A topological duality for monotone expansions of semilattices

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(1) Present a topological description of the canonical extension of a semilattice.

(2) Develop a Stone-type duality for the variety of monotone semilattices.

(3) Characterize the congruences of semilattices and monotone semilattices by means of lower-Vietoris-type topologies.

Preliminaries

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- A subset $F \subseteq A$ is a filter if it is increasing, $1 \in F$ and if $a, b \in F$ then $a \wedge b \in F$.

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- A proper filter F is irreducible if for all $F_1, F_2 \in \mathrm{Fi}(\mathrm{A})$, $F = F_1 \cap F_2$ entails $F = F_1$ or $F = F_2$.

$$\mathrm{X}(\mathrm{A}) = \text{set of irreducible filters of A}$$

Preliminaries

Proposition (Celani-2003)

Let $\mathbf{A}$ *be a semilattice. A proper filter* F *is irreducible if and only if for every* $a, b \notin F$*, there exist* $c \notin F$ *and* $f \in F$ *such that*

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Definition (Celani-González)

Let A *be a semilattice and $F \in \mathrm{Fi}(\mathrm{A})$. A subset $I \subseteq A$ is an* ***F-ideal*** *if it is increasing and for every $a, b \in I$, there exist $c \in I$ and $f \in F$ such that $a \wedge f \leq c$ and $b \wedge f \leq c$.*

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Theorem (Irreducible filter)

Let A *a be a semilattice. Let $F \in \mathrm{Fi}(\mathrm{A})$ and I be an F-ideal. If $F \cap I = \emptyset$ then there exists $P \in \mathrm{X}(\mathrm{A})$ such that*

$$F \subseteq P \quad y \quad P \cap I = \emptyset.$$

Preliminaries

Let $\langle X, \mathcal{K} \rangle$ be a topological space where $\mathcal{K}$ is a subbase for a topology $\mathcal{T}_{\mathcal{K}}$. Let us consider

$$S(X) = \{U^c \colon U \in \mathcal{K}\} \quad \text{and} \quad C_{\mathcal{K}}(X) = \left\{ \bigcap A \colon A \subseteq S(X) \right\}.$$

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Definition

Let $\langle X, \mathcal{K} \rangle$ be a topological space. Let $Y \subseteq X$. A family $Z \subseteq S(X)$ is an Y-family if for every $A, B \in Z$ there exist $H, C \in S(X)$ such that

$$Y \subseteq H, \quad C \in Z, \quad A \cap H \subseteq C \quad y \quad B \cap H \subseteq C.$$

Preliminares

Definition (Celani-González)

A topological space $\langle X, \mathcal{K} \rangle$ *is an* *S-space* *if:*

(1) $\langle X, \mathcal{K} \rangle$ *is* T_0 *and* $X = \bigcup \mathcal{K}$,

(2) $\mathcal{K}$ *is a subbase of compact open sets, closed under finite unions and* $\emptyset \in \mathcal{K}$,

(3) *For every* $U, V \in \mathcal{K}$, *if* $x \in U \cap V$ *then there exist* $W, D \in \mathcal{K}$ *such that* $x \notin W$ *and* $x \in D \subseteq (U \cap V) \cup W$,

(4) *If* $Y \in C_{\mathcal{K}}(X)$ *and* $Z \subseteq S(X)$ *is an* Y*-family such that* $Y \cap V^c \neq \emptyset$ *for every* $V \in Z$, *then* $Y \cap \bigcap \{V^c \colon V \in Z\} \neq \emptyset$.

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Proposition

Let $\langle X, \mathcal{K}\rangle$ *be an S-space. Then:*

(1) $\langle S(X), \cap, X\rangle$ *is a semilattice.*

(2) *The posets* $\langle \mathrm{Fi}(S(X)), \subseteq\rangle$ *y* $\langle C_{\mathcal{K}}(X), \subseteq\rangle$ *are dually isomorphic.*

Preliminaries

If A is a semilattice then

$$\langle \mathrm{X}(\mathsf{A}), \mathcal{K}_{\mathsf{A}} \rangle$$

is an S-space, where

$$\mathcal{K}_{\mathsf{A}} = \{\beta(a)^c : a \in A\}$$

is a subbasis for a topology over $\mathrm{X}(\mathsf{A})$ and $\beta\colon A \to \mathrm{Up}(\mathrm{X}(\mathsf{A}))$ is defined by

