BLAST

9-13 June 2021
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online

Lattices
Universal Algebra
Boolean Algebras
Set Theory
Topology

Volume of abstracts

BLAST is a conference series focusing on **B**oolean Algebras, **L**attices, Algebraic and Quantum logic, Universal **A**lgebra, **S**et Theory, Set-theoretic and Point-free **T**opology. The series is based in the mountain/western/midwest region of the US, and circulates between different universities. The central BLAST web page, with links to past meetings, can be found here:

http://math.colorado.edu/blast/

In 2021, BLAST was held at New Mexico State University. Due to the pandemic, the conference was entirely online.

https://math.nmsu.edu/blast-2021/

This volume contains the papers presented at BLAST 2021. The program includes invited lectures, tutorial lectures, contributed talks, and two special sessions:

- A special session dedicated to the memory of W. Charles Holland (1935-2020) and Jorge Martinez (1945-2020)

  The Black Swamp Problem Book by W. Charles Holland is now available online at:
  https://math.nmsu.edu/blast-2021/black-swamp/BlackSwampBook.pdf

- A special session on Stone and Priestley dualities

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GENERAL SESSION

Tutorials

MINOR IDENTITIES AND PRIMITIVE POSITIVE CONSTRUCTIONS

MANUEL BODIRSKY

Abstract. Primitive positive (pp) constructions, introduced by Barto, Opršal and Pinsker, are a powerful concept to study the complexity of constraint satisfaction problems. They have a beautiful algebraic theory: a finite structure B pp constructs another finite structure A if and only if the polymorphism clone of B has a minor-preserving map to the polymorphism clone of A. Hence, studying finite structures up to pp constructability amounts to studying sets of minor identities of clones, which is of independent interest in universal algebra. In this tutorial I will explain in great detail the basics of pp constructions.

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Tutorial by Alan Dow for Blast 2021

Title: Forcing, Stone spaces, and converging sequences

A number of methods have emerged that have the common theme of producing or avoiding the existence of converging sequences (or ω_1-sequences) in natural classes of Stone spaces. These arose, for example, in the investigations related to still unresolved Efimov problem and problems associated with the Moore-Mrowka problem.

Topics surveyed will include:

(1) constructing Boolean algebras with a tree based recursion (Koszmider's T-algebras)
(2) applications of T-algebra constructions to Efimov's problem and the Moore-Mrowka problem
(3) forcing copies of the ordinal space ω_1 in a Stone space [Balogh] with proper posets having elementary submodels as side-conditions [Todorcevic]
(4) forcing converging ω_1-sequences via Luzin gaps
(5) connections to the minimum and maximum character of the Stone space.

Some topics in Domain Theory

Achim Jung

This tutorial will discuss Dana Scott's domains, starting from his 1969 manuscript "A type theoretic alternative to ISWIM, CUCH, and OWHY". We will look at the challenge of denotational semantics of programming languages and the role that domains play in it. We will review the quest for a "convenient category of domains" and meet some of the problems that have been open for over 25 years. Domains and domain-like structures feature prominently in Stone duality, and this will also be a topic for this course.

Invited talks

CLONOIDS

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Abstract. While the functions in a clone are closed under arbitrary compositions, a number of weaker closure properties have been studied by many authors including, e.g., Couceiro, Foldes, Harnau, Lehtonen, and Pippenger. In 2014, P. Mayr and the author introduced *clonoids*; a clonoid is a set of finitary functions from a set A into an algebra $\mathbf{B}$ that is closed under taking minors, and under the basic operations of $\mathbf{B}$. The proofs of the following results use clonoids: Every subvariety of a finitely generated variety with cube term is finitely generated (Aichinger, Mayr 2014). There are infinitely many not finitely generated clones on $\mathbb{Z}_p \times \mathbb{Z}_p$ containing $+$ (Kreinecker 2020). A finite abelian group has finitely many term-inequivalent expansions if and only if it is of squarefree order (Fioravanti 2020). In addition, clonoids have provided a new proof of a theorem by A. Pinus that on a finite set there are only finitely many algebraic geometries that are closed under union (Aichinger, Rossi, Sparks 2020).

We will discuss these results and state open problems that involve clones and clonoids.

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Esakia's theorem in the monadic setting

Luca Carai

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The Gödel translation allows us to translate fully and faithfully the intuitionistic propositional calculus IPC into the propositional modal logic S4. We say that a normal extension M of S4 is a modal companion of a superintuitionistic logic L if L can be translated fully and faithfully into M via the Gödel translation. Esakia's theorem states that the largest modal companion of IPC is the Grzegorczyk modal logic Grz. In this talk I will present the challenges to extend this result to the monadic setting and how to overcome them. I will also discuss how similar obstacles appear when we try to extend the Blok-Esakia theorem to the monadic setting.

This is based on joint work with Guram Bezhanishvili.

Bitopological duality for some subordination Boolean algebras including compingent and de Vries algebras

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A **subordination algebra** is a pair $(A, \prec)$ where A is a Boolean algebra and $\prec$ is a binary relation defined on A called *proximity* or *subordination* relation, satisfying the following conditions:

(S1) $0 \prec 0$ and $1 \prec 1$,

(S2) $a \prec b, c \Rightarrow a \prec b \wedge c$,

(S3) $a, b \prec c \Rightarrow a \vee b \prec c$,

(S4) $a \leq b \prec c \leq d \Rightarrow a \prec d$.

Subordination algebras are equivalent to the quasi-modal algebras introduced in [2] and the precontact algebras defined in [4]. Subordination algebras and contact algebras come from different sources. One of the sources is the de Vries duality [3]. The relations on arbitrary Boolean algebras that satisfy the conditions of the definition of de Vries proximity relations are known as compingent relations. Deleting some of the conditions of the definition we have the subordination relations of [1].

More precisely, we will say that a subordination algebra $(A, \prec)$ is a $S4$**-subordination** algebra or **quasi-topological** modal algebra if it satisfies the additional conditions:

(S5) $a \prec b \Rightarrow a \leq d$.

(S6) $a \prec b \Rightarrow$ there exists $c \in A$ $(a \prec c \prec b)$.

A $S5$**-subordination** algebra or **quasi-monadic** algebra is a $S4$-subordination algebra $(A, \prec)$ satisfying the conditions
(S7) $a \prec b \Rightarrow \neg b \prec \neg a$.

A **compingent** algebra is $S5$-subordination algebra $(A, \prec)$ satisfying

(S8) $a \neq 0 \Rightarrow \exists b \neq 0\ (b \prec a)$.

Finally, a de **Vries algebra** $(A, \prec)$ is a **complete** compingent algebra.

For the class of de Vries algebras several representations and dualities are known. In 1962 [3] H. de Vries showed that the category **DeV** of de Vries algebras is dually equivalent to the category **KHaus** of compact Hausdorff spaces. It is also known that the category **KRFrm** of compact regular frames is dually equivalent to **KHaus** [5]. On the other hand, in [1] it was proved that the category **Gle** of Gleason spaces is dually equivalent to the category **Dev**. This last duality is a "modal-like" duality, because the Gleason spaces are extremally disconnected Stone spaces endowed with a binary relation satisfying adequate conditions.

In this talk we present a bitopological representation for the class of $S4$-subordination algebras. In particular, we obtain a simple representation for the classes of $S5$-subordination algebras, compingent algebras and de Vries algebras. Considering different types of maps between $S4$-subordination algebras we will obtain different categories of $S4$-subordination algebras. We will see that each of these categories is dual to a suitable category of bitopological spaces.

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SPECTRA AND DEFINABILITY

VERA FISCHER

In this talk, we will consider two aspects in the study of extremal sets of reals, sets like maximal families of eventually different functions, maximal cofinitary groups, or maximal independent families. On one side, we will discuss their spectra, defined as the set of cardinalities of such families and on the other, the existence of witnesses of optimal projective complexity. We will emphasize recent developments in the area and indicate interesting remaining open questions.

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▸ MAI GEHRKE, *Preserving joins at primes.*

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URL Address: `https://math.unice.fr/ mgehrke/`. This is a talk on topological duality for additional operations on Boolean algebras and distributive lattices as applied to various topics related to logic.

Many connectives in logic preserve or reverse at least one of conjunction and disjunction in each input coordinate. This property for additional operations on Boolean algebras and lattices is key to the Jònsson-Tarski duality [9]. Preserving or reversing *both* is more special but, for join and meet, this is distributivity (clearly an important property), and in many-valued logics, where semantics are given by chains, this is the case for any monotone or antitone operation [7, 8, 3]. A slightly weaker (non-equational) property is that of preserving joins (or meets) at primes. This property plays a central rôle in two applications of topological duality in computer science: domain theory [1, 2] and automata theory [5, 4].

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Katětov order

Michael Hrusak

We shall discuss recent developments concerning the Katětov order on Borel ideals with applications to classification of ultrafilters and MAD families on countable sets.

Dualities and logical aspects of Baire functions

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In this talk we discuss the algebraic geometry of Baire functions via Łukasiewicz logic, one of the most studied many-valued logics. More precisely, we will work on *Riesz MV-algebras*. These algebras stand to MV-algebras (the algebraic semantics of Łukasiewicz logic) like vector spaces over $\mathbb{R}$ stand to groups, see [4].

Riesz MV-algebras have proven to be a quite interesting and useful class of algebras in themselves. They provide the semantics for a conservative expansion of Łukasiewicz logic, but more importantly they are successful in overcoming some of the limitations of MV-algebras.

In this presentation we will not deal with the totality of Riesz MV-algebras. Indeed, our starting point will be the class of algebras considered in [2]: the infinitary variety of σ-complete Riesz MV-algebras, denoted by $\mathbf{RMV}_\sigma$. These algebras have a quite concrete representation, since they can be thought as intervals of some Banach lattices. A Banach lattice is a lattice-ordered vector space over $\mathbb{R}$ which is norm-complete. In the special case of a vector space with a strong order unit, one can consider the class of those Banach lattices that are complete with respect to a norm that is induced by the strong unit. These spaces turned out to be all $\mathbb{R}$-valued algebras of continuous functions over compact and Hausdorff topological spaces. Any Dedekind σ-complete lattice-ordered vector space with a strong order unit is norm-complete, and therefore, when in addition one requires suitable topological properties, these algebras of functions turn out to be equivalent "à la Mundici" to σ-complete Riesz MV-algebras.

The focus of this presentation is to discuss the infinitary variety $\mathbf{RMV}_\sigma$ from two points of view: description of its free objects and categorical dualities.

