DIRECTED PARTIAL ORDERS OVER NON-ARCHIMEDEAN FIELDS

Zhipeng Xu and Yuehui Zhang

Shanghai Jiao Tong University, China

2021.6.10/BLAST(USA)

Birkhoff-Pierce problem: Orders over $\mathbb{C}$?

- 1956, Birkhoff-Pierce problem:

  Does $\mathbb{C}$ admit any lattice orders?
- 1963, L. Fuchs:

  Does $\mathbb{R}$ admit any lattice orders besides the usual one?
- In particular, is 1 always positive in $\mathbb{C}$ or in $\mathbb{R}$?

Is 1 always positive as a complex number?

- So far, none of lattice order on $\mathbb{C}$ is known.
- So: HAVE TO look for partial orders weaker than lattice order on $\mathbb{C}$ over non-archimedean o-fields.

Directed partial Orders (DPO)

- DPO = Directed partial Order: Every two elements a, b enjoy an upper bound and a lower bound.
- A lattice order is a DPO.

DPO – the "BEST" orders on $\mathbb{C}$?

- A positive cone P of a po-ring R is a DPO if and only if it is a partial order and $P - P = R$.
- Theorem(R. DeMarr, A. Steger, 1972).
  $\mathbb{C}$ admits no DPO as an $\mathbb{R}$ algebra.
- DPOs on $\mathbb{C}$ are irrelevant to $\mathbb{R}$.

Archimedean property of $\mathbb{R}$

- Let G be an o-group or an o-ring. Let $x, y \in G^+$. Then y is "infinite relative to x", written as $y \gg x$ (or $x \ll y$) if $nx < y$ for $\forall n > 0$. If $y \gg 1$ ($x \ll 1$), then y is "infinite" (x is "infinitesimal").
- The o-group (o-ring) G is "archimedean" if there is no x, y such that $y \gg x$.

The "BEST" orders on $\mathbb{C}$

- Schwarz-Yang Theorem, 2011.[1]
  The field $\mathbb{C}$ admits at least one DPO which is not a lattice order.
- Existence proof. None DPO on $\mathbb{C}$ is constructed so far.

[1] N. Schwartz, Y. Yang, *Fields with directed partial orders*, J. Algebra **336**(2011)342-348. (2011) 342-348.

Bounded semigroup of a non-archimedean o-field F

- Let F be a non-archimedean o-field. A convex additive semigroup S of F^+ containing 0, 1 is called "bounded" if there exists $z \in F^+$ such that $s \ll z$ for all $s \in S$, written as $S \ll z$.
- Let $\mathcal{P}_{1>0} = \{P \subseteq C \,|\, 1 \in P \textit{ is a PDO and } a + bi \in P \textit{ then } b \geq 0\}$
- $\mathcal{S} = \{S \subseteq F^+ \,|\, S \textit{ is a bounded semigroup}\}$.

DPOs are determined by bounded semigroups

- Let $P \in \mathcal{P}_{1>0}$. Define

$$s(P) = \{x \in F^+ \,|\, x \leq a, \forall a \in F^+ \text{ with } a + i \in P\}.$$

Then $s(P)$ is a bounded semigroup of F^+.

- Let S be a bounded semigroup of F^+. Define

$$p(S) = \{a + bi \mid a \in F^+, 0 \leq sb \leq a \text{ for all } s \in S\}.$$

Then $p(S)$ is a DPO on C.

Classification of DPOs with $1 > 0$

- Theorem[2]. Let F be a non-archimedean o-field and $C = F(i)$. Let P be a DPO on C. Then s is a bijection between $\mathcal{P}_{1>0}$ and $\mathcal{S}$ and $s^{-1} = p$.

[2] J. Ma, L. Wu, Y. Zhang, *Directed Partial Orders on $F(i)$ with $1 > 0$*, Order, 35(2018)3, 461-466.

Special convex subsets

Definition

Let $\emptyset \neq V \subseteq F^+$. Then V is **special convex** if

(1) V is convex in F^+;

(2) $1 \ll V$, namely $1 \ll v$ for any $v \in V$;

(3) If $a \sim b, a < b \in V$, then $a \in V$.

Correspondence

- Let V be a special convex subset of F^+. Define

  $$P_V = F^+\{a+i \mid a \in V\} = \{r(a+i) \mid \forall r \in F^+, \forall a \in V\}.$$

  Then P_V is a DPO of C such that $1 \notin P_V$,
- Let P be a DPO on C such that $1 \notin P$. Define

  $$L(P) = \{a \in F^+ \mid a+i \in P\}.$$

  Then $L(P)$ is a special convex subset of F^+.

DPOs on $C = F(i)$ with $1 \not> 0$

- Theorem[3]. Let F be a non-archimedean o-field and $C = F(i)$. Let P be a DPO on C with $1 \notin P$. Then
  (1) $P = P_V$ for some special convex subset of F^+; or
  (2) $P = (P_V)^*$ for some special convex subset of F^+.

[3] J. Ma, L. Wu, Y. Zhang, *Directed partial orders on the field of generalized complex numbers with* $1 \not> 0$, Positivity, 2019(24)3,1001-1007.

Doubly convex set of non-archimedean o-field

Definition

Let $\emptyset \neq V \subseteq F$. Then V is **doubly convex** if
(1) $V \neq \{0\}$.
(2) $|V| \ll 1$, namely $\forall v \in V, |v| \ll 1$.
(3) $\forall a, b \in V, \forall r, s \in F^+$ and $1 \leq r + s \leq 2$, $ra + sb \in V$.

- Remark. A doubly convex set of F is convex, and in general the inverse is not true.