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Theorem (Celani-González)

Let A *be a semilattice and let* $\langle X, \mathcal{K} \rangle$ *be an* S*-space. Then*

$$\mathsf{A} \cong S(\mathrm{X}(\mathsf{A})) \quad \textit{and} \quad X \cong \mathrm{X}(S(X)).$$

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Let A *be a semilattice. A* completion *of* A *is a pair* $\mathrm{E} = \langle E, e\rangle$ *where* E *is a complete lattice and* $e\colon A \to E$ *is an embedding.*

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- $x \in E$ *is closed if there exists* $F \in \mathrm{Fi}(\mathrm{A})$ *such that* $x = \bigwedge e(F)$. *We write* $K(\mathrm{E}) = \{x \in E \colon x \text{ closed}\}$.

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- *The completion* $\mathbf{E}$ *is dense if for every* $x \in E$*:*

$$x = \bigwedge \{y \in O(\mathbf{E}) \colon x \leq y\} = \bigvee \{y \in K(\mathbf{E}) \colon y \leq x\}.$$

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- *The completion* E *is compact if for every non-empty and dually directed subset* $D \subseteq A$ *and every non-empty and directed subset* $U \subseteq A$ *such that* $\bigwedge e(D) \leq \bigvee e(U)$, *there exist* $x \in D$ *and* $y \in U$ *such that* $x \leq y$.

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Definition

A completion $\mathbf{E}$ *is a* canonical extension *if it is dense and compact.*

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Let $\langle X, \mathcal{K}\rangle$ be an S-space. Let $Z \subseteq X$. Then Z is called subbasic saturated if there exists a dually directed family $L \subseteq \mathcal{K}$ such that

$$Z = \bigcap \{W \colon W \in L\}.$$

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Let A be a semilattice. We call $E(\mathrm{X}(\mathrm{A}))$ to the closure system on $\mathrm{Up}(X(\mathrm{A}))$ generated by $\mathcal{Z}(X(\mathrm{A}))$:

$$E(\mathrm{X}(\mathrm{A})) = \left\{\bigcap\{U^c \colon U \in \mathcal{B}\} \colon \mathcal{B} \subseteq \mathcal{Z}(\mathrm{X}(\mathrm{A}))\right\}.$$

Let us consider $\Lambda \colon \mathrm{Up}(X(\mathrm{A})) \to \mathrm{Up}(X(\mathrm{A}))$ be the closure operator associated with $\mathcal{Z}(X(\mathrm{A}))$. I.e.,

$$\Lambda(Y) = \bigcap\{U^c \colon U \in \mathcal{Z}(X(\mathrm{A})) \text{ and } Y \subseteq U^c\}.$$

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Theorem

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Conversely, if $Z \subseteq \mathrm{X}(\mathrm{A})$ is subbasic saturated then we define $I_{\mathrm{A}} : \mathcal{Z}(\mathrm{X}(\mathrm{A})) \to \mathrm{Id}(\mathrm{A})$ by

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Corollary

$$K(E(\mathrm{X}(\mathrm{A}))) = C_{\mathcal{K}_{\mathrm{A}}}(\mathrm{X}(\mathrm{A})) \quad \textit{and} \quad O(E(\mathrm{X}(\mathrm{A}))) = \{Z^c \colon Z \in \mathcal{Z}(\mathrm{X}(\mathrm{A}))\}.$$

Canonical extension

As usual, A^σ denotes the canonical extension of A. Let A and B be semilattices and $f\colon A \to B$ be a monotone map. Let us consider $f^\sigma, f^\pi\colon \mathrm{A}^\sigma \to \mathrm{B}^\sigma$ given by

$$f^\sigma(u) = \bigvee \left\{ \bigwedge \{f(p)\colon x \leq p \in P\}\colon u \geq x \in K(P^\sigma) \right\},$$

and

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Lema

Let A *and* B *be semilattices. Let* $f\colon A \to B$ *be a monotone map and* $f^\sigma, f^\pi\colon E(\mathrm{X}(\mathrm{A})) \to E(\mathrm{X}(\mathrm{B}))$*. Then* $f^\sigma(\beta_{\mathrm{A}}(a)) = f^\pi(\beta_{\mathrm{A}}(a)) = \beta_{\mathrm{B}}(f(a))$ *for every* $a \in A$*. Moreover, for every* $Y \in C_{\mathcal{K}_{\mathrm{A}}}(\mathrm{X}(\mathrm{A}))$, $Z \in \mathcal{Z}(\mathrm{X}(\mathrm{A}))$ *and* $V \in E(\mathrm{X}(\mathrm{A}))$*:*