After introducing the needed preliminary notions, we dive into characterizing the free objects of $\mathbf{RMV}_\sigma$ in the case of an arbitrary set of generators. We denote by $IRL(X)$ the algebra of term functions $p : [0,1]^X \to [0,1]$ in the language of σ-complete Riesz MV-algebras. By the results of [2] (mainly the fact that $\mathbf{RMV}_\sigma$ is the infinitary variety generated by $[0,1]$) and standard results in universal algebra, it is easily seen that $IRL(X)$ is the Lindenbaum-Tarski algebra of $\mathcal{IRL}$ build upon $|X|$-propositional variables, and it is also isomorphic with the free $|X|$-generated σ-complete Riesz MV-algebra. We call its elements IRL-polynomials.

Moreover, if we consider the hypercube $[0,1]^X$ endowed with its natural product topology, by Baire set we mean a subset of the σ-algebra generated by the zeroset of the continuous functions $f : [0,1]^X \to [0,1]$. In symbols, B is a Baire set if, and only if, there exists a continuous $f : [0,1]^X \to [0,1]$ such that $B = f^{-1}(\{0\})$. A function $p : [0,1]^X \to [0,1]$ is Baire-measurable if the preimage of a Baire set of $[0,1]$ is a Baire set of $[0,1]^X$.

With these definitions, the following holds.

Theorem 1. *For an arbitrary non-empty set X, $IRL(X)$ is the algebra of all $[0,1]$-valued and Baire-measurable functions defined over $[0,1]^X$.*

After having obtained this characterization of the free algebra in $\mathbf{RMV}_\sigma$, we will focus on obtaining a categorical duality for a special class of σ-complete Riesz MV-algebras, that is the one of σ-semisimple σ-complete Riesz MV-algebras.

Definition 2. An algebra $A \in \mathbf{RMV}_\sigma$ is called σ-semisimple if

$$\bigcap \{M \mid M \in Max(A) \cap Id_\sigma(A)\} = \{0\},$$

where $Id_\sigma(A)$ is the set of MV-ideals of A that are closed under countable suprema, and $Max(A)$ is the set of all maximal MV-ideals of A.

For the content of this duality, following [1], our category of σ-complete Riesz MV-algebras (denoted by $\mathbf{RMV}_\sigma^\mathbf{p}$) will have as objects *presented* algebras, that is, pairs $(IRL(X), I)$ such that $IRL(X)/I \in \mathbf{RMV}_\sigma$. Moreover, we shall consider the category **Hyper** whose objects are subsets of hypercubes $[0,1]^X$, for an arbitrary X. Note that arrows between objects of **Hyper** are tuples of functions in the free object $IRL(X)$ for a suitable X.

Theorem 3. *The full subcategory* $\mathbf{ssRMV}_\sigma^\mathbf{p}$ *of* $\mathbf{RMV}_\sigma^\mathbf{p}$ *whose objects are σ-semisimple algebras, and the full subcategory* **IRL** *of* **Hyper** *whose objects are intersections of Baire sets, are equivalent.*

Furthermore, the duality can be restricted to finitely presented objects, as follows.

Corollary 4. *The duality restricts between the category* $\mathbf{RMV}_\sigma^\mathbf{fp}$ *of finitely presented σ-complete RMV-algebras and full subcategory* **Baire** *of* **Hyper** *whose objects are Baire subsets of finite-dimensional hypercubes.*

Finally, we shall see how the fundamental Stone-Krein-Kakutani-Yosida duality is translated in this setting. Indeed, any σ-complete Riesz MV-algebra A is isomorphic to the algebra $C(X)$, where $X = Max(A)$ is a basically disconnected compact Hausdorff space. This result is made into a duality in [3], where morphisms on topological spaces are defined as follows: A function $f : X \to Y$ between compact Hausdorff and basically disconnected spaces is *cozero-closed* if for every countable union U of clopens of Y, we have $f^{-1}(\overline{U}) = \overline{f^{-1}(U)}$. Then, we have the following.

Proposition 5. *The algebraic category* $\mathbf{RMV}_\sigma$ *is dual to the category* **BDKH** *whose objects are basically disconnected, compact, Hausdorff spaces and whose morphisms are continuous and cozero-closed functions.*

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SEMILATTICE ORDERED ALGEBRAS

ANNA ZAMOJSKA-DZIENIO

Let $\mho$ be the variety of all algebras (A, Ω) of a (fixed) finitary type $\tau\colon \Omega \to \mathbb{N}^+$ and let $\mathcal{V} \subseteq \mho$ be a subvariety of $\mho$.

An algebra $(A, \Omega, +)$ is called a ***semilattice ordered $\mathcal{V}$-algebra*** (or briefly ***semilattice ordered algebra***, i.e. SLO algebra) if

- (A, Ω) belongs to a variety $\mathcal{V}$,
- $(A, +)$ is a (join) semilattice (with semilattice order $\leq$, i.e. $x \leq y \ \Leftrightarrow\ x + y = y$),
- the operations from the set Ω distribute over the operation $+$, i.e. for each n-ary operation $\omega \in \Omega$, and $x_1, \ldots, x_i, y_i, \ldots, x_n \in A$

$$\omega(x_1, \ldots, x_i + y_i, \ldots, x_n) = \omega(x_1, \ldots, x_i, \ldots, x_n) + \omega(x_1, \ldots, y_i, \ldots, x_n)$$

  for any $1 \leq i \leq n$.

The above definition can be also formulated for semilattice ordered algebras with constants. Such constants may be of two types. The first one may consist of some special elements in the semilattice $(A, +)$ and the second one may refer to the algebra $(A, \Omega) \in \mathcal{V}$. In particular, we can consider semilattice algebras with neutral element with respect to the operation $+$ or with unit elements with respect to operations in Ω.

The basic role in the theory is played by extended power algebras of (non-empty) subsets and extended algebras of (non-empty) subalgebras.

The aim of this talk is to discuss the properties of semilattice ordered $\mathcal{V}$-algebras, mainly in relation to the variety $\mathcal{V}$ we started from. Among the others, we describe free objects in an arbitrary variety $\mathcal{S}$ of semilattice ordered algebras and in the quasivariety of Ω-subreducts of SLO algebras in $\mathcal{S}$. We investigate the identities satisfied by SLO algebras. In each case, we study the relations between the SLO algebras with the signatures including and excluding constants. We apply the results to some particular (idempotent) varieties of algebras.

Presented results were obtained together with Agata Pilitowska [1]–[6].

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Contributed talks

Quasivarieties of commutative residuated lattices

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The class of all subquasivarieties of a given quasivariety V is a lattice under inclusion, called the **lattice of subquasivarieties** of Q and denoted by $\Lambda_q(\mathsf{Q})$.

Lattices of subquasivarieties are in general very complex. A quasivariety Q is **Q-universal** if for any other quasivariety Q' of finite type, $\Lambda_q(\mathsf{Q}')$ is a homomorphic image of a sublattice of $\Lambda_q(\mathsf{Q})$.

Lemma 0.1. *For every Q-universal quasivariety* Q

- *the free lattice on* ω *generators is embeddable in* $\Lambda_q(\mathsf{Q})$*;*
- $|\Lambda_q(\mathsf{Q})| = 2^{\aleph}_0$.

So the lattice of subquasivarieties of a Q-universal quasivariety is horribly complex and unfortunately Q-universal quasivarieties are ubiquitous.

On the other end of the spectrum there are the primitive quasivarieties. A quasivariety Q is **primitive** if every subquasivariety of Q is equational relative to Q. Clearly primitivity is downward hereditary and a variety V is primitive if and only if every subquasivariety of V is a variety.

An algebra $\mathbf{A}$ is **weakly projective** in a class K of algebras (of the same type) if for all $\mathbf{B} \in \mathsf{K}$, if $\mathbf{A} \in \mathbf{H}(\mathbf{B})$ then $\mathbf{A}$ is embeddable in $\mathbf{B}$. We denote by $[\mathsf{K} : \mathbf{A}] = \{\mathbf{B} \in \mathsf{K} : \mathbf{A} \text{ is not embeddable in } \mathbf{B}\}$.

Theorem 0.2. *(Gorbunov1998) Let* Q *a locally finite quasivariety of finite type. Then the following are equivalent:*

1. Q *is primitive;*
2. *if* $\mathbf{A}$ *is a finite relative subdirectly irreducible algebra in* Q*, then* $\mathsf{Q} : \mathbf{A}$ *is a quasivariety that is equational relative to* Q*;*
3. *every finite relative subdirectly irreducible algebra* $\mathbf{A} \in \mathsf{Q}$ *is weakly projective in* Q*;*
4. *every finite relative subdirectly irreducible algebra* $\mathbf{A} \in \mathsf{Q}$ *is weakly projective in the class of finite members of* Q*.*

Moreover if Q *is primitive, then* $\Lambda_q(\mathsf{Q})$ *is a distributive lattice.*

This talk is a very preliminary report of an ongoing investigation. We will mainly discuss quasivarieties of Wajsberg hoops and basic hoops, where some results can be obtained using well-established techniques.

On The Networks of Large Embeddings

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Let K be a class of similar algebras. We define the *network* $\mathcal{N}(\mathrm{K})$ *of large embeddings in* K to be the graph (possibly infinite) whose vertices are the members of K, and which has two types of edges: red dashed edges connecting the different isomorphic members of K, and blue edges connecting any two algebras of K if they are not isomorphic and one of them is a large subalgebra of the other one. A large subalgebra of an algebra $\mathfrak{B}$ is a proper subalgebra that needs only one extra element to generate the whole $\mathfrak{B}$. Here is an example: Figure 1 illustrates the network of large embeddings in the class of all subalgebras of an atomic Boolean algebra whose atoms are a, b and c.

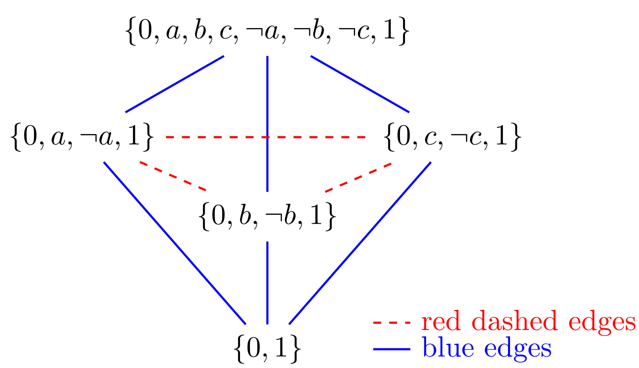

Figure 1: A small network of Boolean algebras

Investigating networks of large embeddings in general is thought-provoking and may lead to quite interesting results, e.g., [4] and [5]. We introduce a notion of distance that conceivably counts the minimum number of "dissimilarities" between two given structures in K. Formally, this distance counts the minimum number of blue edges among all paths connecting the given structures in the network $\mathcal{N}(\mathrm{K})$; with the possibility that this distance may take the value ∞. It turns out that this distance notion can have potential applications in various disciplines. For instance, one can see [1], [2] and [3] for applications in the class of Lindenbaum–Tarski algebras of first-order theories.

