On the Cantor and Hilbert cube frames

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BLAST 2021

June 2021

The Cantor frame

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(Q$_1$) $B_r(a) \wedge B_s(b) = 0$ whenever $|a-b|_p \geq r$ and $s \leq r$.
(Q$_2$) $1 = \bigvee \left\{B_r(a) : a \in \mathbb{Q},\ r \in |\mathbb{Q}|\right\}$.
(Q$_3$) $B_r(a) = \bigvee \left\{B_s(b) \colon |a-b|_p < r,\ s < r,\ b \in \mathbb{Z},\ s \in |\mathbb{Q}|\right\}$.

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- $\mathcal{L}(\mathbb{Q}_p)$ satisfies the following:
  (1) Zero-dimensional.
  (2) Completely regular.
  (3) Continuous.
  (4) Uniform.

The Cantor frame

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- $\mathcal{B}(\mathbb{Z}) = \{B_r(a) \mid r \in |\mathbb{Z}|, a \in \mathbb{Z}\}$
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Theorem

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$$\mathcal{L}(\mathbb{Z}_p) \cong \uparrow B_p(0)'.$$

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With this $B_p(0)'$ (the complement of $B_p(0)$).
Completely regularity and continuity of $\mathcal{L}(\mathbb{Q}_p)$ gives

$$B_r(a) \prec B_p(0) \ll 1$$

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Proposition

$\mathcal{L}(\mathbb{Z}_p)$ is a compact frame.

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For each natural number n, set

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- $$U_{n+1}B_r(a) = \bigvee\{B_s(b) \mid s = p^{-n-i}, i \in \mathbb{N}, B_s(b) \wedge B_r(a) \neq 0\} \leq B_r(a)$$

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$\lim(X) = \{\text{non-isolated points of } X\}$
$\lim^{\infty}(X) = \{\text{ perfect part of } X\}$
Therefore a space is perfect if

$$\lim(S) = S$$

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Proposition

The frame $\mathcal{L}(\mathbb{Z}_p)$ satisfies

$$cbd_{\mathcal{L}(\mathbb{Z}_p)}(0) = 0.$$

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Theorem

A topological space is a Cantor space if and only if it is non-empty, perfect, compact, totally disconnected, and metrizable.

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Theorem

Let A be a frame. For the prime $p = 2$ we have,

$$A \cong \mathcal{L}(\mathbb{Z}_2)$$

$\Leftrightarrow$ A is zero-dimensional, compact, and metrizable, with $cdb_A(0) = 0$.

The uses of $\mathcal{L}(\mathbb{Z}_p)$

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The frame $\mathcal{L}(\mathbb{R})$ is the free frame of the partial ordered set of open intervals (p, q) with $p, q \in \mathbb{Q}$, modulo some relations (see [PP12] for details) :

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Theorem

The frame of the reals $\mathcal{L}(\mathbb{R})$ is generated by $(p, -)$ and $(-, q)$ with $p, q \in \mathbb{Q}$ subject to the following relations.

(1) $(p, -) \wedge (-, q) = 0$ whenever $p \geq q$.

(2) $(p, -) \vee (-, q) = 1$ whenever $p < q$.

(3) $(p, -) = \bigvee\{(r, -) \mid r > p\}$.

(4) $(-, q) = \bigvee\{(-, s) \mid s < q\}$.

(5) $\bigvee\{(p, -) \mid p \in \mathbb{Q}\} = 1$.

(6) $\bigvee\{(-, q) \mid q \in \mathbb{Q}\} = 1$.

Recall that $\mathcal{L}[0, 1]$ is defined as $\uparrow ((\_, 1) \vee (0, \_))$ in $\mathcal{L}(\mathbb{R})$.

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$$D = \{p/q \mid \text{ with } q = 2^n \text{ for some nonnegative integer } n.\}$$

The uses of $\mathcal{L}(\mathbb{Z}_p)$

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Proposition

There is an injective frame morphism

$$\widetilde{\phi}\colon \mathcal{L}[0,1] \to \mathcal{L}(\mathbb{Z}_2)$$

$$\widetilde{\phi}(-, u/2^g) = \bigvee \left\{ B(a, g+k) \mid \phi(a) < \frac{u}{2^g} - \frac{1}{2^{g+k}} \text{ and } k \geq 0 \right\}$$

$$\widetilde{\phi}(u/2^g, -) = \bigvee \left\{ B(a, g+k) \mid \phi(a) > \frac{u}{2^g} + \frac{1}{2^{g+k}} \text{ and } k \geq 0 \right\}$$

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There is an injective frame morphism (induced by $\widetilde{\phi}$)

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Say $\mathcal{H} = {}^{\mathbb{N}}\mathcal{L}[0,1]$ *The Hilbert cube frame.*

The uses of $\mathcal{L}(\mathbb{Z}_2)$

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As in the topological case we have:

Proposition

The following are equivalent for a frame L:

a) L is regular and has a countable basis.

b) L is a quotient of a $\mathcal{H}$.

Beyond $\mathcal{L}(\mathbb{Z}_2)$

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In recent times we get the definition of $\mathcal{L}(R)$ for R an non-Archimedean ring.

In fact there is a notion of *non-Archimedean frame*

Let A be a frame, we will say A is *non-Achimedean* if is regular and has a basis $\mathcal{B}$ such that for any $a, b \in \mathcal{B}$, then either:

(1) $a \wedge b = 0$ or.

(2) $a \leq b$ or.

(3) $b \leq a$.

Beyond $\mathcal{L}(\mathbb{Z}_2)$

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This notion is totally inspired in the topological setting ([Nyi99]), that is, a topological space S is non-Archimedean if it Hausdorff and has a basis $\mathcal{B}$ such that if $B_1, B_2 \in \mathcal{B}$, then either

(1) $B_1 \cap B_2 = \emptyset$ or.

(2) $B_1 \leq B_2$ or.

(3) $B_2 \leq B_1$.

Beyond $\mathcal{L}(\mathbb{Z}_2)$

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For those spaces the following holds ([Nyi99]):

Theorem

Every non-Archimedean space has a base which is a tree by the reverse inclusion.

Theorem

A space is non-Archimedean if and only if, it can be embedded into the branch space of a tree.

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THANK YOU!