(1) $f^\sigma(Y) = \bigcap \{ \beta_{\mathrm{B}}(f(a))\colon a \in A, Y \subseteq \beta_{\mathrm{A}}(a) \}$,

(2) $f^\pi(Z^c) = \bigvee \{ \beta_{\mathrm{B}}(f(a))\colon a \in A, \beta_{\mathrm{A}}(a) \subseteq Z^c \}$,

(3) $f^\sigma(V) = \bigvee \{ f^\sigma(Y)\colon V \supseteq Y \in C_{\mathcal{K}_{\mathrm{A}}}(\mathrm{X}(\mathrm{A})) \}$,

(4) $f^\pi(V) = \bigcap \{ f^\pi(Z^c)\colon V^c \supseteq Z \in \mathcal{Z}(\mathrm{X}(\mathrm{A})) \}$.

Canonical extension

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Let A *and* B *be semilattices. Let* $f\colon A \to B$ *be a monotone map. Then*

$$f^{\pi}(Z^c) = \{P \in \mathrm{X}(\mathsf{B})\colon f^{-1}(P) \cap I_{\mathsf{A}}(Z) \neq \emptyset\}$$

for every $Z \in \mathcal{Z}(\mathrm{X}(\mathsf{A}))$.

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Given $f\colon A \to B$ monotone, we define $R_f \subseteq \mathrm{X}(\mathrm{B}) \times \mathcal{Z}(\mathrm{X}(\mathrm{A}))$ by

$$(P, Z) \in R_f \iff f^{-1}(P) \cap I_{\mathrm{A}}(Z) = \emptyset.$$

Canonical extension

Lema

Let A *and* B *be semilattices. Let* $f\colon A \to B$ *be a monotone map. Then*

$$f^{\pi}(Z^{c}) = \{P \in \mathrm{X}(\mathrm{B})\colon f^{-1}(P) \cap I_{\mathrm{A}}(Z) \neq \emptyset\}$$

for every $Z \in \mathcal{Z}(\mathrm{X}(\mathrm{A}))$.

Given $f\colon A \to B$ monotone, we define $R_f \subseteq \mathrm{X}(\mathrm{B}) \times \mathcal{Z}(\mathrm{X}(\mathrm{A}))$ by

$$(P, Z) \in R_f \iff f^{-1}(P) \cap I_{\mathrm{A}}(Z) = \emptyset.$$

Proposition

Let A *and* B *be semilattices. Let* $f\colon A \to B$ *be a monotone map. Then*

$$f^{\pi}(V) = \{P \in \mathrm{X}(\mathrm{B})\colon \forall Z \in R_f(P)\ (Z \cap V \neq \emptyset)\}$$

for every $V \in E(\mathrm{X}(\mathrm{A}))$.

Canonical extension

Given $f\colon A \to B$ monotone, we define $G_f \subseteq \mathrm{X}(\mathsf{B}) \times \mathcal{C}_{\mathcal{K}_{\mathsf{A}}}(\mathrm{X}(\mathsf{A}))$ by

$$(P, Y) \in G_f \iff \{U \in S(\mathrm{X}(\mathsf{A}))\colon Y \subseteq U\} \subseteq f^{-1}(P)$$

Canonical extension

Given $f: A \to B$ monotone, we define $G_f \subseteq \mathrm{X}(\mathrm{B}) \times \mathcal{C}_{\mathcal{K}_\mathrm{A}}(\mathrm{X}(\mathrm{A}))$ by

$$(P, Y) \in G_f \iff \{U \in S(\mathrm{X}(\mathrm{A}))\colon Y \subseteq U\} \subseteq f^{-1}(P)$$

Proposition

Let A *and* B *be semilattices. Let $f: A \to B$ be a monotone map. Then*

$$f^\sigma(V) = \bigwedge\left(\bigcup\{G_f^{-1}(Y)\colon V \supseteq Y \in C_{\mathcal{K}_\mathrm{A}}(\mathrm{X}(\mathrm{A}))\}\right)$$

for every $V \in E(\mathrm{X}(\mathrm{A}))$.

