Semilattice ordered algebras

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Semilattice ordered algebras

$\mho$ - the variety of all algebras (A, Ω) of a fixed finitary type.
$\mathcal{V} \subseteq \mho$ - a subvariety of $\mho$

Definition

An algebra $(A, \Omega, +)$ is a **semilattice ordered $\mathcal{V}$-algebra**, if

- $(A, \Omega) \in \mathcal{V}$
- $(A, +)$ is a (join) **semilattice** (with semilattice order $\leqslant$, i.e. $x \leqslant y \ \Leftrightarrow \ x + y = y$)
- the operations from the set Ω **distribute** over the operation $+$ i.e. for each n-ary operation $\omega \in \Omega$, and $x_1, \ldots, x_i, y_i, \ldots, x_n \in A$

$$\omega(x_1, \ldots, x_i + y_i, \ldots, x_n) = \omega(x_1, \ldots, x_i, \ldots, x_n) + \omega(x_1, \ldots, y_i, \ldots, x_n)$$

for any $1 \leqslant i \leqslant n$.

Basic examples

Basic examples are given by

- distributive lattices
- semilattice ordered semigroups: additively idempotent semirings $(S, \cdot, +)$
  1. $(S, \cdot)$ - a semigroup
  2. $(S, +)$ - a semilattice
  3. for $a, b, c \in S$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$
- quantales - semilattice ordered semigroups $(A, \cdot, +)$
  1. $(A, +)$ - complete semilattice
  2. $\cdot$ distributes over arbitrary joins

Basic properties

- For each n-ary operation $\omega \in \Omega$ and $x_{ij} \in A$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant r$

$$\omega(x_{11}, \ldots, x_{n1}) + \ldots + \omega(x_{1r}, \ldots, x_{nr})$$
$$\leqslant \omega(x_{11} + \ldots + x_{1r}, \ldots, x_{n1} + \ldots + x_{nr})$$

- Ω-operations are **monotone** with respect to the semilattice order $\leqslant$
  i.e. if $x_i \leqslant y_i \in A$ for each $1 \leqslant i \leqslant n$, then

$$\omega(x_1, \ldots, x_n) \leqslant \omega(y_1, \ldots, y_n)$$

Modals = semilattice ordered idempotent and entropic algebras

An algebra (M, Ω) is

- **idempotent** if for every n-ary operation $\omega \in \Omega$ and $x \in A$:

$$\omega(x, \dots, x) = x;$$

- **entropic** if for every m-ary $\omega \in \Omega$ and n-ary $\varphi \in \Omega$ operations and $x_{11}, \dots, x_{n1}, \dots, x_{1m}, \dots, x_{nm} \in A$

$$\omega(\varphi(x_{11}, \dots, x_{n1}), \dots, \varphi(x_{1m}, \dots, x_{nm})) =$$
$$\varphi(\omega(x_{11}, \dots, x_{1m}), \dots, \omega(x_{n1}, \dots, x_{nm})).$$

Modes (=idempotent and entropic algebras) and modals:
A. Romanowska, J.D.H. Smith

Entropic modals = semilattice modes: K. Kearnes

One more basic property

Lemma

Let $(A, \Omega, +)$ be a semilattice ordered $\mho$-algebra and let $\omega \in \Omega$ be an $0 \neq n$-ary operation. The algebra $(A, \Omega, +)$ satisfies for any $x \in A$ the condition

$$\omega(x, \ldots, x) \leqslant x,$$

if and only if for any $x_1, \ldots, x_n \in A$ the following holds

$$\omega(x_1, \ldots, x_n) \leqslant x_1 + \ldots + x_n.$$

In particular, if a semilattice ordered $\mho$-algebra $(A, \Omega, +)$ is idempotent then the second condition holds.

Extended power algebras

$\mathcal{P}_{>0}A$ - the family of all non-empty subsets of A

For any n-ary operation $\omega\colon A^n \to A$ we define **the complex operation** $\omega\colon (\mathcal{P}_{>0}A)^n \to \mathcal{P}_{>0}A$:

$$\omega(A_1, \ldots, A_n) := \{\omega(a_1, \ldots, a_n) \mid a_i \in A_i\}.$$

For any algebra $(A, \Omega) \in \mho$:

$$(\mathcal{P}_{>0}A, \Omega, \cup)$$

the **extended power algebra**.

