A hyperarchimedean ℓ-group not embeddable into any hyperarchimedean ℓ-group with strong unit

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In "Settling a number of questions about hyperarchimedean lattice-ordered groups" Conrad and Martinez present the first example of a hyperarchimedean (h.a.) ℓ-group not embeddable into any Archimedean ℓ-group with strong unit. Hager and Johnson, in "Some comments and examples on generation of (hyper-) archimedean ℓ-groups and f-rings," refine the Conrad-Martinez work and exploit their common technique to produce more examples of h.a. ℓ-groups with special properties related to units. This talk will describe an h.a. ℓ-group, not embeddable into an h.a. ℓ-group with unit, built by a different method. This work should become part of an ongoing project with Hager on conditions under which Archimedean ℓ-groups may be embedded in Archimedean ℓ-groups with strong unit.

First note the definition and some characterizations of h.a. ℓ-groups.

Definition. The ℓ-group G is Archimedean just in case for all $x, y \geq 0$ in G, if $nx \leq y$ for all $n \in \mathbb{N}$, then $x = 0$.

All Archimedean ℓ-groups are Abelian. From now on all ℓ-groups will be Abelian.

Definition. An ℓ-group G is hyperarchimedean just in case all of its homomorphic images are Archimedean.

Every h.a. ℓ-group may be embedded in an ℓ-group $\mathbb{R}^I$.

Theorem. If G is an ℓ-group, the following conditions are equivalent:

(i) G is h.a.;

(ii) every linearly ordered homomorphic image of G is Archimedean;

(iii) for all $a \in G$, $G = G(a) \oplus a^{\perp}$;

(iv) G has an embedding into some $\mathbb{R}^I$; and for every embedding $\varphi : G \to \mathbb{R}^J$ and every $a, b \geq 0$ in G, there is a positive integer n with

$$j \in \text{supp}(\varphi(a)) \Rightarrow n\varphi(a)(j) > \varphi(b)(j).$$

If G is any ℓ-group, let

$$\mathsf{Cp}(G) = \{G(a) : a \in G\}.$$

$(\mathsf{Cp}(G), \subseteq)$ is a distributive lattice. If G is h.a., $(\mathsf{Cp}(G), \subseteq)$ is a generalized Boolean algebra. G.B.a.s correspond to Boolean rings possibly without unit. If the g.B.a. $\mathcal{B}$ corresponds to the Boolean ring R without unit, and a unit is adjoined to R in the usual way to get the Boolean ring S, then B is a maximal ideal of S (and of the corresponding B.a.).

If G is an ℓ-group and $u \in G$, u is a weak unit of G just in case

$$u \geq 0 \text{ and for all } x \in G,\ x \wedge u = 0 \Rightarrow x = 0,$$

and u is a strong unit of G just in case

$$u \geq 0 \text{ and for all } x \in G,\ \text{there is } n \in \mathbb{N} \text{ with } x \leq nu.$$

Every strong unit is a weak unit, but the converse fails in general. In an h.a. ℓ-group, every weak unit is a strong unit.

If G is an ℓ-group and $a \in G$, a is basic in G just in case $a > 0$ and $G(a)$ is linearly ordered. $S \subseteq G$ is a basis of G just in case S consists of pairwise disjoint basic elements and $S^{\perp} = (0)$.

If $0 \leq f, g \in \mathbb{R}^{\mathbb{N}}$, say that

$f \leq^* g$ iff $f(n) \leq g(n)$ for all sufficiently large $n \in \mathbb{N}$.

If $S \subseteq (\mathbb{R}^{\mathbb{N}})^+$, say that S is $*$-bounded just in case there is $g \geq 0$ in $\mathbb{R}^{\mathbb{N}}$ with

$$f \leq^* g \text{ for all } f \in S.$$

If $|S| \leq \aleph_0$, S is $*$-bounded: so

$$\mathfrak{b} = \min\{|S| : S \subseteq (\mathbb{R}^{\mathbb{N}})^+ \text{ is not } *-\text{bounded}\}$$

is uncountable, but of course at most $\mathfrak{c} = 2^{\aleph_0}$.

Viewing $\mathfrak{b}$ an an initial ordinal, one may consider the ℓ-group $H = \mathbb{R}^{\omega\times\mathfrak{b}}$. Let $\{b_\alpha : \alpha < \mathfrak{b}\}$ be a $*$-unbounded subset of $(\mathbb{N}\setminus\{0\})^{\mathbb{N}}$. If $n \in \mathbb{N}$ and $\alpha < \mathfrak{b}$, let $\chi_{n,\alpha} \in H$ be the characteristic function of $\{(n,\alpha)\}$, $u_n \in H$ be the characteristic function of $\{n\} \times \mathfrak{b}$, and $c_\alpha \in H$ be given by

$$c_\alpha(n,\beta) = \begin{cases} b_\alpha(n) & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

(so c_α is a copy of b_α on row α of $\omega \times \mathfrak{b}$). Let G be the sub-ℓ-group of H generated by the $\chi_{n,\alpha}$s, u_ns, and c_αs.

<u>Theorem</u>. G is h.a. but is not embeddable into any h.a. ℓ-group with unit.