Topological duality

Definition

A *monotone semilattice* *is a pair* $\langle \mathrm{A}, m\rangle$ *where* A *is a semilattice and* $m\colon A \to A$ *is a monotone map, i.e. if*

$$a \leq b \quad \textit{then} \quad m(a) \leq m(b)$$

for every $a, b \in A$.

Topological duality

Definition

A *monotone semilattice* *is a pair* $\langle \mathrm{A}, m\rangle$ *where* A *is a semilattice and* $m\colon A \to A$ *is a map such that*

$$m(a \wedge b) \leq m(a) \wedge m(b)$$

for every $a, b \in A$.

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Remark

The class of monotone semilattices is a variety.

Topological duality

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for every $a, b \in A$.

Proposition

Let $\langle \mathrm{A}, m\rangle$ *be a monotone semilattice. Then*

$$m^{\pi}(V) = \{P \in \mathrm{X}(\mathrm{A})\colon \forall Z \in R_m(P)\ (Z \cap V \neq \emptyset)\}$$

for every $V \in E(\mathrm{X}(\mathrm{A}))$, *where* $R_m \subseteq \mathrm{X}(\mathrm{A}) \times \mathcal{Z}(\mathrm{X}(\mathrm{A}))$ *is defined by*

$$(P, Z) \in R_m \iff m^{-1}(P) \cap I_{\mathrm{A}}(Z) = \emptyset.$$

Topological duality

Let $\langle X, \mathcal{K}\rangle$ be an S-space. For every $U \in S(X)$ we define $L_U \subseteq \mathcal{Z}(X)$ by

$$L_U = \{Z \in \mathcal{Z}(X) \colon Z \cap U \neq \emptyset\}.$$

Definition

An *mS-space* *is an structure* $\langle X, \mathcal{K}, R\rangle$ *where* $\langle X, \mathcal{K}\rangle$ *is an* S*-space and* $R \subseteq X \times \mathcal{Z}(X)$ *is a relation such that*

(1) *For every* $U \in S(X)$,

$$m_R(U) = \{x \in X \colon \forall Z \in R(x)\ (Z \cap U \neq \emptyset)\} \in S(X).$$

(2) *For every* $x \in X$,

$$R(x) = \bigcap \{L_U \colon U \in S(X), \ \ x \in m_R(U)\}.$$

Topological duality

Let $\langle X, \mathcal{K} \rangle$ be an S-space. For every $U \in S(X)$ we define $L_U \subseteq \mathcal{Z}(X)$ by

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Definition

An mS-space is an structure $\langle X, \mathcal{K}, R \rangle$ where $\langle X, \mathcal{K} \rangle$ is an S-space and $R \subseteq X \times \mathcal{Z}(X)$ is a relation such that

(1) *For every $U \in S(X)$,*

$$m_R(U) = \{x \in X \colon R(x) \subseteq L_U\} \in S(X).$$

(2) *For every $x \in X$,*

$$R(x) = \bigcap \{L_U \colon U \in S(X), \ \ x \in m_R(U)\}.$$

Topological duality

Proposition

If $\langle X, \mathcal{K}, R\rangle$ *an mS-space then*

$$\langle S(X), m_R\rangle$$

is a monotone semilattice.

Proposition

If $\langle \mathrm{A}, m\rangle$ *is a monotone semilattice then*

$$\langle \mathrm{X}(\mathrm{A}), \mathcal{K}_{\mathrm{A}}, R_m\rangle$$

is an mS-space. Moreover, $m_{R_m} = m^{\pi}$.