B. Jónsson and A. Tarski proved that complex operations distribute over the union $\cup$.

The algebra $(\mathcal{P}^{<\omega}_{>0}A, \Omega, \cup)$ of all finite non-empty subsets of A is a subalgebra of $(\mathcal{P}_{>0}A, \Omega, \cup)$.

Universality Property for Semilattice Ordered Algebras

Let $\mathcal{V} \subseteq \mho$ and $(F_{\mathcal{V}}(X), \Omega)$ be the free algebra over a set X in the variety $\mathcal{V}$.

$\mathcal{S}_{\mathcal{V}}$ denote the variety of all semilattice ordered $\mathcal{V}$-algebras

Theorem (Universality Property for Semilattice Ordered Algebras)

Let X be an arbitrary set and $(A, \Omega, +) \in \mathcal{S}_{\mathcal{V}}$. Each mapping $h \colon X \to A$ can be extended to a unique homomorphism $\overline{\overline{h}} \colon (\mathcal{P}^{<\omega}_{>0} F_{\mathcal{V}}(X), \Omega, \cup) \to (A, \Omega, +)$ such that $\overline{\overline{h}}|_X = h$.

BUT: $(\mathcal{P}^{<\omega}_{>0} F_{\mathcal{V}}(X), \Omega, \cup)$ needn't belong to $\mathcal{S}_{\mathcal{V}}$

Corollary

Let $(F_{\mho}(X), \Omega)$ be the free algebra over a set X in the variety $\mho$. The extended power algebra $(\mathcal{P}^{<\omega}_{>0} F_{\mho}(X), \Omega, \cup)$ is free over X in the variety $\mathcal{S}_{\mho}$ of all semilattice ordered $\mho$-algebras.

Identities in varieties generated by power algebras

Let $\mathcal{V} \subseteq \mho$ be a variety of algebras and consider

$$\mathcal{V}\Sigma_{>0} := \mathrm{HSP}(\{(\mathcal{P}_{>0}A, \Omega) \mid (A, \Omega) \in \mathcal{V}\}),$$

and its subvariety

$$\mathcal{V}\Sigma_{>0}^{<\omega} := \mathrm{HSP}(\{(\mathcal{P}_{>0}^{<\omega}A, \Omega) \mid (A, \Omega) \in \mathcal{V}\})$$

of power algebras of finite subsets.

In fact, both varieties coincide.

Theorem (G. Grätzer and H. Lakser 1988)

Let $\mathcal{V}$ be a variety of algebras. The variety $\mathcal{V}\Sigma_{>0}$ satisfies precisely those identities which can be obtained from the linear identities true in $\mathcal{V}$ through identification of variables.

Linear identities

A term t of the language of a variety $\mathcal{V}$ is **linear**, if every variable occurs in t at most once. An identity $t \approx u$ is **linear**, if both terms t and u are linear.

The definition of a complex operation extends to each linear derived operation t:

$$t(A_1, \ldots, A_n) := \{t(a_1, \ldots, a_n) \mid a_i \in A_i\}.$$

For a variety $\mathcal{V}$ let $\mathcal{V}^*$ be the variety defined by all linear identities satisfied in $\mathcal{V}$. Obviously, $\mathcal{V}^*$ contains $\mathcal{V}$ as a subvariety.

Summarizing:

- For any subvariety $\mathcal{V} \subseteq \mho$ the algebra $(\mathcal{P}_{>0}^{<\omega} F_{\mathcal{V}}(X), \Omega)$ belongs to $\mathcal{V}^*$, but it does not belong to any of its proper subvarieties.
- $\mathcal{V} = \mathcal{V}\Sigma_{>0}^{<\omega}$ if and only if $\mathcal{V} = \mathcal{V}^*$ if and only if $\mathcal{V}$ is closed under power construction.