In this talk, we take an exciting tour between the networks of large embeddings and we shed light on the interesting distance notion mentioned above. We give examples from different areas of

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algebra, e.g., group theory, lattice theory, theory of Boolean algebras, theory of monounary algebras, etc. We also point up some connections with other notions from universal algebra.

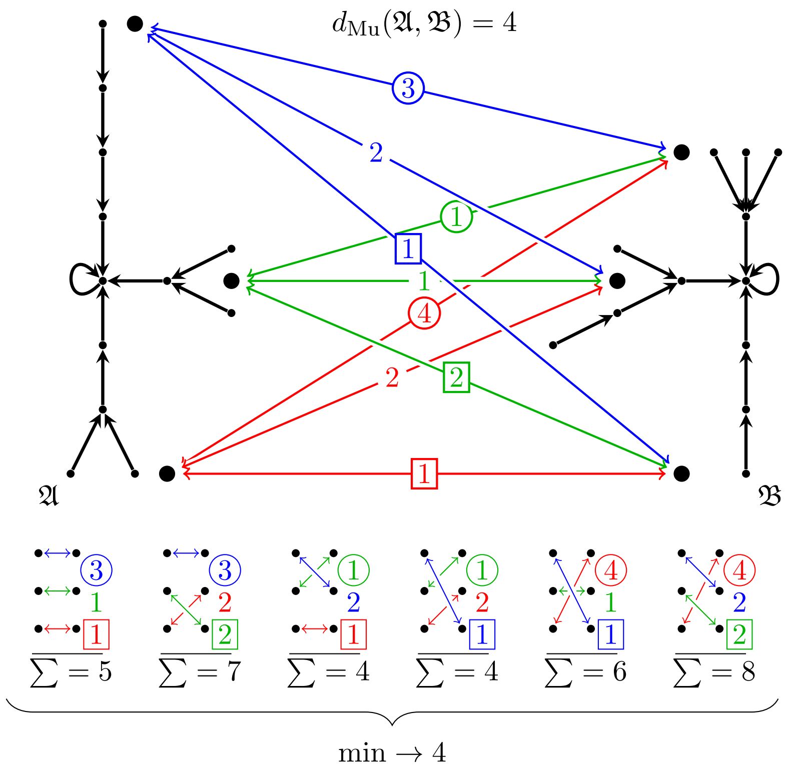

Figure 2: Illustration of calculating the distance between two finite monounary algebras

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A Point-Free Version of the Alexandroff-Hausdorff Theorem

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ABSTRACT. The Alexandroff-Hausdorff Theorem states that every compact metric space is a continuous image of the Cantor set. With the introduction of the Cantor frame $\mathcal{L}(\mathbb{Z}_2)$, we give a point-free version of that theorem: every compact metric frame embeds in $\mathcal{L}(\mathbb{Z}_2)$.

To this end, we introduce an equivalent description of the frame of the real numbers, then we define an injective morphism from $\mathcal{L}[0,1]$ to $\mathcal{L}(\mathbb{Z}_2)$ which is lifted to an injective morphism in their coproducts. Later, we show that every countable coproduct of $\mathcal{L}(\mathbb{Z}_2)$ is isomorphic to $\mathcal{L}(\mathbb{Z}_2)$, and we use the frame theoretic version of the Urysohn's metrization theorem to see that each metric compact frame is a closed quotient of the Hilbert cube frame. We then revise the notion of retraction in the point-free setting and note that every non-trivial closed quotient of $\mathcal{L}(\mathbb{Z}_2)$ is a retract of itself.

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On the Cantor and the Hilbert cube frames

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ABSTRACT. Frames (locales, complete Heyting algebras) are complete lattices $(L, \leq \bigvee, \wedge, \top, \bot)$ such that the following distributivity holds:

$$a \wedge (\bigvee X) = \bigvee \{a \wedge x \mid x \in X\}$$

for each $a \in L$ and $X \subseteq L$.

Classical examples of frames are topologies, and there are many other non-topological examples Banaschewski introduces the frame of the real numbers $\mathcal{L}(\mathbb{R})$ generated by the poset of ordered pairs (p, q) with $p, q \in \mathbb{Q}$. This constructive approach has led to many interesting situations e.g. continuous real-valued functions in the point-free setting; and the characterization of real-complete normal frames (see [PP12, XIV.8] for details).

The frame of p-adic numbers $\mathcal{L}(\mathbb{Q}_p)$ has been introduce in [Ávi20], a modification of this construction leads to the frame of the p-adic integers $\mathcal{L}(\mathbb{Z}_p)$, in particular we define $\mathcal{L}(\mathbb{Z}_2)$ as the Cantor frame.

In this talk we explore some properties of the frame $\mathcal{L}(\mathbb{Z}_p)$ and we provide a characterization for the Cantor frame $\mathcal{L}(\mathbb{Z}_2)$.

In addition, we review some properties of coproducts of compact regular frames and then we introduce the Hilbert cube frame $\mathcal{H}$ as the coproduct ${}^{\mathbb{N}}\mathcal{L}[0, 1]$. We give a point-free counterpart of the Urysohn's Metrization Theorem: Every regular second countable space embeds in the Hilbert cube. This fact is crucial to understand the relationship between any compact metrizable frame L and the Cantor frame $\mathcal{L}(\mathbb{Z}_2)$.

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BLAST ABSTRACT: CLUB STATIONARY REFLECTION AND THE SPECIAL ARONSZAJN TREE PROPERTY

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A fruitful line of research in set theory investigates the tension between so-called compactness or reflection principles on the one hand and incompactness principles on the other. Instances of compactness and incompactness principles are often inconsistent, and consequently it is of interest when principles in these categories are in fact jointly consistent. In a recent result with Omer Ben-Neria, we have established such a joint consistency result. Namely, we showed that Club Stationary Reflection (a strong form of reflection; see [6]) is consistent with the Special Aronszajn Tree property (a strong form of incompactness; see [5]) on the cardinal ω_2.

At a general level, the poset which produces the final model follows the collapse of a weakly compact cardinal first with an iteration of club adding and that in turn with an iteration specializing the desired Aronszajn trees. (Consequently, our club-adding posets use "anticipation" techniques such as those of [4].)

To make this strategy work, we needed to show that one can specialize Aronszajn trees on ω_2 after not just the Levy collapse of a weakly compact cardinal, but after a more involved forcing which incorporates the club-adding. To accomplish this, we isolated a property of posets called $\mathcal{F}_{\mathrm{WC}}$-*Strongly Proper* (where $\mathcal{F}_{\mathrm{WC}}$ is the weakly compact filter); a $\mathcal{F}_{\mathrm{WC}}$-Strongly Proper poset uses systems of continuous reside functions (such as in [3]) in order to witness strong genericity for many models.

Our first main result is that one obtains a κ-c.c. iteration of specializing after using an $\mathcal{F}_{\mathrm{WC}}$-Strongly Proper poset as a preparatory forcing; this provides a strong generalization of the classic result [5] of Laver and Shelah. We next prove a theorem that the composition of Levy collapsing a weakly compact and club-adding with anticipation fits into this class. This theorem is reminiscent of Abraham's use of "guiding reals" ([1]).

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Finally, to complete our result, we prove new theorems on Aronszajn tree specialization under certain distributivity assumptions, rather than stronger closure assumptions (generalizing [7]), and we prove new results about preserving stationarity by non-closed quotients of specializing forcings (generalizing [2]).

In this talk, we will introduce the notion of an $\mathcal{F}_{\mathrm{WC}}$-Strongly Proper poset, and outline the proof that one can iterate to specialize Aronszajn trees after forcing with a poset from this class. Then we will outline how to construct $\mathcal{F}_{\mathrm{WC}}$-Strongly Proper posets using the composition of Levy collapsing and club-adding with anticipation. We will close by indicating the main details of our new results on stationary set and Aronszajn tree preservation.

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States on Płonka sums of Boolean algebras

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Abstract

Probability theory is grounded on the notion of "event". Events are traditionally interpreted as elements of a (σ-complete) Boolean algebra. Intrinsically, this means that classical propositional logic is the most suitable formal language to speak about events. The direct consequence of this standard assumption is that any event can either happen (to be the case) or not happen, and, thus, its negation is taking place. One could claim that this criterion does not encompass all situations: certain events might not be either true or false (simply happen or not happen). Think about the coin toss to decide which one, among two tennis players, is choosing whether to serve or respond at the beginning of a match. Although being statistically extremely rare, the coin may fall on the edge, instead of on one face. Pragmatically, the issue is solved re-tossing the coin (hoping to have it landing on one face). Theoretically, one should admit that there are circumstances in which the event "head" (and so also its logical negation "tail") could be indeterminate (or, undefined). Interestingly enough, not all kinds of events are well modeled by classical logic. This is the case, for instance, of assertions about the properties of quantum systems (which motivated Von Neumann to introduce quantum logic [3]), or assertions that can be neither true nor false (like paradoxes), or also propositions regarding vague, or fuzzy properties (like "begin tall" or "being smart"). Yet, we are convinced that adopting (some) non-classical logical formalism to describe certain situations is not a good objection to renounce to measure their probability. On the contrary, we endorse the idea of those who think that it constitutes a good reason to look beyond classical probability, namely to render probability when events under consideration do not belong to classical propositional logic.

The idea of studying probability maps over algebraic structures connected to non-classical logic formalisms is at the heart of the theory of *states*. A theory that is well developed for different structures involved in the study of fuzzy logics, including Łukasiewicz [6], Gödel-Dummett [1] logic and product logic [5]. Within the same strand of research, probability maps have been defined and studied also for other algebras of logic, such as Heyting algebras [10], De Morgan algebras [8], orthomodular lattices [2] and effect algebras [7].

The present contribution aims at further extending the theory of states to non-classical events; in particular, to weak Kleene logics, whose algebraic semantics is played by the variety of involutive bisemilattices (see [4]). The peculiarity

of such variety is that each of its members has a representation in terms of Płonka sums of Boolean algebras. This abstract construction, originally introduced in universal algebra by J. Płonka [9], is performed over direct systems of algebras whose index set is a semilattice. The axiomatisation of states we propose, which is motivated by the logic PWK (Paraconsistent Weak Kleene), allows to "break" a state into a family of (finitely additive) probability measures over the Boolean algebras in the Płonka sum representation of an involutive bisemilattice. In other words, the proposed notion of state accounts for (and is strictly connected to) all the Boolean algebras in the Płonka sum. Moreover, we show that states over an involutive bisemilattice $\mathbf{B}$ are in bijective correspondence with finitely additive probability measures over the Boolean algebra $\mathbf{A}_\infty$ constructed as the direct limit of the algebras in the (semilattice) direct system of the representation. This allows to prove that each state corresponds to an integral over the dual space of the direct limit (the inverse limit of the dual spaces). The relation between $\mathbf{B}$ and the direct limit $\mathbf{A}_\infty$ gets deeper when $\mathbf{B}$ and $\mathbf{A}_\infty$ are topologised via the (pseudo)metric induced by a strictly positive state and the correspondent regular probability measure (on $\mathbf{A}_\infty$). Interestingly enough, we show (among other results) that the Boolean algebras of open regular sets of the topological spaces $\mathbf{B}$ and $\mathbf{A}_\infty$ are isomorphic.