Topological duality

Let $\langle X_1, \mathcal{K}_1 \rangle$ and $\langle X_2, \mathcal{K}_2 \rangle$ be S-spaces. A $\wedge$-relation is a subset $T \subseteq X_1 \times X_2$ such that:

Topological duality

Let $\langle X_1, \mathcal{K}_1 \rangle$ and $\langle X_2, \mathcal{K}_2 \rangle$ be S-spaces. A $\wedge$-relation is a subset $T \subseteq X_1 \times X_2$ such that:

(1) For every $U \in S(X_2)$,

$$\Box_T(U) = \{x \in X_1 \colon T(x) \subseteq U\} \in S(X_1).$$

Topological duality

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It is clear that $\Box_T \colon S(X_2) \to S(X_1)$ is a homeomorphism between the semilattices $S(X_2)$ and $S(X_1)$.

Topological duality

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It is clear that $\Box_T \colon S(X_2) \to S(X_1)$ is a homeomorphism between the semilattices $S(X_2)$ and $S(X_1)$.
If $h \colon A \to B$ is a homeomorphism then the relation $R_h \subseteq \mathrm{X(B)} \times \mathrm{X(A)}$ defined by

$$(P, Q) \in R_h \iff h^{-1}(P) \subseteq Q$$

is a $\wedge$-relation.

Topological duality

Definition

Let $\langle X_1, \mathcal{K}_1, R_1 \rangle$ and $\langle X_2, \mathcal{K}_2, R_2 \rangle$ be mS-spaces. A $\wedge$-relation $T \subseteq X_1 \times X_2$ is called monotone *if the following diagram commutes:*

$$\begin{array}{ccc} S(X_2) & \xrightarrow{\Box_T} & S(X_1) \\ {\scriptstyle m_{R_2}} \downarrow & & \downarrow {\scriptstyle m_{R_1}} \\ S(X_2) & \xrightarrow[\Box_T]{} & S(X_1) \end{array}$$

Topological duality

Definition

Let $\langle X_1, \mathcal{K}_1, R_1\rangle$ and $\langle X_2, \mathcal{K}_2, R_2\rangle$ be mS-spaces. A $\wedge$-relation $T \subseteq X_1 \times X_2$ is called monotone *if the following diagram commutes:*

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Proposition

Let $\langle \mathrm{A}, m\rangle$ and $\langle \mathrm{B}, n\rangle$ be monotone semilattices. Let $h\colon A \to B$ a homomorphism. Then h satisfies the condition

$$h(m(a)) = n(h(a))$$

for every $a \in A$ if and only if R_h is a monotone $\wedge$-relation, i.e., $\Box_{R_h} m^{\pi} = n^{\pi} \Box_{R_h}$.

Topological duality

So far, we have the following:

$$\mathrm{mSp} \quad = \quad \textsf{\textit{mS}-spaces} \quad + \quad \textsf{monotone } \wedge\textsf{-relations}$$

and

$$\mathrm{SM} \quad = \quad \textsf{monotone semilattices} \quad + \quad \textsf{monotone homomorphisms}$$

Topological duality

So far, we have the following:

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Theorem

The categories mSp *and* SM *are dually equivalent.*

Congruences and lower-Vietoris-type topologies

Definition

Let $\langle X_1, \mathcal{K}_1 \rangle$ and $\langle X_2, \mathcal{K}_2 \rangle$ be S-spaces. Let $R \subseteq X_1 \times X_2$ be a $\wedge$-relation. We say that R is 1-1 if

$$\forall x \in X_1, \forall U \in S(X_1) \text{ such that } x \notin U, \exists V \in S(X_2) \text{ such that } U \subseteq \Box_R(V) \text{ and } x \notin \Box_R(V).$$

Congruences and lower-Vietoris-type topologies

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Theorem

Let $\langle X_1, \mathcal{K}_1 \rangle$ and $\langle X_2, \mathcal{K}_2 \rangle$ be S-spaces. Let $R \subseteq X_1 \times X_2$ be a $\wedge$-relation. Then

$$\Box_R \colon S(X_2) \to S(X_1) \text{ is surjective if and only if } R \text{ is 1-1.}$$

Congruences and lower-Vietoris-type topologies

Definition

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Theorem

Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be S-spaces. Let $R \subseteq X_1 \times X_2$ be a $\wedge$-relation. Then

$$\Box_R \colon S(X_2) \to S(X_1) \text{ is surjective if and only if } R \text{ is 1-1.}$$

Corollary

Let A and B be semilattices. Let $h \colon A \to B$ be a homomorphism. Then

$$h \text{ is surjective if and only if } R_h \subseteq \mathrm{X}(\mathrm{B}) \times \mathrm{X}(\mathrm{A}) \text{ is a } \wedge\text{-relation 1-1.}$$