Theorem

Let $\mathcal{V}$ be a variety defined by a set of linear identities. The extended power algebra $(\mathcal{P}_{>0}^{<\omega} F_{\mathcal{V}}(X), \Omega, \cup)$ is free over X in the variety $\mathcal{S}_{\mathcal{V}}$ of all semilattice ordered $\mathcal{V}$-algebras.

Let $\mathcal{S}$ be a non-trivial subvariety of $\mathcal{S}_{\mathcal{V}}$ and X be a set.

The congruence

$$\Phi_{\mathcal{S}}(X) := \bigcap\{\phi \in Con(\mathcal{P}^{<\omega}_{>0}F_{\mathcal{V}}(X), \Omega, \cup) \mid (\mathcal{P}^{<\omega}_{>0}F_{\mathcal{V}}(X)/\phi, \Omega, \cup) \in \mathcal{S}\}$$

is the $\mathcal{S}$-**replica congruence** of $(\mathcal{P}^{<\omega}_{>0}F_{\mathcal{V}}(X), \Omega, \cup)$.

$(\mathcal{P}^{<\omega}_{>0}F_{\mathcal{V}}(X)/\Phi_{\mathcal{S}}(X), \Omega, \cup)$ is the $\mathcal{S}$-**replica** of $(\mathcal{P}^{<\omega}_{>0}F_{\mathcal{V}}(X), \Omega, \cup)$.

Theorem

The $\mathcal{S}$-replica of the algebra $(\mathcal{P}^{<\omega}_{>0}F_{\mathcal{V}}(X), \Omega, \cup)$ is free over a set X in the variety $\mathcal{S} \subseteq \mathcal{S}_{\mathcal{V}}$.

Applications to idempotent algebras

Extended power algebras of idempotent algebras are very rarely idempotent.

Remark

If the power algebra $(\mathcal{P}_{>0}A, \Omega)$ *of* (A, Ω) *is idempotent then the algebra* (A, Ω) *must be idempotent too.*

Lemma

The power algebra $(\mathcal{P}_{>0}A, \Omega)$ *of an idempotent algebra* (A, Ω) *is idempotent if and only if each non-empty subset* B *of* A *is a subalgebra of* (A, Ω)*.*

Idempotent replica

(M, Ω) - an idempotent and entropic algebra

$(\mathcal{P}_{>0}^{<\omega} M, \Omega, \cup)$

$\rho \subseteq \mathcal{P}_{>0}^{<\omega} M \times \mathcal{P}_{>0}^{<\omega} M$:

$A \ \rho \ B \quad \Leftrightarrow$ there exist a k-ary term t and an m-ary term s both of type Ω such that $A \subseteq t(B, B, \dots, B)$ and $B \subseteq s(A, A, \dots, A)$

Remark

ρ is the idempotent replica congruence of $(\mathcal{P}_{>0}^{<\omega} M, \Omega, \cup)$

ρ - the idempotent replica congruence of $(\mathcal{P}^{<\omega}_{>0} M, \Omega, \cup)$

Theorem

$$(\mathcal{P}^{<\omega}_{>0} M/\rho, \Omega, \cup) \cong (\{\langle A \rangle : A \in \mathcal{P}^{<\omega}_{>0} M\}, \Omega, +)$$

$\langle A \rangle$ - the subalgebra of (M, Ω) generated by the set A

For each n-ary $\omega \in \Omega$ and $\emptyset \neq A_1, \ldots, A_n \subseteq M$

$$\omega(\langle A_1 \rangle, \ldots, \langle A_n \rangle) = \langle \omega(A_1, \ldots, A_n) \rangle$$

$$\langle A_1 \rangle + \langle A_2 \rangle = \langle A_1 \cup A_2 \rangle$$

Theorem

$\mathcal{M}$ - the variety of all $\mho$-modes

$$(\{\langle A \rangle : A \in \mathcal{P}^{<\omega}_{>0} F_{\mathcal{M}}(X)\}, \Omega, +)$$

is free over a set X in the variety $\mathcal{S}_{\mathcal{M}}$ of all modals.

Varieties generated by modes of submodes

Let $(M, \Omega) \in \mathcal{M}$.

$S(M)$ is the set of all (non-empty) subalgebras of (M, Ω).