The results obtained insofar explore, on the one hand, the possibility of defining probability measures over the algebraic counterpart of certain non-classical logics; on the other, it shows how (finitely additive) probability measures can be lifted from Boolean algebras to the Płonka sum of Boolean algebras.

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Title: Covering versus partitioning with Polish spaces

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Abstract:

Is there a completely metrizable space that can be covered with fewer Polish spaces than it can be partitioned into? The answer to this question turns out not only to be independent of ZFC, but also to depend on large cardinal axioms.

Given a completely metrizable space X, let $\mathfrak{par}(X)$ denote the minimum size of a partition of X into Polish spaces, and $\mathfrak{cov}(X)$ the minimum size of a covering of X with Polish spaces. Observe that $\mathfrak{cov}(X) \leq \mathfrak{par}(X)$ for every X, because every partition of X is also a covering.

In this talk, I will outline a proof that it is consistent relative to a huge cardinal that the strict inequality $\mathfrak{cov}(X) < \mathfrak{par}(X)$ can hold for some completely metrizable space X. Using large cardinals is necessary for obtaining this strict inequality, because if $\mathfrak{cov}(X) < \mathfrak{par}(X)$ for any completely metrizable X, then $0^\dagger$ exists.

Twist structures and Nelson conuclei

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Abstract

Motivated by Kalman residuated lattices, Nelson residuated lattices and Nelson paraconsistent residuated lattices, we provide a common natural generalization of all of them in the context of residuated lattices. Nelson conucleus algebras unify these examples and further extend them to the non-integral and non-commutative setting. We study their structure and establish a representation theorem for them in terms of twist structures and conuclei, and explore situations where this is actually an isomorphism. In the latter case, the categorical adjunction is elevated to a categorical equivalence. By applying this representation to the original motivating special cases we bring to the surface their underlying similarities.

A *residuated lattice* is an algebra $\mathbf{A} = (A, \vee, \wedge, \cdot, \backslash, /, e)$ such that $(A, \cdot, e)$ is a monoid, $(A, \vee, \wedge)$ is a lattice and the residuation condition

$$x \cdot y \leq z \text{ iff } y \leq x\backslash z \text{ iff } x \leq z/y$$

holds for all x, y and z in A, where $\leq$ is the order given by the lattice structure. A residuated lattice is called a *commutative*, if it satisfies $x \cdot y = y \cdot x$; in such case $x\backslash y = y/x$ and we denote the common value by $x \to y$. We say that the residuated lattice $\mathbf{A}$ is *distributive* if the lattice $(A, \vee, \wedge)$ is distributive. An *involutive residuated lattice* is an expansion with an dualizing element f: it satisfies $\sim\sim x = x$, where $\sim x = x \to f$.

Residuated lattices arise in many contexts in general and ordered algebra. Examples of residuated lattices include lattice-ordered groups, the ideal latices of rings and relation algebras. At the same time they serve as algebraic semantics for substructural logics, including linear, relevance and many valued logics. As a result, their algebraic semantics form further examples of residuated lattices and include MV, Heyting and Boolean algebras. Here we investigate a construction of involutive residuated lattices that has interesting applications to models of paraconsistent logics and use it to provide a unified approach to these models.

Given a lattice $\mathbf{L}$, the *twist-structure over* $\mathbf{L}$ is obtained by considering the cartesian product of $\mathbf{L}$ and its order-dual $\mathbf{L}^\partial$. The resulting lattice has a natural De Morgan involution given by

$$\sim (x, y) = (y, x)$$

for all $(x, y) \in L \times L^\partial$. This construction was used by Kalman in 1958, while the modifier "twist" is due to Kracht. Although Kalman only worked with the lattice structure, several other authors considered expansions of $\mathbf{L}$ with additional operations which induce new and interesting operations on the twist-structure.

In particular, Tsinakis and Wille, inspired by Chu's work on category theory and its specialization to quantales, considered the twist-structure over a residuated lattice $\mathbf{L}$ having a greatest element $\top$ and endowed it with a residuated lattice structure with unit $(e, \top)$, such that the pair $(\top, e)$ is the dualizing element for the natural involution. Busaniche and Cignoli proved that the logical systems of Nelson constructive logic with strong negation, and its paraconsistent analogue have as algebraic semantics residuated lattices whose lattice reducts are twist-structures. The monoid and residuum operators coincide with the ones proposed by Tsinakis and Wille, but the unit of the residuated lattice is not the same in all the cases, so these structures do not fall directly under the existing framework.

First we observe that the twist product construction applies also to residuated lattices $\mathbf{L}$ that may lack a top element. The general twist-product $\mathbf{Tw}(\mathbf{L})$ is then an involutive residuated lattice-ordered semigroup (a residuated lattice that may lack an identity element), with an extra unary operation of involution. By localizing to a specific positive idempotent element of $\mathbf{Tw}(\mathbf{L})$ (induced by an arbitrary fixed element $\imath$ of $\mathbf{L}$), we obtain a subalgebra $\mathbf{Tw}(\mathbf{L}, \imath)$ of $\mathbf{Tw}(\mathbf{L})$ that is a residuated lattice; this localization is done by the double-division conucleus given by Galatos and Jipsen, which focuses on the local submonoid of the positive idempotent element. This generalization allows us to work with twist products of residuated lattices without a top element and subsumes all of our motivating examples. Thus we accomplish our first goal: we put into the same framework Nelson residuated lattices and Paraconsistent Nelson residuated lattices.

Having established this first theoretical framework, we describe the class of involutive residuated lattices that have a representation as a twist-product over a residuated lattice, which requires us to focus on a reverse construction to the twist product. We show that the desired residuated lattice is also obtained by a conucleus on the involutive residuated lattice, which is of a very different nature than the double-division one, and we call it a *Nelson conucleus*. The main representation result shows that pairs of the form $(\mathbf{A}, \mathbf{n})$, where $\mathbf{A}$ is a cyclic involutive residuated lattice and $\mathbf{n}$ is a Nelson conucleus, are representable by a twist-product over a residuated lattice defined on $\mathbf{n}[\mathbf{A}]$. We call these algebras Nelson conucleus algebras and denote the variety they form by $\mathcal{NCA}$. As a corollary we provide an adjunction between the algebraic category given by $\mathcal{NCA}$ and a catergory whose objects are pairs of the form $(\mathbf{L}, \imath)$ where $\mathbf{L}$ is a residuated lattice and $\imath \in L$ is a cyclic element. Along the way of proving the representation, we generalize the original construction of Rasiowa on Nelson algebras and their representation by twist-structures: Rasiowa's homomorphic image construction can be replaced by a conucleus construction, internal to the original algebra.

Our motivating examples share some extra common features: they are commutative residuated lattices and the Nelson conucleus $\mathbf{n}$ is definable by a term function; therefore they actually form varieties of commutative involutive residuated lattices. First we identify the subvariety of $\mathcal{NCA}$ whose elements are term equivalent to Kalman lattices. Secondly we prove that Nelson residuated lattices and Paraconsistent Nelson residuated lattices form classes term equivalent to subvarieties of $\mathcal{NCA}$, thus the representation result applies to them. Furthermore, we show that both of these subvarieties are contained in $\mathcal{NT}$, a subvariety of $\mathcal{NCA}$, up to term equivalence, whose elements we call *Nelson-type algebras*. Following Sendlewski's representation for Nelson algebras and the paraconsistent analogue given by Odintsov, we improve the representation by identifying each Nelson-type algebra with a subalgebra of the twist-product on $\mathbf{n}[\mathbf{A}]$. As a consequence we get a categorical equivalence between the algebraic category of Nelson-type algebras and the category whose objects are triples of the form $(\mathbf{H}, i, F)$, where $\mathbf{H}$ is a Browerian algebra, $\imath$ is a cylcic element in $\mathbf{H}$ and $F \subseteq \mathbf{H}$ is a Boolean filter. This then restricts to categorical equivalences for the two subvarieties corresponding to Nelson residuated lattices and Paraconsistent Nelson residuated lattices.

On Finitely-Generated Johansson Algebras

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Abstract

Kuznetsov's Theorem about finitely-generated Heyting algebras is extended to Johansonn algebras in the following way: if $\mathbf{A} = (\mathsf{A}; \wedge, \vee, \to, \mathbf{1}, \mathbf{f})$ is a Johansson algebra, by the rank of element $\mathsf{a} \in \mathsf{A}$, we understand the cardinality of the set $\{\mathsf{b} \in \mathbf{A} \mid \mathsf{b} \leq \mathsf{a}\}$, and we prove that if $\mathbf{A}$ is finitely-generated, then for each element $\mathsf{a} \in \mathsf{A}$ of a finite rank, algebra $(\{\mathsf{b} \in \mathbf{A} \mid \mathsf{a} \leq \mathsf{b}\}; \wedge, \vee, \to, \mathbf{1}, \mathbf{f}')$, where $\mathbf{f}' = \mathsf{a} \vee \mathbf{f}$, is finitely-generated as well.

In [3], Kuznetsov - without a proof - announced a theorem stating that given a finitely-generated Heyting algebra $\mathbf{A}$, a sublattice of all dense elements of $\mathbf{A}$ contains the smallest element and can be regarded as a Heyting algebra, which is also finitely-generated. This theorem entails that any infinite finitely-generated Heyting algebra contains an infinite linearly ordered (alias chain) subalgebra and hence, if a variety of Heyting algebras does not contain an infinite chain algebra, this variety is locally finite. A proof was given in [2] (cf., also, [1]). We generalize this theorem and extend it to the Johansson algebras.

Johansson algebra (J-algebra for short) is an algebra $(\mathsf{A}; \to, \wedge, \vee, \mathbf{1}, \mathbf{f})$, where $(\mathsf{A}; \to, \wedge, \vee, \mathbf{1})$ is an implicative lattice with a top element $\mathbf{1}$ and a constant $\mathbf{f}$. *Heyting* algebras can be viewed as J-algebras in which $\mathbf{f}$ is the smallest element ($\mathbf{f} \to x \approx \mathbf{1}$ holds), while *Brouwerian* algebras (or implicative lattices) can be viewed as J-algebras in which $\mathbf{f}$ is the largest element ($\mathbf{f} \approx \mathbf{1}$ holds).