Congruences and lower-Vietoris-type topologies

Definition

Let X be a set. Let $\mathcal{F}$ be a family of non-empty subsets of X and $\mathcal{O}$ be a topology on $\mathcal{F}$. For each $U \subseteq X$ let us consider

$$U^-_{\mathcal{F}} = \{Y \in \mathcal{F} \colon Y \cap U \neq \emptyset\}.$$

Let $\mathcal{M}_{\mathcal{F}} = \{U^-_{\mathcal{F}} \colon U \subseteq X\}$. We say that $\mathcal{O}$ is a lower-Vietoris-type topology on $\mathcal{F}$ if $\mathcal{O} \cap \mathcal{M}_{\mathcal{F}}$ is a subbase for $\mathcal{O}$.

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Let A be a semilattice and $\langle X, \mathcal{K} \rangle$ be a S-space. Let $R \subseteq X \times \mathbb{X}(\mathrm{A})$ be a $\wedge$-relation 1-1. Let us consider the family $\mathcal{F}_R = \{R(x) \colon x \in X\}$ and for each $a \in A$ let us take

$$H_a = \{R(x) \colon R(x) \cap \beta(a)^c \neq \emptyset\}.$$

Congruences and lower-Vietoris-type topologies

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$$H_a = \{R(x) \colon R(x) \cap \beta(a)^c \neq \emptyset\}.$$

Proposition

The family

$$\mathcal{M} = \{H_a \colon a \in A\}$$

is a subbase for a lower-Vietoris-type topology $\mathcal{T}_{\mathcal{M}}$ over $\mathcal{F}_R$. Moreover, $\langle \mathcal{F}_R, \mathcal{M}\rangle$ is an S-space.

Theorem

Let A *and* B *be semilattices. If* $h\colon A \to B$ *is a surjective homomorphism then* $\langle \mathcal{F}_{R_h}, \mathcal{M} \rangle$ *is an* S*-space homeomorphic to the* S*-space* $\langle \mathrm{X}(\mathrm{B}), \mathcal{K}_{\mathrm{B}} \rangle$.

Congruences and lower-Vietoris-type topologies

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Definition

Let $\langle X, \mathcal{K}\rangle$ *be an* S*-space. A non-empty family* $\mathcal{F} \subseteq C_{\mathcal{K}}(X)$ *is a* *lower-Vietoris-type family* *if the pair* $\langle \mathcal{F}, \mathcal{M}\rangle$ *is an* S*-space, where* $\mathcal{M} = \{U^{-}_{\mathcal{F}} \colon U \in \mathcal{K}\}$. *We write* $\mathcal{V}(X)$ *for the set of lower-Vietoris-type families of* $\langle X, \mathcal{K}\rangle$.

Congruences and lower-Vietoris-type topologies

Theorem

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Definition

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Lema

Let A *a semilattice and* $\mathcal{F} \subseteq C_{\mathcal{K}}(X(A))$ *a lower-Vietoris-type topology. Then the relation* $R_{\mathcal{F}} \subseteq \mathcal{F} \times \mathrm{X}(\mathrm{A})$ *defined by*

$$(Y, P) \in R_{\mathcal{F}} \iff P \in Y$$

is a $\wedge$*-relation 1-1.*

Congruences and lower-Vietoris-type topologies

Lema

Let A *be a semilattice and* $\langle X, \mathcal{K} \rangle$ *be an* S*-space.*

(1) *If* $R \subseteq X \times \mathrm{X}(\mathrm{A})$ *is a* $\wedge$*-relation 1-1 then*

$$(x, P) \in R \iff (R(x), P) \in R_{\mathcal{F}_R}.$$

(2) *If* $\mathcal{F} \subseteq C_{\mathcal{K}}(X)$ *is a lower-Vietoris-type family then* $\mathcal{F} = \mathcal{F}_{R_{\mathcal{F}}}$.

Congruences and lower-Vietoris-type topologies

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The homomorphic images of a semilattice can be characterized through lower-Vietoris-type families.