Then $(S(M), \Omega) \leqslant (\mathcal{P}_{>0}M, \Omega)$ and $(S(M), \Omega) \in \mathcal{M}$.

Theorem

Let $\mathcal{M}$ be a variety of modes and let the variety $\mathcal{M}\Sigma_{>0}$ be locally finite. The variety

$$\mathcal{M}\mathcal{S} := \mathrm{HSP}(\{(S(M), \Omega) \mid (M, \Omega) \in \mathcal{M}\})$$

satisfies precisely the consequences of the idempotent and linear identities true in $\mathcal{M}$.

Conjecture

An idempotent variety $\mathcal{V}$, in which every algebra has the algebra of subalgebras, coincides with $\mathcal{V}\mathcal{S}$ if and only if $\mathcal{V}$ has a basis consisting of idempotent and linear identities.

Sketch of proof

- Take a mode (M, Ω).
- Make the power algebra (of finite subsets): $(\mathcal{P}^{<\omega}_{>0} M, \Omega)$.
- Extend the language, i.e. add set-theoretical union: $(\mathcal{P}^{<\omega}_{>0} M, \Omega, \cup)$.
- Take the least congruence which give an idempotent factor. Here it is the congruence ρ which glues together subsets which generate the same subalgebras of M.
- Make a factor algebra $(\mathcal{P}^{<\omega}_{>0} M/\rho, \Omega, \cup)$.
- Restrict the language, i.e. forget about $\cup$.
- Consider the variety

$$\mathrm{HSP}(\{(\mathcal{P}^{<\omega}_{>0} M/\rho, \Omega) \mid (M, \Omega) \in \mathcal{M}\}).$$

It coincides with $\mathcal{MS}$. It also satisfies only linear and idempotent identities.

Remarks

Local finiteness is CRUCIAL in the proof!

QUESTION Is the result true in more general case?

HOPE We know examples where the conclusion of the theorem holds, while the variety of power algebras is not locally finite.

For non-idempotent varieties the conjecture is **FALSE**!

Constants in semilattice ordered algebras

Definition

$(A, \Omega, +, 0)$ is called a **0-semilattice ordered $\mathcal{V}$-algebra** if

- $(A, \Omega, +)$ is a semilattice ordered $\mathcal{V}$-algebra
- $(A, +, 0)$ is a semilattice with the least element 0
- for each $\omega \in \Omega$ and $x_1, \ldots, x_i, \ldots, x_n \in A$

$$\omega(x_1, \ldots, x_i, \ldots, x_n) = 0$$

whenever there is $1 \leqslant i \leqslant n$ such that $x_i = 0$

Units in semilattice ordered algebras

Definition

$1 \in A$ is a **unit for an algebra** (A, Ω) if for each operation $\omega \in \Omega$ and every $x \in A$

$$\omega(x, 1, \ldots, 1) = \omega(1, x, 1, \ldots, 1) = \ldots = \omega(1, \ldots, 1, x) = x.$$

$\mathcal{U}_1$ - the variety of algebras $(A, \Omega, 1)$ with the unit 1 such that $(A, \Omega) \in \mathcal{U}$ and $\mathcal{V}_1 \subseteq \mathcal{U}_1$ - a subvariety of $\mathcal{U}_1$

Definition

$(A, \Omega, +, 1)$ is a **semilattice ordered $\mathcal{V}_1$-algebra** if $(A, \Omega, +)$ is a semilattice ordered $\mathcal{V}$-algebra and 1 is a unit for (A, Ω).

Definition

$(A, \Omega, +, 0, 1)$ is a 0-**semilattice ordered $\mathcal{V}_1$-algebra** if $(A, \Omega, +, 0)$ is a 0-semilattice ordered $\mathcal{V}$-algebra and 1 is a unit for (A, Ω).