DPO and Doubly convex sets over $F(i)$

- Theorem[4] Let $\mathcal{V}$ be the set of all doubly convex sets of F. Let $\mathcal{P}$ be the set of all directed partial orders on $C = F(i)$. Then there is a bijection between $\mathcal{V}$ and $\mathcal{P}$.
- Corollary. Suppose V is a doubly convex set of F. Then
  1. If $0 \in V$, then P_V is a DPO on C with $1 \in P_V$.
  2. If $0 \notin V$, then P_V is a DPO on C with $1 \notin P_V$.

[4] Zhipeng XU, Yuehui Zhang, Directed Partial Orders over Non-Archimedea o-Fields, Positivity, 24(2)2020, 1279 - 1291.

$H = F + Fi + Fj + Fk$

- Let $\alpha = (a_1, a_2, a_3), \beta = (b_1, b_2, b_3) \in F^3$. Put $I = (i, j, k)$. Define

$$\alpha \cdot I = a_1 i + a_2 j + a_3 k$$

and

$$\alpha \times \beta = (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2) i$$

$$+ (a_0 b_2 + a_2 b_0 + a_3 b_1 - a_1 b_3) j + (a_0 b_3 + a_3 b_0 + a_1 b_2 - a_2 b_1) k.$$

- Let $q_1 = a_0 + \alpha \cdot I, \quad q_2 = b_0 + \beta \cdot I$. Then

$$q_1 q_2 = a_0 b_0 - \alpha \cdot \beta + (a_0 \beta + b_0 \alpha + \alpha \times \beta) \cdot I.$$

Doubly convex sets over F^3

Definition

A subset $\emptyset \neq V \subseteq F^3$ is **doubly convex** provided
(1) V contains at least 2 linearly independent elements.
(2) $|V| \ll 1$, that is $\forall \alpha = (a_1, a_2, a_3) \in V, |\alpha| \ll 1$.
(3) V is convex, i.e. $\forall \alpha, \beta \in V$, $\forall r, s \in F^+, r + s = 1$, then $r\alpha + s\beta \in V$.
(4) $\forall \alpha, \beta \in V, \dfrac{\alpha + \beta + \alpha \times \beta}{1 - \alpha \cdot \beta} \in V, \dfrac{\alpha + \beta - \alpha \times \beta}{1 - \alpha \cdot \beta} \in V$.

DPO and Doubly convex sets over $H = F + Fi + Fj + Fk$, Xu-Z, 2020

Theorem

Let $\mathcal{V}$ be the set of all doubly convex sets of F^3. Let $\mathcal{P}$ be the set of all directed partial orders on H. Then there is a bijection between $\mathcal{V}$ and $\mathcal{P}$.

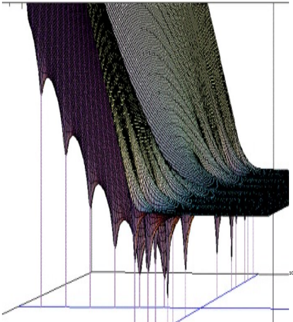
Trivial zeros of Riemann ζ-function

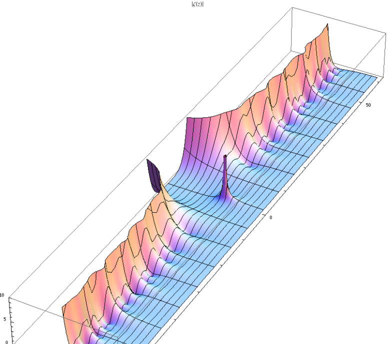
- Riemann Hypothesis, 1859. All nontrivial zeros of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots + \frac{1}{n^s} + \ldots$$

lie on $Re(s) = \frac{1}{2}$.

- What are "trivial zeros"?





Analytic continuation & Riemann function equation

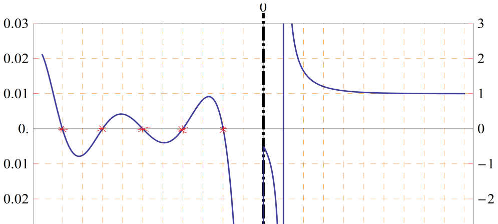
- Riemann function equation.

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{s\pi}{2} \Gamma(1-s)\zeta(1-s), \quad s \neq 0, 1$$

- $\zeta(-1) = -\dfrac{1}{12}, \zeta(-2) = 0$, or informally

$$1 + 2 + 3 + 4 + ... + n + ... = -\frac{1}{12}$$

$$1 + 2^2 + 3^2 + 4^2 + ... + n^2 + ... = 0$$



Ordered structure of $\mathbb{C}$ and trivial zeros of ζ-function

- Unknown the topology of $1^2 + 2^2 + 3^2 + ... + n^2 + ... = 0$.
- The 2-adic topology makes the following

$$2^0 + 2^1 + 2^2 + 2^3 + ... + 2^n + ... = -1$$

since $2^n \to 0(n \to +\infty)$ under 2-adic topology.
- n^2 is not always positive in $\mathbb{C}$, the "positivity disproof" of $1^2 + 2^2 + 3^2 + ... + n^2 + ... = 0$ is incorrect.

References

1. G.Birkhoff, R.S.Pierce, An. Acad. Brasil. Cienc. 28(1956).
2. R.DeMarr, A.Steger, PAMS 31(1972).
3. N.Schwartz, Y.Yang, JA 336(2011).
4. Y.Yang, JA, 2006
5. W.Rump, Y.Yang, JA, 2014
6. MWZ, DPO over NA Fields, Order, 2017
7. MWZ, DPO on $F(i)$ with $1 > 0$, Order, 2018
8. MWZ, DPO on $F(i)$ with $1 \not> 0$, Positivity, 2019
9. XZ, Directed Partial Orders over Non-Archimedean o-Fields, Positivity, 2020

Thank you for your attention!