Theorem

Every semilattice can be represented trough an S*-space associated a lower-Vietoris-type family.*

Congruences and lower-Vietoris-type topologies

Let A be a semilattice, $\theta \in \mathrm{Con}(\mathsf{A})$ and let us take the canonical homomorphism $q_\theta \colon A \to A/\theta$.

Since q_θ is surjective then $R_{q_\theta} \subseteq \mathrm{X}(\mathsf{A}/\theta) \times \mathrm{X}(\mathsf{A})$ is a $\wedge$-relation 1-1.

Therefore,

$$\mathcal{F}_{R_{q_\theta}} = \{R_{q_\theta}(P) \colon P \in \mathrm{X}(\mathsf{A}/\theta)\}$$

is a lower-Vietoris-type family.

Congruences and lower-Vietoris-type topologies

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Therefore,

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is a lower-Vietoris-type family.

Theorem

Let A *be a semilattice and let* $\mathcal{F} \subseteq C_{\mathcal{K}_\mathrm{A}}(\mathrm{X}(\mathrm{A}))$ *be a lower-Vietoris-type family. Then*

$$\theta_{\mathcal{F}} = \{(a, b) \in A^2 \colon [\beta(a)^c]^-_{\mathcal{F}} = [\beta(b)^c]^-_{\mathcal{F}}\}$$

is a congruence of A, *where* $[\beta(a)^c]^-_{\mathcal{F}} = \{Y \in \mathcal{F} \colon Y \cap \beta(a)^c \neq \emptyset\}$. *Moreover,*

$$Y \in \mathcal{F} \iff \exists P \in \mathrm{X}(\mathrm{A}/\theta) \text{ such that } Y = R_{q_{\theta_\mathcal{F}}}(P).$$

Congruences and lower-Vietoris-type topologies

Summarizing,

Congruences and lower-Vietoris-type topologies

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$$\mathcal{F} \rightsquigarrow \theta_{\mathcal{F}} \rightsquigarrow q_{\theta_{\mathcal{F}}} \rightsquigarrow R_{q_{\theta_{\mathcal{F}}}} \rightsquigarrow \mathcal{F}_{R_{q_{\theta_{\mathcal{F}}}}}$$

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Congruences and lower-Vietoris-type topologies

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Analogously,

$$\theta \rightsquigarrow q_{\theta} \rightsquigarrow R_{q_{\theta}} \rightsquigarrow \mathcal{F}_{R_{q_{\theta}}} \rightsquigarrow \theta_{\mathcal{F}_{R_{q_{\theta}}}}$$

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Congruences and lower-Vietoris-type topologies

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and

$$\theta = \theta_{\mathcal{F}_{R_{q_{\theta}}}}$$

Theorem

The congruences of a semilattice can be characterized through lower-Vietoris-type families.

Summarizing,

$$\mathcal{F} \rightsquigarrow \theta_{\mathcal{F}} \rightsquigarrow q_{\theta_{\mathcal{F}}} \rightsquigarrow R_{q_{\theta_{\mathcal{F}}}} \rightsquigarrow \mathcal{F}_{R_{q_{\theta_{\mathcal{F}}}}}$$

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and

$$\theta = \theta_{\mathcal{F}_{R_{q_{\theta}}}}$$

Let $\langle X, \mathcal{K} \rangle$ be an S-space and let $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{V}(X)$.

$$\mathcal{F}_1 \leq \mathcal{F}_2 \Longleftrightarrow \forall U, V \in \mathcal{K}(U^{-}_{\mathcal{F}_2} = V^{-}_{\mathcal{F}_2} \Longrightarrow U^{-}_{\mathcal{F}_1} = V^{-}_{\mathcal{F}_1}).$$

Congruences and lower-Vietoris-type topologies

Summarizing,

$$\mathcal{F} \rightsquigarrow \theta_{\mathcal{F}} \rightsquigarrow q_{\theta_{\mathcal{F}}} \rightsquigarrow R_{q_{\theta_{\mathcal{F}}}} \rightsquigarrow \mathcal{F}_{R_{q_{\theta_{\mathcal{F}}}}}$$

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Analogously,

$$\theta \rightsquigarrow q_{\theta} \rightsquigarrow R_{q_{\theta}} \rightsquigarrow \mathcal{F}_{R_{q_{\theta}}} \rightsquigarrow \theta_{\mathcal{F}_{R_{q_{\theta}}}}$$

and

$$\theta = \theta_{\mathcal{F}_{R_{q_{\theta}}}}$$

Theorem

Let $\langle X, \mathcal{K}\rangle$ be an S-space. Then $\langle \mathcal{V}(X), \leq\rangle$ is a complete lattice which is dually isomorphic to the lattice of all congruences of $\mathsf{S}(X)$.