Extended power algebra of $(A, \Omega, 1)$

$\mathcal{P}A$ - the family of all subsets of A
The complex operation $\omega\colon (\mathcal{P}A)^n \to \mathcal{P}A$ for any $\omega \in \Omega$

$$\omega(A_1, \ldots, A_n) := \{\omega(a_1, \ldots, a_n) \mid a_i \in A_i\}, \text{ if } \emptyset \neq A_1, \ldots, A_n \subseteq A$$

$$\omega(A_1, \ldots, A_n) := \emptyset, \text{ if for some } A_i = \emptyset$$

Example

$(\mathcal{P}_{>0}A, \Omega, \cup, \{1\})$ - a semilattice ordered algebra with the unit $\{1\}$
$(\mathcal{P}A, \Omega, \cup, \emptyset, \{1\})$ - a $\emptyset$-semilattice ordered algebra with the unit $\{1\}$

$(\mathcal{P}A, \cdot, \cup, \emptyset)$ for a semigroup $(A, \cdot)$ - a quantale
$(\mathcal{P}A, \cdot, \cup, \emptyset, \{1\})$ for a monoid $(A, \cdot, 1)$ - a unital quantale

Free SLO algebras with constants

$\mho_1$ - the variety of algebras $(A, \Omega, 1)$ with the unit 1 such that $(A, \Omega) \in \mho$ and $\mathcal{V}_1 \subseteq \mho_1$
$(F_{\mathcal{V}_1}(X), \Omega)$ - the free $\mathcal{V}_1$-algebra over a set X
$\mathcal{S}^0_{\mathcal{V}}$ - the variety of all 0-semilattice ordered $\mathcal{V}$-algebras
$\mathcal{S}_{\mathcal{V}_1}$ - the variety of all semilattice ordered $\mathcal{V}_1$-algebras
$\mathcal{S}^0_{\mathcal{V}_1}$ - the variety of all 0-semilattice ordered $\mathcal{V}_1$-algebras

Theorem

- $(\mathcal{P}^{<\omega} F_{\mathcal{V}}(X), \Omega, \cup, \emptyset)$ *is free over a set* X *in the variety* $\mathcal{S}^0_{\mathcal{V}}$ *if and only if* $(\mathcal{P}^{<\omega} F_{\mathcal{V}}(X), \Omega, \cup, \emptyset) \in \mathcal{S}^0_{\mathcal{V}}$
- $(\mathcal{P}^{<\omega}_{>0} F_{\mathcal{V}_1}(X), \Omega, \cup, \{1\})$ *is free over a set* X *in the variety* $\mathcal{S}_{\mathcal{V}_1}$ *if and only if* $(\mathcal{P}^{<\omega}_{>0} F_{\mathcal{V}_1}(X), \Omega, \cup, \{1\}) \in \mathcal{S}_{\mathcal{V}_1}$
- $(\mathcal{P}^{<\omega} F_{\mathcal{V}_1}(X), \Omega, \cup, \emptyset, \{1\})$ *is free over a set* X *in the variety* $\mathcal{S}^0_{\mathcal{V}_1}$ *if and only if* $(\mathcal{P}^{<\omega} F_{\mathcal{V}_1}(X), \Omega, \cup, \emptyset, \{1\}) \in \mathcal{S}^0_{\mathcal{V}_1}$.

Free algebras in $\mathcal{S}^0_{\mathcal{M}}$

Assuming that

$$\langle \emptyset \rangle = \emptyset$$

Theorem

$\mathcal{M}$ - the variety of all idempotent and entropic Ω-algebras

$$(\{\langle A \rangle \colon A \in \mathcal{P}^{<\omega} F_{\mathcal{M}}(X)\}, \Omega, +, \emptyset)$$

is free over a set X in the variety $\mathcal{S}^0_{\mathcal{M}}$.

$\mathcal{M}_1$ - the variety of all idempotent and entropic Ω-algebras with 1

Assume that $\forall(\omega \in \Omega)$ and $\forall(x_1, \ldots, x_n \in M)$ $(M, \Omega, 1) \in \mathcal{M}_1$ satisfies:

$$\omega(x_1, \ldots, x_n) = 1 \quad \Rightarrow \quad \forall(1 \leqslant i \leqslant n) \quad x_i = 1 \qquad (1)$$

