Kechris, Pestov, & Todorčević 2005 $\mathcal{K}'$ has the Ramsey property $\implies$ $\mathrm{Aut}(L')$ is extremely amenable, i.e., its universal minimal flow is a singleton.

Kechris & Rosendal 2007 $\mathcal{K}'$ has the Hrushovski property, i.e., for every $A \in \mathcal{K}'$ there is $B \in \mathcal{K}'$ such that every partial isomorphism of A extends to an automorphism of B $\implies$ $\mathrm{Aut}(L')$ is amenable, i.e., every $\mathrm{Aut}(L')$-flow has an invariant Borel probability measure.

$\vdots$

Amenability and combinatorics

Lemma (Kechris & Sokić 2012)

For a Fraïssé class $\mathcal{C}$, $\mathrm{Aut}(\lim_{Fraïssé}\mathcal{C})$ is not amenable if there are:

- *a reasonable order expansion $\mathcal{C}^*$ closed under substructures,*
- *$A, A' \in \mathcal{C}$,*
- *a family of embeddings $\{\iota_< : A \hookrightarrow A' \mid (A, <) \in \mathcal{C}^*\}$*
- *an admissible ordering $<'$ on A'*

s.t.:

1. *For no admissible $<$ on A, $\iota_<$ embeds $(A, <) \hookrightarrow (A', <')$*
2. *For every distinct admissible $<_1, <_2$ on A and every admissible $<''$ on A', $\iota_{<_i}$ fails to embed $(A, <_i) \hookrightarrow (A', <'')$ for one of $i = 1, 2$.*
3. *$\mathrm{Aut}(A')$ acts transitively on the set of admissible orderings on A'.*

A reasonable order expansion of $\mathcal{K}$

Definition

- H : a finite nondegenerate Heyting algebra
- $I(b)$: the set of join-prime elements below or equal to $b \in H$
- $\prec$: any linear extension of the induced partial order on $I(1)$

We define a total order $\prec^{\text{alex}}$ on H extending $\prec$:

$$a \prec^{\text{alex}} a' \iff \max_{\prec}(I(a) \triangle I(a')) \in I(a').$$

We call this a natural ordering on H.
Let $\mathcal{K}^*$ be the class of finite nondegenerate Heyting algebras expanded with natural orderings on them.

Lemma (Y.)

$\mathcal{K}^*$ is a reasonable order expansion of $\mathcal{K}$ closed under substructures.

Aut(L) is not amenable

Corollary (Y.)

Aut(L) *is not amenable.*

Proof.

1. $A := F^{\mathbf{BA}}_{ab}$ and $B := F^{\mathbf{BA}}_{xyz}$.
2. For $<_1$ which extends $a <_1 b$:

$$\pi_{<_1}(a) := x, \qquad \pi_{<_2}(b) := y \vee z$$

3. For $<_2$ which extends $b <_2 a$:

$$\pi_{<_2}(a) := y, \qquad \pi_{<_2}(b) := x \vee z$$

4. Let $<'$ be defined as extending $z <' y <' x$.

These satisfy the hypotheses of the Lemma by Kechris and Sokić (who used these to show the non-amenability of Aut(B_∞)). □

Toward the study of extreme amenability I

Definition

For a topological group G:

1. A G-flow is minimal if it contains no G-invariant compact nonempty proper subset.
2. $M(G)$, the universal minimal flow of G, is the G-flow universal wrt the homomorphisms between minimal G-flows.
3. G is extremely amenable if $M(G)$ is a singleton.

Toward the study of extreme amenability II

Theorem (Kechris-Pestov-Todorčević 2005)

For a Fraïssé class $\mathcal{C}$, TFAE:

1. $\mathrm{Aut}(\lim_{Fraïssé} \mathcal{C})$ *is extremely amenable.*
2. $\mathcal{C}$ *has the* Ramsey property*: for every $k < \omega$ and $A, B \in \mathcal{C}$, there exists $C \in \mathcal{C}$ s.t. for every k-coloring of copies of A in C, there is a copy of B in C in which all copies of A are monochromatic.*

$\mathcal{K}$ can't have this property, but an order expansion of it might.
This is the case with the class of finite Boolean algebras.
If the order extension is reasonable and has the order property, then we can moreover calculate $M(\mathrm{Aut}(\lim_{\mathrm{Fraïssé}}(\mathcal{K})))$.

Recipe for calculating minimal universal flows

For the class $\mathcal{C}$ of finite Boolean algebras or the class $\mathcal{C}$ of finite dimensional vector spaces for a fixed finite field:

1. Consider a reasonable order-forgetful Fraïssé order expansion $\mathcal{C}^*$:

$$(A, <) \cong (A', <') \in \mathcal{C}^* \iff A \cong A'.$$

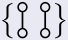
2. It will trivially have the order property.
3. If $\mathcal{C}$ has the Ramsey property, so will $\mathcal{C}^*$. (Kechris-Pestov-Todorčević 2005).

An obstacle to an easy results on extreme amenability

Proposition (Y.)

$\mathcal{K}$ does not have a Fraïssé order expansion that is order-forgetful.

Proof.