If $\mathbf{A}$ is a J-algebra and $\mathsf{a} \in \mathbf{A}$, by $[\mathsf{a})$ we denote a *principal filter* of $\mathbf{A}$ generated by a, that is $[\mathsf{a}) = \{\mathsf{b} \in \mathbf{A} \mid \mathsf{a} \leq \mathsf{b}\}$. Similarly to Heyting algebras, any coset $\mathsf{C} \in \mathbf{A}/[\mathsf{a})$ contains the largest element denoted by m_C, and for any $\mathsf{C} \in \mathbf{A}/[\mathsf{a})$, $card(\mathsf{C}) \leq card([\mathsf{a}))$.

By a *rank* of element $\mathsf{a} \in \mathbf{A}$, we understand the cardinality of $\mathbf{A}/[\mathsf{a})$, that is, the rank of a is the cardinality of the ideal $\{\mathsf{b} \in \mathbf{A} \mid \mathsf{b} \leq \mathsf{a}\}$. In addition, we can convert $[\mathsf{a})$ into J-algebra (denoted by $\mathbf{A}[\mathsf{a}]$) by letting $\mathbf{f} = \mathsf{a} \vee \mathbf{f}_\mathbf{A}$. Thus, if $\mathsf{a} \in \mathbf{A}$ is an element of finite rank, then, $\mathbf{A}$ is finite as long as $\mathbf{A}[\mathsf{a}]$ is finite.

Theorem 1. *Let* $\mathbf{A}$ *be a J-algebra generated by elements* $\mathsf{G} \subseteq \mathbf{A}$, *and* $\mathsf{a} \in \mathbf{A}$. *Then J-algebra* $\mathbf{A}[\mathsf{a}]$ *is generated by elements*

$$\mathsf{G}' \rightleftharpoons \{\mathsf{a}, \mathsf{a} \vee \mathsf{g}, \mathsf{a} \vee \mathsf{m}_\mathsf{C} \mid \mathsf{g} \in \mathsf{G}, \mathsf{C} \in \mathbf{A}/[\mathsf{a})\}.$$

Immediately from Theorem 1, we obtain the Generalized Kuznetsov Theorem.

Theorem 2. *If* $\mathbf{A}$ *is a finitely-generated J-algebra and* $\mathsf{a} \in \mathbf{A}$ *is an element of a finite rank, then J-algebra* $\mathbf{A}[\mathsf{a}]$ *is finitely-generated.*

Every finitely-generated J-algebra has the smallest element denoted by $\mathsf{m}_\mathbf{A}$, and we let $\neg\mathsf{a} := \mathsf{a} \to \mathsf{m}_\mathbf{A}$. Thus, $(\mathsf{A}; \to, \wedge, \vee, \mathbf{1}, \mathsf{m}_\mathbf{A})$ is a Heyting algebra. Accordingly, an element a of a finitely-generated J-algebra $\mathbf{A}$ is *dense* if $\neg\mathsf{a} = \mathsf{m}_\mathbf{A}$. All dense elements of a finitely-generate J-algebra $\mathbf{A}$ form a filter denoted by $\mathsf{D}(\mathbf{A})$.

Corollary 3 (Kuznetsov's Theorem [3])**.** *If* $\mathbf{A}$ *is finitely-generated J-algebra, then filter* $\mathsf{D}(\mathbf{A})$ *contains the smallest element* $\mathsf{d}_\mathbf{A}$ *and algebra* $\mathbf{A}[\mathsf{d}_\mathbf{A}]$ *is finitely-generated.*

Corollary 4. *If* a *is an atom of a finitely-generated J-algebra* $\mathbf{A}$*, then algebra* $\mathbf{A}[\mathsf{a}]$ *is finitely-generated. Moreover, if elements* $\mathsf{m}_{\mathbf{A}}$ *and* a *generate algebra* $\mathbf{A}$*, then* $\mathbf{A}[\mathsf{a}]$ *is generated by elements* a *and* $\mathsf{a} \vee \neg\mathsf{a}$*. If* $\mathbf{A}$ *is a Heyting algebra generated by element* a*, then* $\mathbf{A}[\mathsf{a}]$ *is generated by a single element* $\mathsf{a} \vee \neg\mathsf{a}$*.*

Using Corollary 4, by simple induction (and without use of the Nishimura Theorem) one can prove that for every $n > 1$ there is a unique modulo isomorphism single-generated Heyting algebra of cardinality n.

Suppose that $\mathbf{A}$ is a J-algebra and $\mathsf{a}, \mathsf{b} \in \mathbf{A}$. Element a is *strongly smaller* than element b (in symbols $\mathsf{a} \ll \mathsf{b}$) if $\mathsf{a} \leq \mathsf{b}$ and $(\mathsf{b} \to \mathsf{a}) = \mathsf{a}$. It is not hard to see that $\mathsf{a} \ll \mathsf{b} \iff (\mathsf{b} \to \mathsf{a}) \to \mathsf{b} = \mathbf{1}$. *Refined height* of element a (in symbols, $\mathfrak{h}(\mathsf{a})$) is a maximal length of a strongly descending chain of elements $\mathsf{a}_1 \ll \cdots \ll \mathsf{a}_n \leq \mathsf{a}$ smaller than a, and *refined height* of algebra is the refined height of $\mathbf{1}$. Note that $\mathfrak{h}(\mathsf{a}) > 0$ if and only if a is dense.

There is close relation between the strongly descending chains in J-algebra and in its quotients.

Proposition 5. *Let* $\mathbf{A}$ *be a* J*-algebra, and* $\mathsf{F} \subseteq \mathbf{A}$ *be a filter. Then, for any* $\mathsf{a}, \mathsf{b} \in \mathbf{A}$ *such that* $\mathsf{a} \notin \mathsf{F}$*,*

(a) if $\mathsf{a} \ll \mathsf{b}$*, then* $\mathsf{a}/\mathsf{F} \ll \mathsf{b}/\mathsf{F}$*;*

(b) if $\mathsf{a}/\mathsf{F} \ll \mathsf{b}/\mathsf{F}$*, then there are* $\mathsf{a}' \in \mathsf{a}/\mathsf{F}$ *and* $\mathsf{b}' \in \mathsf{b}/\mathsf{F}$ *such that* $\mathsf{a}' \ll \mathsf{b}'$*.*

Remark 1. *Condition (a) entails that* $\mathfrak{h}(\mathsf{a}) \leq \mathfrak{h}(\mathsf{a}/\mathsf{F})$ *for any filter* F *not containing* a*. But it is possible that* $\mathfrak{h}(\mathsf{a}/\mathsf{F}) > \mathfrak{h}(\mathsf{a})$*. Take as an example Heyting algebra* $\mathbf{B} := \mathbf{C} \times \mathbf{2}$*, where* $\mathbf{C} := (\{\mathbf{0} < \mathsf{a}_1 < \mathsf{a}_2 < \cdots < \mathsf{b} < \mathbf{1}\}; \wedge, \vee, \to, \mathbf{0}, \mathbf{1})$ *and* $\mathbf{2}$ *is two-element Boolean algebra. Element* $(\mathsf{b}, \mathbf{0})$ *is not dense and thus,* $\mathfrak{h}((0, \mathsf{b})) = 0$*, but in* $\mathbf{B}/[(\mathbf{0}, \mathbf{1}))$*, element* $(\mathsf{b}, \mathbf{0})/[(\mathbf{0}, \mathbf{1}))$ *has infinite refined height.*

J-algebra $\mathbf{A}$ is *finitely approximable* if for any distinct from $\mathbf{1}$ element $\mathsf{a} \in \mathbf{A}$, there is a filter $\mathsf{F} \subseteq \mathbf{A}$ such that $\mathsf{a} \notin \mathsf{F}$ and the quotient $\mathbf{A}/\mathsf{F}$ is finite.

Theorem 6. *If* J*-algebra* $\mathbf{A}$ *is finitely-generated, then,* $\mathbf{A}$ *is finitely approximable if and only if every distinct from* $\mathbf{1}$ *element of* $\mathbf{A}$ *has a finite refined height.*

Remark 2. *In Theorem 6 the condition that* J*-algebra* $\mathbf{A}$ *is finitely-generated is essential: take as an example free Heyting algebra* $\mathbf{F}_{\mathbf{H}}(\omega)$ *which is finitely approximable, but contains elements of infinite refined height: algebra* $\mathbf{C}$ *from Remark 1 can be embedded in* $\mathbf{F}_{\mathbf{H}}(\omega)$*, and* $\mathfrak{h}(\mathsf{b}) = \infty$*.*

Theorem 7. *A finitely-generated* J*-algebra* $\mathbf{A}$ *is finite if and only if the height of* $\mathbf{A}$ *is finite.*

It is clear that an element of finite rank has a finite refined height. The converse needs not to be true: consider Heyting algebra $\mathbf{B}$ from Remark 1: element $(\mathbf{1}, \mathbf{0})$ has an infinite rank while on the other hand, its refined height is 0.

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The Equational Theory of Distributive Lattice-Ordered Monoids

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For any totally ordered set (briefly, chain) Ω, the monoid $\mathrm{End}\,\Omega$ of its order-preserving endomorphisms ordered pointwise is a distributive lattice-ordered monoid (briefly, distributive ℓ-monoid), in the sense that the monoid operation distributes over meet and join, and that the lattice reduct is distributive. In 1984, Marlow Anderson and Constance Edwards showed that any distributive ℓ-monoid is in fact an ℓ-monoid of order-preserving endomorphisms of a chain [1]—thereby extending Charles Holland's representation theorem for lattice-ordered groups (briefly, ℓ-groups) [6]. Hence, the variety of distributive ℓ-monoids is generated by the class of ℓ-monoids of order-preserving endomorphisms of chains.

We refine this result as follows:

Theorem 1. The variety of distributive ℓ-monoids is generated by the class of (finite) distributive ℓ-monoids of order-endomorphisms of finite chains.

As a consequence, we get the following decidability result.

Corollary 2. The equational theory of distributive ℓ-monoids is decidable.

If an ℓ-group is an algebra in the signature $\{\wedge, \vee, \cdot, ^{-1}, \mathrm{e}\}$—where e is the group identity—a distributive ℓ-monoid is an algebra in the inverse-free language $\{\wedge, \vee, \cdot, \mathrm{e}\}$. Our main contribution to the theory of distributive ℓ-monoids is the fact that the equations holding in all distributive ℓ-monoids are precisely the inverse-free equations holding in all ℓ-groups. Equivalently:

Theorem 3. The variety of distributive ℓ-monoids is generated by the inverse-free reduct of $\mathrm{Aut}\,\mathbb{Q}$, where $\mathrm{Aut}\,\mathbb{Q}$ is the ℓ-group of order-preserving bijections of the rational line.