Then

Theorem

- $(\{\langle A\rangle : 1 \notin A \text{ and } A \in \mathcal{P}^{<\omega}_{>0} F_{\mathcal{M}}(X)\} \cup \{\langle A \cup \{1\}\rangle : 1 \notin A \text{ and } A \in \mathcal{P}^{<\omega}_{>0} F_{\mathcal{M}}(X)\} \cup \{\{1\}\}, \Omega, +, \{1\})$ *is free over a set* X *in the variety* $\mathcal{S}_{\mathcal{M}_1}$
- $(\{\langle A\rangle : 1 \notin A \text{ and } A \in \mathcal{P}^{<\omega} F_{\mathcal{M}}(X)\} \cup \{\langle A \cup \{1\}\rangle : 1 \notin A \text{ and } A \in \mathcal{P}^{<\omega} F_{\mathcal{M}}(X)\}, \Omega, +, \emptyset, \{1\})$ *is free over a set* X *in the variety* $\mathcal{S}^0_{\mathcal{M}_1}$

Entropic algebra and a unit

Remark

If there is a unit 1 *in an entropic algebra* (M, Ω) *then the algebra is* ***symmetric****, i.e. for every n-ary operation* $\omega \in \Omega$ *and* $x_1, \ldots, x_n \in M$ *the following identity holds:*

$$\omega(x_1, \ldots, x_n) = \omega(x_{\pi(1)}, \ldots, x_{\pi(n)})$$

for each permutation π *of the set* $\{1, \ldots, n\}$.

Commutative double idempotent semirings $(S, \cdot, +, 0, 1)$

I. Chajda, H. Länger 2018
$(S, +, 0)$ - join semilattice with the least element 0
$(S, \cdot, 1)$ - meet semilattice with the greatest element 1
for each $x \in S$:

$$x \cdot 0 = 0 \cdot x = 0$$

Remark

Commutative double idempotent semirings $(S, \cdot, +, 0, 1)$ *semirings are exactly* 0*-semilattice ordered semilattices with a unit* 1*.*

Commutative double idempotent semirings $(S, \cdot, +, 0, 1)$

$\mathcal{SL}_1$ - be the variety of all semilattices with a unit 1 (monoids)
$(F_{\mathcal{SL}}(X), \cdot)$ - the free semilattice in $\mathcal{SL}$ generated by a set X

- The condition

$$x_1 \cdot x_2 = 1 \quad \Rightarrow \quad x_1 = x_2 = 1$$

  is satisfied in any idempotent monoid
- Commutative semigroups are entropic

Theorem

$$(\{\langle A\rangle : 1 \notin A \in \mathcal{P}^{<\omega} F_{\mathcal{SL}}(X)\} \cup \{\langle A\rangle \cup \{1\} : 1 \notin A \in \mathcal{P}^{<\omega} F_{\mathcal{SL}}(X)\},$$
$$\cdot, +, \emptyset, \{1\})$$

is free over a set X in the variety $\mathcal{S}^0_{\mathcal{SL}_1}$

Remark

X - a finite set

$$|F_{\mathcal{S}^0_{\mathcal{SL}_1}}(X)| = 2|\{(A, \cdot)\colon (A, \cdot) \leqslant F_{\mathcal{SL}}(X)\}|$$

Example

For $X = \emptyset$

$$F_{\mathcal{S}^0_{\mathcal{SL}_1}}(\emptyset) \cong (\{\emptyset, \{1\}\}, \cdot, \cup, \emptyset, \{1\})$$

For $X = \{x\}$

$$F_{\mathcal{S}^0_{\mathcal{SL}_1}}(\{x\}) \cong (\{\emptyset, \{x\}, \{1\}, \{x, 1\}\}, \cdot, \cup, \emptyset, \{1\})$$

Example

$X = \{x, y\}$
The free semilattice $F_{\mathcal{SL}}(X)$ on two generators has three elements:

$$x, y, xy$$

and 7 subalgebras (including the empty set):

$$\emptyset, \{x\}, \{y\}, \{xy\}, \{x, xy\}, \{y, xy\}, \{x, y, xy\}$$

Hence $|F_{\mathcal{S}^0_{\mathcal{SL}_1}}(\{x, y\})| = 14$.

THANK YOU FOR YOUR ATTENTION