1. Consider such an order expansion $\mathcal{K}^*$.
2. $\forall H \in \mathcal{K}$, $\mathrm{Aut}(H)$ acts on the set of binary relations on H.
3. The set of admissible orderings on it is a single orbit.
4. Let H' be the dual of a  b $(a, b \in H')$.
5. Let $H \hookrightarrow H'$ be dual to the p-morphism that collapses a.
6. H is rigid; a, b are conjugates by an automorphism ϕ on H'.
7. Let $\prec$ be admissible on H' with $a \prec b$.
8. $\mathcal{K}^*$ is a Fraïssé class, so $\prec \cap H^2, \prec^\phi \cap H^2$ are admissible on H'.
9. They are distinct as $b \prec^\phi a$.
10. They cannot belong to the same orbit as H is rigid. □

Toward simplicity: Stronger variants of the AP

$\mathcal{K}$ also has the strong amalgamation property.
It does not has the free amalgamation property.
It has the superamalgamation property: for every diagram $A_1 \hookleftarrow A_0 \hookrightarrow A_2$ of inclusion maps in $\mathcal{K}$, the AP of $\mathcal{K}$ is witnessed by a diagram $A_1 \hookrightarrow A \hookleftarrow A_2$ of inclusion maps s.t. $A_1 \downarrow_{A_0} A_2$, where

$$S \underset{U}{\downarrow} T \iff \forall a \in S\, \forall b \in T \begin{Bmatrix} a \leq b & \implies & \exists c \in U\, a \leq c \leq b \\ b \leq a & \implies & \exists c \in U\, b \leq c \leq a \end{Bmatrix}$$

for $S, T, U \subseteq A$.

$$\text{Free AP} \overset{\text{vacuously}}{\implies} \text{Super-AP} \implies \text{Strong AP} \implies \text{AP}$$

The superamalagamation property characterizes propositional logics admitting the interpolation theorem.

Simplicity

Our new goal: to derive the simplicity of $\mathrm{Aut}(L)$ from the super-AP of $\mathcal{K}$.

Theorem (Tent & Ziegler)

If M is countable and has a stationary independence relation, and $g \in \mathrm{Aut}(M)$ moving almost maximally, then any element of $\mathrm{Aut}(M)$ is the product of 16 conjugates of g.

Stationary independence relation

Definition (Tent & Ziegler)

A ternary relation for finite subsets of M is a stationary independence relation if

Invariance Whether $A \mathop{\downarrow}_C B$ or not depends only on the "type" of ABC.

Monotonicity $A \mathop{\downarrow}_B CD \implies A \mathop{\downarrow}_B C,\ A \mathop{\downarrow}_{BC} D$.

Transitivity $A \mathop{\downarrow}_B C,\ A \mathop{\downarrow}_{BC} D \implies A \mathop{\downarrow}_B D$.

Symmetry $A \mathop{\downarrow}_B C \implies C \mathop{\downarrow}_B A$.

Existence If p is a "type" over B, and C is a finite set, then there is $\bar{a} \models p$ s.t. $\bar{a} \mathop{\downarrow}_C B$.

Stationarity If $\bar{a}_0$ and $\bar{a}_1$ have the same "type" over B, and $\bar{a}_i \mathop{\downarrow}_C B$ $(i = 0, 1)$, then $\bar{a}_0$ and $\bar{a}_1$ have the same "type" over BC.

Two ternary relations on subsets of L

1. $A \perp_C B \iff \langle AC \rangle \otimes_{\langle C \rangle} \langle BC \rangle \cong \langle ABC \rangle$.
   This satisfies Invariance, Symmetry, Existence, & Stationarity (but not Monotonicity or Transitivity).
   This is true in general for $\otimes$ witnessing the AP canonically and functorially.
2. $\mathop{\smile\!\!\!\!\!\!\;\;\,|}$ coming from the super-AP.
   This satisfies Invariance, Monotonicity, Transitivity, Symmetry, & Existence (but not Stationarity).

Proofs from Tent & Ziegler

Upon examining proofs in Tent & Ziegler:

Theorem

If M has an "independence" relation satisfying Existence and Stationarity, then $\mathrm{Aut}(M)$ has a dense conjugacy class.

Theorem

Suppose:

- $\mathrm{Aut}(M)$ *has a dense conjugacy class;*
- M *has an "independence" relation satisfying Invariance, Monotonicity, Transitivity, Symmetry, & Existence;*
- $g \in \mathrm{Aut}(M)$ *moves almost maximally.*

Then, any element of $\mathrm{Aut}(M)$ *is the product of* 16 *conjugates of* g.

Automorphisms moving almost maximally I

Definition

$g : M \xrightarrow{\sim} M$ moves almost maximally with respect to $\mathop{\smile\!\!\!\!\!\mid}$

$\iff$

every realized 1-type over a finite set B has a realization a s.t. $a \mathop{\smile\!\!\!\!\!\mid}_B g(a)$.

Lemma (Y.)

- *M countable & ultrahomogeneous*
- Age(M) *has the super-AP*

Then there is $g : M \xrightarrow{\sim} M$ moving almost maximally with respect to $\mathop{\smile\!\!\!\!\!\mid}$ coming from the super-AP.

This is proved by a back-and-forth construction.

Corollary

Corollary (Y.)

Aut(L) *is simple.*

This method appears to be applicable to other classes of lattices with the superamalgamation property.

Open problems

Problem

Is Aut(L) *Roelcke precompact?*

Problem

Does there exist a Fraïssé order expansion of $\mathcal{K}$ *that has the order property?*

Problem

Can the Ramsey property of $\mathcal{K}^*$ *or that of another order expansion of* $\mathcal{K}$ *be proved directly?*

Problem

Identify a pair of a property P *of a logic and a property* P' *of a topological group such that* Aut(L) *has* P' *whenever* L *is the Fraïssé limit of finite (or f.g.) algebraic models of a logic satisfying* P.

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