This is especially interesting in view of the fact that, in contrast to the situation for ℓ-groups, finite distributive ℓ-monoids exist in abundance—indeed, as already mentioned, they generate the whole variety.

Sketch of Proof. Let ε be an equation that fails in $\mathrm{End}\,\Omega$ for some chain Ω; each term involved in ε is assigned to an order-preserving endomorphism of Ω. The structure of the subterms of ε can be used to modify such endomorphisms and obtain partial order-preserving injections on $\mathbb{Q}$, each with finite domain. These partial order-preserving injections are then extended to order-preserving bijections of $\mathbb{Q}$ that continue to falsify the original equation. $\square$

Remarkably, it has long been known that the variety of those distributive ℓ-monoids that are commutative strictly contains the variety generated by the inverse-free reduct of $\mathbb{Z}$. This follows from a theorem of Vladimir Repnitskii, who in 1983 proved that the variety generated by the ordered monoid of integers is not finitely based [7]. We extend the negative result here and show that Theorem 3 does not specialize to totally ordered structures.

Theorem 4. There is an inverse-free equation that is valid in all totally ordered groups, but not in all totally ordered monoids.

*Joint work with Nikolaos Galatos (University of Denver), George Metcalfe (University of Bern), Simon Santschi (University of Bern).

We also provide the following axiomatization for the variety of ℓ-monoids generated by totally ordered monoids:

$$z_1y_1z_2 \wedge w_1y_2w_2 \leq z_1y_2z_2 \vee w_1y_1w_2.$$

By Theorem 3, to check whether an equation is valid in all distributive ℓ-monoids, it suffices to check the validity of this same equation in all ℓ-groups. We prove here that a certain converse also holds, namely:

Theorem 5. For any ℓ-group equation ε with variables in a set X, a finite set of inverse-free equations Σ with variables in $X \cup Y$ for some finite set Y can be effectively constructed such that ε is valid in all ℓ-groups if and only if the equations in Σ are valid in all distributive ℓ-monoids.

This result might lead not only to a first calculus for ℓ-groups admitting an algebraic proof of cut elimination (cf. [5]), by importing tools and techniques developed in [4], but also to develop a framework for a more systematic study of the proof theory for ℓ-groups.

Recall that a right order on a monoid M is a total order such that $a \leq b$ entails $ac \leq bc$, for all $a, b, c \in M$. When a monoid M admits such an order, it is said to be right-orderable. It is well-known that the class of right-orderable groups coincides with the class of subgroups of ℓ-groups [3]. Indeed, any monoid M that admits a right order $\leq$ embeds into the distributive ℓ-monoid of its order-preserving endomorphisms, by mapping each $a \in M$ to the order-endomorphism $x \mapsto xa$. However, contrary to the claim made in [1], it is not the case that every submonoid of a distributive ℓ-monoid is right-orderable.

Proposition 6. The monoid $\mathrm{End}\,\Omega$ is not right-orderable for any chain Ω with $|\Omega| \geq 3$.

The relationship between distributive ℓ-monoids and right orders is further studied here in the form of a correspondence between validity of equations in distributive ℓ-monoids and existence of certain right orders on free monoids. A remarkable consequence of this is a neat connection between right orders on free monoids and right orders on free groups.

Theorem 7. Every right order on the free monoid over a set X extends to a right order on the free group over X.

All the results presented here can be found in [2].

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Algorithmic correspondence for relevance logics: The algorithm PEARL and its implementation (Extended abstract)

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We report on work that brings together two important areas of development in non-classical logics, viz. *relevance logics* and *algorithmic correspondence theory*.

(Modal) Algorithmic Correspondence Theory One of the classical results in modal logic since the invention of the possible worlds semantics was the *Sahlqvist-van Benthem theorem*, which makes the following two important claims for all formulae from a certain syntactically defined class (subsequently called *Sahlqvist formulae*), including the modal principles appearing in axioms of the most important systems of normal modal logics: (i) that they define conditions on Kripke frames that are also definable in the corresponding first-order language (FO), and (ii) that all normal modal logics axiomatized with Sahlqvist formulae are complete with respect to the class of Kripke frames that they define, because these logics are *canonical*, i.e., valid in their respective canonical frames.

That result set the stage for the emergence and development of *correspondence theory in modal logic*, cf. [2]. Over the past 20 years that theory expanded significantly in at least three directions: Firstly, the class of formulae covered by Sahlqvist's theorem was extended considerably to the class of so called *inductive formulae* (cf. [4], [8]). Secondly, the method for eliminating propositional variables from modal formulae and computing their first-order equivalents was substantially extended in scope and made algorithmic in a series of papers developing *algorithmic correspondence theory* implemented by the algorithmic procedure SQEMA [5], which not only provably succeeds in computing the first-order equivalents of all inductive (and, in particular, all Sahlqvist) formulae, but also automatically proves their canonicity, just by virtue of succeeding on them. Thirdly, both the traditional and the algorithmic correspondence theory were subsequently developed further and extended significantly. First, SQEMA was generalized to the algorithm ALBA introduced in [6] to cover the inductive formulae for distributive modal logic. This was extended to a wide range of logics ultimately including any logic algebraically captured by classes of normal (possibly non-distributive) lattice expansions [7].

Correspondence theory for Relevance Logic Much work has been done over the years on computing first-order equivalents and proving completeness of a range of specific axioms for relevance logics with respect to the *Routley-Meyer relational semantics* (cf. [9]). Because of the more complex semantics, these kinds of results can be significantly more elaborated than for modal logics with their standard Kripke semantics, which calls for development of a systematic correspondence theory of relevance logics. Until recently that problem remained largely unexplored, with just a couple of works, incl. [10] and [1], defining some classes of Sahlqvist-type formulae for relevance logics and proving correspondence results for them. A general algorithmic correspondence theory of relevance logics has recently been developed in [3], on which the presently reported work is based.

The algorithm PEARL and its implementation A non-deterministic algorithmic procedure PEARL (acronym for Propositional variables Elimination Algorithm for Relevance Logic) for computing first-order equivalents in terms of frame validity of formulae of the language $\mathcal{L}_R$ for relevance logics is developed in [3]. PEARL is an adaptation of the above mentioned procedures SQEMA [5] (for normal modal logics) and ALBA [6, 7] (for distributive and non-distributive modal logics). A large syntactically defined class of *inductive relevance formulae* in $\mathcal{L}_R$ is defined in [3] where it is shown that PEARL succeeds for all such formulae and correctly computes their equivalent with respect to frame validity first-order definable conditions on Routley-Meyer frames.

In this talk we will present a deterministic algorithmic version of the procedure PEARL and will describe an implementation of it in Python. The input is a LaTeX string using the syntax of relevance logic expressions, extended with some residuals and dual residuals of the standard relevance logic connectives. The expression is parsed with a simple top-down Pratt parser using standard rules of precedence. For each well-formed formula an abstract syntax tree (AST) based on Python dictionaries and lists of arguments is created. Short recursive Python functions are then used to transform the AST representation step-by-step according to specific groups of PEARL transformation rules. Several larger strategies are implemented to first transform the AST (if possible) into a pure formula (containing no propositional variables, but only nominals), simplifying it along the way, and then the standard translation is applied to produce a first-order formula on the Routley-Meyer frames, followed by further simplifications. At each stage the internal AST can be examined, translated to LaTeX output and/or displayed in typeset form. The Python code can be used in any Jupyter notebook, with the output displayed in standard mathematical notation. No special installation is needed to use it in a personal Jupyter notebook or in a public cloud-based notebook such as `colab.research.google.com`, and the output can be copied into standard LaTeX documents. Moreover the code can be easily extended to handle the syntax of other suitable logics and lattice-ordered algebras.

The talk will end with brief demonstrations of our implementation of PEARL.

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Dualities for default bilattices and their applications

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Bilattices, which provide an algebraic tool for simultaneously modelling knowledge and truth, were introduced by N. D. Belnap in a 1977 paper entitled *How a computer should think*. Prioritised default bilattices include not only Belnap's four values, for 'true' ($\boldsymbol{t}$), 'false'($\boldsymbol{f}$), 'contradiction' ($\top$) and 'no information' ($\bot$), but also indexed families of default values for simultaneously modelling degrees of knowledge and truth.

In our paper [3] we introduced a new class $\{\, \mathbf{J}_n \mid n \in \omega \,\}$ of default bilattices for use in prioritised default logic. The first of these bilattices, $\mathbf{J}_0$, is Belnap's famous four-element bilattice known as $\mathcal{FOUR}$ [1], while for $n \geqslant 1$, the bilattice $\mathbf{J}_n$ provides a new algebraic structure for dealing with inconsistent and incomplete information. The importance of our prioritised default bilattices in comparison with those previously studied is that in our family there is no distinction between the level at which the contradictions or agreements take place. Any contradictory response that includes some level of truth ($\boldsymbol{t}_i$) and some level of falsity ($\boldsymbol{f}_j$) is registered as a total contradiction ($\top$) and a total lack of consensus ($\bot$). This can lead to improvements in existing applications of default bilattices.

Prioritised default bilattices now have many applications in artificial intelligence. Sakama [8] studied default theories based on a 10-valued bilattice and applications to inductive logic programming. Shet, Harwood and Davis [9] proposed a prioritised multi-valued default logic for identity maintenance in visual surveillance. Encheva and Tumin [5] applied default logic based on a 10-element default bilattice in an intelligent tutoring system as a way of resolving problems with contradictory or incomplete input.

Bilattices are algebras $\mathbf{A} = \langle A; \otimes, \oplus, \wedge, \vee, \neg \rangle$ with two lattice structures, a knowledge lattice $\mathbf{A}_{\mathrm{k}} = \langle A; \otimes, \oplus \rangle$, with associated *knowledge order* $\leqslant_{\mathrm{k}}$, and a truth lattice $\mathbf{A}_{\mathrm{t}} = \langle A; \wedge, \vee \rangle$, with associated *truth order* $\leqslant_{\mathrm{t}}$, along with an involutive negation $\neg$ which is an order automorphism of $\mathbf{A}_{\mathrm{k}}$ and a dual order automorphism of $\mathbf{A}_{\mathrm{t}}$. Our algebra $\mathbf{J}_n$ is a *prioritised default bilattice* as it is equipped with two hierarchies of nullary operations, $\boldsymbol{t}_i$ and $\boldsymbol{f}_i$, that represent, respectively, true and false by default. We refer the reader to [3] for motivation and background on bilattices in general and prioritised default bilattices in particular.