Lemma 1. G is h.a. with basis $\{\chi_{n,\alpha} : (n, \alpha) \in \omega \times \mathfrak{b}\}$.

Proof sketch. The remark about the basis is clear. G is h.a. because if $\varphi : G \to K$ is any homomorphism of G onto an ordered group K, then K is Archimedean, as one may show by considering the following cases:

some $\varphi(\chi_{n,\alpha}) \neq 0$;

every $\varphi(\chi_{n,\alpha}) = 0$ but some $\varphi(u_n) \neq 0$;

every $\varphi(\chi_{n,\alpha}) = 0 = \varphi(u_n)$ but some $\varphi(c_\alpha) \neq 0$.

Lemma 2. G may not be embedded into any h.a. ℓ-group with unit.

Proof. Suppose otherwise. By a fundamental result of Conrad-Martinez—later clarified and refined by Hager-Johnson—there is $v \in \mathbb{R}^{\omega \times \mathfrak{b}}$ such that the sub-ℓ-group K of H generated by $G \cup \{v\}$ is h.a. with unit v. Since K and G share the basis $\{\chi_{n,\alpha} : (n, \alpha) \in \omega \times \mathfrak{b}\}$, v is positive everywhere on $\omega \times \mathfrak{b}$. If $n < \omega$, then since K is h.a., there is an integer $s(n) > 0$ such that

$$v(n, \alpha) \leq s(n)u_n(n, \alpha) = s(n) \text{ for every } \alpha < \mathfrak{b}.$$

Let $s = (n \in \mathbb{N} \mapsto s(n)) \in (\mathbb{N} \setminus \{0\})^{\mathbb{N}}$. If $\alpha < \mathfrak{b}$, then since K is h.a. and v is a unit of K, v is a strong unit of K, and there is an integer $k(\alpha) > 0$ with

$$c_\alpha \leq k(\alpha)v.$$

So for all $n \in \mathbb{N}$

$$b_\alpha(n) = c_\alpha(n, \alpha) \leq k(\alpha)v(n, \alpha) \leq k(\alpha)s(n)$$

and $b_\alpha \leq k(\alpha)s$. The at most countable set $\{k(\alpha)s : \alpha < \mathfrak{b}\}$ has a $*$-upper bound $t \in \mathbb{R}^{\mathbb{N}}$, and then $b_\alpha \leq^* t$ always: contradiction.

What is the source of the new example?

Remember that when M is an h.a. ℓ-group, Cp(M) is a generalized Boolean algebra. Bounded Boolean powers of $\mathbb{Q}$ allow one to show that any g.B.a. is isomorphic to a g.B.a. Cp(M) with M an h.a. ℓ-group embeddable in an h.a. ℓ-group with unit.

Definition. The g.B.a. $\mathcal{B}$ is said to be unit-embeddable just in case every h.a. ℓ-group H with Cp$(H) \cong \mathcal{B}$ may be embedded into an h.a. ℓ-group with unit.

So the interesting question here is: are there unit-embeddable g.B.a.s?

Lemma. If $\mathcal{B}$ is disjointly generated—i.e., there is a pairwise disjoint subset of B that generates $\mathcal{B}$ as an ideal—then $\mathcal{B}$ is unit-embeddable.

Note that countably generated $\mathcal{B}$ are disjointly generated, but not conversely in general.

G.B.a.s are the maximal ideals of Boolean algebras, and for any ideal of a Boolean algebra one may ask whether it is disjointly generated as an ideal.

Lemma (Heindorf). Let $\mathcal{P}$ be a Boolean algebra with Stone space X, $\mathcal{I}$ be an ideal of $\mathcal{P}$, and U be the open subset of X corresponding to $\mathcal{I}$.

(i) $\mathcal{I}$ is disjointly generated just in case U is paracompact.

(ii) $\mathcal{I}$ is countably generated just in case U is Lindelöf.

In "Boolean algebras whose ideals are disjointly generated" Heindorf considers "paracompact" Boolean algebras—Boolean algebras, all of whose ideals are disjointly generated—as well as "Lindelöf" Boolean algebras: Boolean algebras, all of whose ideals are countably generated. Let $Z(\omega)$ ($Z(\omega_1)$) be the one-point compactification of the discrete space ω (ω_1), with $p(\omega)$ ($p(\omega_1)$) the point at infinity, and let $p = (p(\omega), p(\omega_1))$. Let $\mathcal{B}(\omega)$ ($\mathcal{B}(\omega_1)$) be the clopen algebra of $Z(\omega)$ ($Z(\omega_1)$). Heindorf points out that though $\mathcal{B}(\omega)$ is Lindelöf and $\mathcal{B}(\omega_1)$ is paracompact, $\mathcal{B}(\omega) * \mathcal{B}(\omega_1)$ is not paracompact, since this is the clopen algebra of $Z = Z(\omega) \times Z(\omega_1)$, and $Z \setminus \{p\}$ is neither normal nor paracompact.

Under CH, $\mathfrak{b} = \omega_1$, and if all the b_αs are strictly increasing, there is an h.a. ℓ-group M, of extended real-valued functions on $Z \setminus \{p\}$, with Cp(M) the ideal corresponding to $Z \setminus \{p\}$ and G resulting from M by removing the last row and column from the domain of every element of M.

Some references—

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