Definition

Let $\langle X_1, \mathcal{K}_1, R_1\rangle$ and $\langle X_2, \mathcal{K}_2, R_2\rangle$ be mS-spaces. We shall say that a monotone $\wedge$-relation $T \subseteq X_1 \times X_2$ is 1-1 if as a $\wedge$-relation it is 1-1.

Congruences and lower-Vietoris-type topologies

Definition

Let $\langle X_1, \mathcal{K}_1, R_1\rangle$ and $\langle X_2, \mathcal{K}_2, R_2\rangle$ be mS-spaces. We shall say that a monotone $\wedge$-relation $T \subseteq X_1 \times X_2$ is 1-1 if as a $\wedge$-relation it is 1-1.

Theorem

Let $\langle X_1, \mathcal{K}_1, R_1\rangle$ and $\langle X_2, \mathcal{K}_2, R_2\rangle$ be mS-spaces. Let $R \subseteq X_1 \times X_2$ be a monotone $\wedge$-relation. Then, the homomorphism $\Box_R \colon \langle S(X_2), m_{R_2}\rangle \to \langle S(X_1), m_{R_1}\rangle$ is onto if and only if R is 1-1.

Congruences and lower-Vietoris-type topologies

Definition

Let $\langle X_1, \mathcal{K}_1, R_1\rangle$ and $\langle X_2, \mathcal{K}_2, R_2\rangle$ be mS-spaces. We shall say that a monotone $\wedge$-relation $T \subseteq X_1 \times X_2$ is 1-1 if as a $\wedge$-relation it is 1-1.

Theorem

Let $\langle X_1, \mathcal{K}_1, R_1\rangle$ and $\langle X_2, \mathcal{K}_2, R_2\rangle$ be mS-spaces. Let $R \subseteq X_1 \times X_2$ be a monotone $\wedge$-relation. Then, the homomorphism $\Box_R \colon \langle S(X_2), m_{R_2}\rangle \to \langle S(X_1), m_{R_1}\rangle$ is onto if and only if R is 1-1.

Corollary

Let $\langle \mathrm{A}, m\rangle$ and $\langle \mathrm{B}, n\rangle$ be monotone semilattices and $h\colon A \to B$ be a homomorphism. Then, h is onto if and only if $R_h \subseteq \mathrm{X}(\mathrm{B}) \times \mathrm{X}(\mathrm{A})$ is a one-to-one meet-relation.

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Let λ be the homeomorphism between the S-spaces $\langle \mathcal{F}_{R_h}, \mathcal{M} \rangle$ and $\langle X(\mathrm{B}), \mathcal{K}_{\mathrm{B}} \rangle$.

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Corollary

Let $\langle \mathrm{A}, m \rangle$ and $\langle \mathrm{B}, n \rangle$ be monotone semilattices. Let $h \colon \mathrm{A} \to \mathrm{B}$ be an onto homomorphism. Let $T \subseteq \mathcal{F}_{R_h} \times \mathcal{Z}(\mathcal{F}_{R_h})$ be the relation defined as follows:

$$(R_h(P), Z) \in T \Leftrightarrow (P, \lambda^{-1}[Z]).$$

Then $\langle \mathcal{F}_{R_h}, \mathcal{M}, T \rangle$ is an mS-space which is isomorphic to $\langle \mathrm{X}(\mathrm{B}), \mathcal{K}_{\mathrm{B}}, R_m \rangle$.

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Corollary

Every monotone semilattice can be represented as a mS-space associated to a lower-Vietoris-type topology.

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Every monotone semilattice can be represented as a mS-space associated to a lower-Vietoris-type topology.

Corollary

Let $\langle X, \mathcal{K}, R \rangle$ be an mS-space. Then $\langle \mathcal{V}(X), \leq \rangle$ is a complete lattice which is dually isomorphic to the lattice of all congruences of $\mathrm{S}(X)$.

Thanks !

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