In our approach we address mathematical rather than logical aspects of our prioritised default bilattices. The lack of the much-used product representation in our context led us to develop a concrete representation via the theory of natural dualities (in the sense of [2]). In [3], we presented a natural duality between the variety $\mathcal{V}_n$ generated by $\mathbf{J}_n$ and a category $\mathfrak{X}_n$ of multi-sorted topological structures. At the beginning of our talk we briefly recall this duality. Our first main aim in this talk is to describe the dual category $\mathfrak{X}_n$. We begin by giving an axiomatisation of the multi-sorted category $\mathfrak{X}_n$ and then describe an isomorphic category $\mathfrak{Y}_n$ of single-sorted topological structures. The objects of $\mathfrak{Y}_n$ are Priestley spaces endowed with a continuous retraction in which the order has a natural ranking. Then we describe the Priestley dual $\mathrm{H}(\mathbf{A}^\flat)$ of the underlying bounded distributive lattice $\mathbf{A}^\flat$ of an algebra $\mathbf{A}$ in $\mathcal{V}_n$. As an application of we show that the size of the free algebra $\mathbf{F}_{\mathcal{V}_n}(1)$ is given by a polynomial in n of degree 6. This result is presented below:

Theorem 1. *Let $n \in \omega\backslash\{0\}$. Then the cardinality of the 1-generated free algebra in the variety $\mathcal{V}_n = \mathrm{Var}(\mathbf{J}_n)$ is*

$$|F_{\mathcal{V}_n}(1)| = \tfrac{1}{2}\big(n^6 + 10n^5 + 42n^4 + 102n^3 + 157n^2 + 148n + 72\big).$$

.

Except the result presenting the natural duality between the variety $\mathcal{V}_n$ generated by $\mathbf{J}_n$ and the category $\mathfrak{X}_n$ of multi-sorted topological structures from [3], the results presented here are based on our second paper [4] that was submitted in November 2020.

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The V-logic Multiverse and MAXIMIZE

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In this paper, I argue that classical set theory, $ZFC(+LCs)$, is *restrictive* over the V-logic multiverse (a novel set theoretic multiverse conception developed by the present author and Claudio Ternullo). This multiverse conception is based upon Friedman's hyperuniverse and Steel's set-generic multiverse: like the hyperuniverse, it uses the infinitary V-logic as the background logic and admits all kinds of outer models of V (produced by set-generic, class-generic, hyperclass forcing, etc.). Like Steel's set-generic multiverse, it is recursively axiomatisable and is rooted on a ground universe that satisfies ZFC. For this proof, I compare $ZFC + LCs$ and the V-logic multiverse, characterised as $ZFC + LCs+$ the multiverse axioms, following Maddy's methodological principle MAXIMIZE, as described in Maddy (1997). According to this principle, when choosing between two foundational theories we ought to prefer the one that can prove more isomorphisms types. I claim that the V-logic multiverse, as opposed to $ZFC + LCs$, does exactly that. This is because in the V-logic multiverse theory we can prove the existence of proper, uncountable, extensions of V, that we cannot have in $ZFC + LCs$. In turn, this extra object means we can realise more isomorphisms types, since in the V-logic multiverse we can prove the existence of iterable class sharps and, more importantly, maps between them. This opens up a new realm of isomorphisms types that are not available in $ZFC + LCs$. Thus, this latter theory is restrictive over the V-logic multiverse theory: the V-logic multiverse, characterised as $ZFC + LCs+$ Multiverse Axiom Schema, and with V-logic as the background logic, proves more isomorphism types than classical set theory ($ZFC + LCs$), and thus we can say that classical set theory is *restrictive* over it.

First of all I need to precise the terms of this comparison. On the one hand, I am taking classical set theory in its usual axiomatization ZFC plus the addition of large cardinals axioms, as instantiated by the the cumulative hierarchy V. This, together with the usual interpretation of forcing as applicable only to countable transitive models is the foundational framework that universists defend, and claim that it is enough for set theoretic practice. On the other hand, the V-logic Multiverse is characterised as $ZFC + LCs+$ the Multiverse Axiom Schema (there are also other axioms, but they are not really needed right now). Note that, as usually argued by the universist, the addition of the Multiverse Axiom Schema does not add any "real" power to $ZFC + LCs$, since everything we need is already in the latter theory, at least according to universists.

We can now proceed to the first step of my argument, i.e. proving that the V-logic multiverse can prove the existence of an extra object that it is unavailable in $ZFC + LCs$. This object is a proper, uncountable, outer model of V. Such an object cannot exists in the universist's framework of $ZFC + LCs$: indeed, the application of forcing in that usual setting is done only to countable transitive models. This is because to do it we need the existence of generic filters, and for the universist there are no V-generic filters.

However, in the V-logic multiverse framework we can prove the following theorem:

Theorem 1. *Let φ be a V-logic sentence (for instance, a sentence which says "$Con(T)$" for some V-logic theory T). The following are equivalent:*

1. *φ is consistent in V-logic.*

2. *φ is consistent in $\mathfrak{V}$-logic.*

3. *$\mathfrak{V}$ has an outer model, $\mathfrak{W}$, such that $\mathfrak{W} \models \varphi$.*

4. *(since $\mathfrak{W}$ is elementarily equivalent to $\mathfrak{V}^*$) $\mathfrak{V}^* \models \varphi$.*[1]

What this theorem says is that, in the V-logic multiverse, even if we start with a countable model of ZFC inside V, we can then end up with a proper, uncountable outer model of an *uncountable* V[2]

Consequently we have, in the V-logic Multiverse, an object that cannot be found in the universist's framework. We now need to prove that this new object realises a new isomorphism type. And this is exactly my claim.

To see this, consider the technique of #-generation.[3] As stated by Antos, Barton and Friedman, this method is very useful in encapsulating several large cardinals consequences of reflection properties. It is based upon the existence of *class-iterable sharps*: these are transitive structures that are amenable, with a normal measure and iterable in the sense that all successive ultrapower iterations along class well-orders are well founded.[4] If such an object exists, then we could have *class iterated sharp generated* models, i.e. models that arise through collecting together each level indexed by the largest cardinal of the model that result from the iteration of a class-iterable sharp.[5] Finally, we could claim that V is such class iterably sharp generated, and enjoy all advantages of this fact. Sadly, we cannot, since we cannot find, in V, a class-iterable sharp. If it were the case, then we would be able to prove the existence of a cardinal that is both regular *and* singular, but this is impossible. So in the classical set theoretic framework all of the above is unattainable.

This situation is fundamentally different in the V-logic multiverse. Indeed, since in the V-logic multiverse we can have proper, uncountable, extensions of V, we can also have, in these extensions, a class-iterable sharp! And thus, in the V-logic multiverse, we can claim that V is, in fact, class iterably sharp generated! This result opens a new realm of isomorphisms types between all the various iterated ultrapowers, and models of different heights that are provided by #-generation.

Thus, in conclusion, we can claim that $ZFC + LCs$ is restrictive over the V-logic multiverse, since in the latter we can find a new object that realises a new isomorphism type.

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[2] The V-logic multiverse is not the only multiverse conceptions that claims the existence of proper outer models of V, the other being the Hyperuniverse. However, the latter assume that V is countable, thus semplifying the setting by a lot.

[3] See Antos, N. Barton, and S.-D. Friedman (nd) for a discussion of it.

[4] Here I am following the definition from Antos, N. Barton, and S.-D. Friedman (nd). The original definition in S. Friedman (2016) is slightly different, however nothing important rests on this difference.

[5] Again, the precise definition can be found in Antos, N. Barton, and S.-D. Friedman (nd).

BIG RAMSEY DEGREES OF UNIVERSAL INVERSE LIMIT STRUCTURES

NATASHA DOBRINEN AND KAIYUN WANG

In [2], Huber, Geschke and Kojman showed that the profinite graph $\mathcal{G}$, the universal inverse limit of the Fraïssé class of finite ordered graphs with topology inherited as a subspace of the reals, has elegant partition properties. Precisely, they proved that for each finite ordered graph $\mathbf{A}$, there exists a positive integer T such that for any finite Baire measurable partition of the copies of $\mathbf{A}$ in $\mathcal{G}$, there is a subcopy $\mathcal{G}' \subseteq \mathcal{G}$ such that all copies of $\mathbf{A}$ in $\mathcal{G}'$ lie in at most T many pieces of the partition. The least such number T is denoted $T(\mathbf{A})$ and called the *big Ramsey degree* of $\mathbf{A}$ in $\mathcal{G}$. Moreover, they showed that the probability that $T(\mathbf{A}) = 1$ for a randomly chosen finite graph $\mathbf{A}$ on n vertices tends to 1 as n goes to infinity. However, this left open the question of exactly characterizing the numbers $T(\mathbf{A})$ for each finite ordered graph $\mathbf{A}$.

Zheng in [3] built a topological Ramsey space of infinite trees so that the set of branches through any tree in the space forms a profinite graph. She then used these spaces to find a shorter proof recovering the theorem of Huber, Geschke and Kojman. Using Zheng's construction method, for any Fraïssé class of finite ordered structures on finitely many binary relations, we build in [1] a topological Ramsey space of trees so that the set of branches through any tree in the space forms a universal limit (or profinite) structure. The proof relies on the Halpern-Läuchli and Nešetřil-Rödl Theorems. It follows that big Ramsey degrees exist for finite Baire measurable colorings of these universal limit structures. Furthermore, we characterize the exact big Ramsey degrees inside the universal limit structures for the classes of finite ordered graphs, digraphs, tournaments, and k-clique-free graphs, for any $k \geq 3$.

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On spectra of BCK-algebras

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In 1966, Imai and Iséki introduced BCK-algebras as the algebraic sematnics of a non-classical logic. The set of irreducible ideals of a BCK-algebra can be endowed with the Zariski topology giving a topological space we call the *spectrum* of the algebra. Meng and Jun showed that when the algebra is bounded and commutative, the spectrum is a spectral space (that is, homeomorphic to the prime spectrum of some commutative ring). In my dissertation, I showed that when the algebra is commutative (not necessarily bounded), the spectrum is a generalized spectral space (that is, homeomorphic to the prime spectrum of some distributive lattice with 0). In this talk I discuss what properties the spectrum has when the algebra is not assumed to be either bounded or commutative.

Generalized divisibility and ordinal sums

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The ordinal sum construction has played an important role in the study of residuated structures, in particular of BL-algebras and their 0-free subreducts, basic hoops. BL-algebras can be characterized as those FL_{ew}-algebras (bounded commutative integral residuated lattices) that are representable (subdirect products of totally ordered structures) and satisfy the divisibility condition. Aglianò and Montagna in a paper of 2003 show that every subdirectly irreducible basic hoop (or BL-algebra) can be decomposed into an ordinal sum of sum-indecomposable components (in which the first one is bounded) [1]. Such sum-indecomposable components coincide with totally ordered Wajsberg hoops, the 0-free subreducts of MV-algebras. This Birkhoff-like representation theorem in terms of ordinal sums has shown to be extremely useful for studying the lattice of subvarieties of BL-algebras and basic hoops. In particular, the authors are able to use this representation to axiomatize varieties of basic hoops (or BL-algebras) generated by ordinal sums with at most a certain number of components, and with each component belonging to a specific variety of Wajsberg hoops.

We show that if divisibility is dropped altogether, one cannot obtain the same kind of results, even though subdirectly irreducible MTL-algebras (representable FL_{ew}-algebras) have been shown to have a maximal decomposition in terms of ordinal sums [2].

However, we show how one can weaken the divisibility condition and obtain an analogous representation theorem of the subdirectly irreducible algebras in terms of ordinal sums. In particular, we introduce a hierarchy of varieties of MTL-algebras that we call n-divisible, for each $n \in \mathbb{N}$. The totally ordered members in these varieties can be characterized in terms of ordinal sums of what we shall call n-Wajsberg hoops. Moreover, as in the case of BL-algebras, we can find equations that characterize n-divisible algebras with at most a certain number of components.

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TOWARDS A CORRESPONDENCE THEORY IN REGION-BASED THEORIES OF SPACE

RAFAŁ GRUSZCZYŃSKI
(JOINT WORK WITH ANDRZEJ PIETRUSZCZAK)

Since the beginning of the XXth century Boolean contact algebras (BCAs for short) are the standard setting for doing region-based topology.[1] A Boolean contact algebra is a Boolean algebra $\mathfrak{R} = \langle R, \sqcap, \sqcup, -, \mathsf{0}, \mathsf{1}\rangle$ (the elements of R are called *regions*) extended with a binary *contact* relation C that satisfies the following constraints:

$$\neg(\mathsf{0}\ \mathsf{C}\ x), \tag{C0}$$

$$x \leq y \wedge x \neq \mathsf{0} \longrightarrow x\ \mathsf{C}\ y, \tag{C1}$$

$$x\ \mathsf{C}\ y \longrightarrow y\ \mathsf{C}\ x, \tag{C2}$$

$$x \leq y \longrightarrow \forall_{z\in R}(z\ \mathsf{C}\ x \longrightarrow z\ \mathsf{C}\ y)\,, \tag{C3}$$

$$x\,\mathsf{C}\,y \sqcup z \longrightarrow x\,\mathsf{C}\,y \vee x\,\mathsf{C}\,z\,. \tag{C4}$$

Additional constraints are considered as well:

$$(\forall x \neq 0)(\exists y \neq 0)\, y\ \not\mathsf{C}\ -x\,, \tag{C5}$$

$$\forall x \forall y\, (x\ \not\mathsf{C}\ y \longrightarrow \exists z (x\ \not\mathsf{C}\ -z \wedge z\ \not\mathsf{C}\ -y))\,, \tag{C6}$$

$$(\forall x \notin \{\mathsf{0}, \mathsf{1}\})\, x\,\mathsf{C} -x\,. \tag{C7}$$

Various constructions of points as sets of regions give rise to different topological spaces and different representation and duality theorems for BCAs. Maximal *round* filters (called *ends* as well)[2] in De Vries algebras[3] are points of compact Hausdorff spaces. Utilizing techniques from proximity spaces[4], the so-called *clusters* give rise to representation theorems for BCAs satisfying the basic axioms (C0)–(C4).[5] Slightly lesser known *Grzegorczyk points* from (Grzegorczyk, 1960) serve as building blocks of *concentric spaces*, a subclass of the class of regular spaces.[6] It is also well known that ultrafilters may be points of those BCAs in which contact collapses to the overlap relation:

$$x \bigcirc y \ :\longleftrightarrow\ x \cdot y \neq \mathsf{0}\,. \tag{df $\bigcirc$}$$

Geometrically motivated, Grzegorczyk points are sets of regions that are to mimic the idea of a point as s system of «shrinking» regions of space. To define the notion, first Grzegorczyk representatives are defined as non-empty sets of regions that satisfy the following conditions:

$$\mathsf{0} \notin Q\,, \tag{r0}$$

$$\forall_{u,v\in Q}(u = v \vee u \ll v \vee v \ll u)\,, \tag{r1}$$

$$\forall_{u\in Q}\exists_{v\in Q}\ v \ll u\,, \tag{r2}$$

$$\forall_{x,y\in R}\big(\forall_{u\in Q}(u \bigcirc x \wedge u \bigcirc y) \longrightarrow x\ \mathsf{C}\ y\big)\,. \tag{r3}$$

where $\ll$ is a well-inside relation introduced via:

$$x \ll y \ :\longleftrightarrow\ x\ \not\mathsf{C}\ -y\,. \tag{df $\ll$}$$

A *Grzegorczyk point* (a *G-point*) is any filter that is generated by some G-representative. Let **Grz** be the set of G-points of a given BCA.

[1] The standard reference for BCAs is Bennett and Düntsch (2007).

[2] A filter $\mathscr{F}$ is round iff for every $x \in \mathscr{F}$ there is $y \in \mathscr{F}$ such that $y\ \not\mathsf{C}\ -x$.

[3] De Vries algebras were originally presented in (de Vries, 1962) as algebras with the relation of being *well-inside* as basic. However the approach via contact relation with the axioms (C5) and (C6) is definitionally equivalent.

[4] See (Naimpally and Warrack, 1970).

[5] See (Dimov and Vakarelov, 2006).

[6] See (Gruszczyński and Pietruszczak, 2018, 2019).

The fact that there are many different characterizations of points of BCAs gives rise to a natural question about the mutual dependencies between second-order monadic statements about points and properties of BCAs. For example, it is known that in every BCA every G-point is an end: $\mathbf{Grz} \subseteq \mathbf{End}$, but the reverse inclusion does not have to hold in general.

In the talk, I would like to present initial results concerning such dependencies. I would particularly like to focus on relations between G-points and filters of BCAs, and draw attention to correspondences between the statements about points and either first- or higher-order properties of BCAs.

I will prove that in certain subclasses of the class of BCAs, the so-called Grzegorczyk contact algebras, the following theorems hold:

Theorem 1. *Every G-point is an ultrafilter iff every region is well-inside itself* (*equivalently: contact collapses to overlap, or in one more formulation: the algebra satisfies the negation of* (C7)).

Theorem 2. *Every ultrafilter is a G-point iff the BCA is finite.*

I will also show that in the case of atomic contact algebras, the statement "the co-finite filter is a Grzegorczyk point" lets us prove that the topological space of Grzegorczyk points is a continuous image of a Stone space.

Apart from this, I will point to open problems solutions of whose could contribute to developing the correspondence theory between second-order properties of points and familiar, either first- or higher-order, properties of Boolean contact algebras.

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Local Automorphisms of Cartesian Products

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For R a field and P a set let $\Pi = \Pi_{x\in P} Re_x$ denote the full cartesian product of $|P|$ copies of R. We will consider the R-algebra automorphisms $\mathrm{Aut}(\Pi)$ of Π and the local automorphisms $\mathrm{LAut}(\Pi)$ of Π, where an R-linear map $\eta : \Pi \to \Pi$ is called a *local automorphism* if for every $a \in \Pi$ there exists some $\varphi \in \mathrm{Aut}(\Pi)$ with $\eta(a) = \varphi(a)$. We are going to answer two basic questions:

1. What are the R-algebra automorphisms of Π?

2. Is every local automorphism of Π an R-algebra automorphism?

Surprisingly, the answer to the second question will depend on the chosen model of set theory. We will indicate how this result relates to recent work of the Baylor algebra group on local automorphisms of incidence algebras. This is joint work with M. Dugas and J. Courtemanche.

A REPRESENTATION THEOREM FOR BOTH EVEN AND ODD INVOLUTIVE COMMUTATIVE RESIDUATED CHAINS BY MEANS OF DIRECT SYSTEMS OF ABELIAN o-GROUPS

SÁNDOR JENEI

ABSTRACT. For involutive, commutative residuated chains, where either the residual complement operation leaves the unit element fixed or the unit element is the cover of its residual complement, a representation theorem will be presented in this talk by means of a direct system of totally ordered abelian groups equipped with further structure.

An *FL_e-algebra* (aka. pointed commutative residuated lattice) $(X, \wedge, \vee, \circledast, \to_{\circledast}, t, f)$ is a structure such that $(X, \wedge, \vee)$ is a lattice, $(X, \leq, \circledast, t)$ is a commutative, residuated monoid and f is an arbitrary constant. One defines the *residual complement operation* by $x' = x \to_{\circledast} f$ and calls an FL_e-algebra *involutive* if $(x')' = x$ holds. Call an involutive FL_e-algebra *odd* if the residual complement operation leaves the unit element fixed, and *even* if the following quasi-identity holds: $x < t \Rightarrow x \leq f$. The former condition is equivalent to $f = t$, while the latter condition is equivalent to assuming that f is the unique lower cover of t if chains (or more generally conical algebras) are considered. In the involutive setting $t' = f$ holds and it means that the two constants are positioned symmetrically with respect to $'$ in the underling lattice. Thus, one extreme situation is the integral case, when t is the top element of X and hence f is its bottom element. This case has been deeply studied. But despite the extensive literature there are still very few results that effectively describe their structure, and most of the effective descriptions postulate, besides integrality, the naturally ordered condition (or its dual notion, divisibility), too [1, 3, 6, 7, 11, 12, 14, 16]. The other extreme situation, when the two constants are both "in the middle", (the odd and even case) is a much less studied scenario, and hence non-integral residuated structures and consequently, substructural logics without the weakening rule, are far less understood at present than their integral counterparts. Prominent examples of odd involutive FL_e-algebras are lattice-ordered abelian groups and odd Sugihara monoids. The former constitutes an algebraic semantics of Abelian Logic [2, 15, 17] while the latter constitutes an algebraic semantics of a logic at the intersection of relevance logic and many-valued logic [4].

Giving structural description of involutive FL_e-chains seemed out of reach several years ago, since *only the idempotent ones* have been treated to some extent, see e.g. [19, 4, 5]. Recently a representation theorem were presented in [9, 8] for those odd involutive FL_e-chains where the number of idempotent elements of the algebra is finite, by means of partial sublex products of abelian o-groups which are well understood mathematical objects that are much more regular than what had been expected to need for describing these particular FL_e-chains.

Our main result shows that every even or odd involutive commutative residuated chain can be represented uniquely by some direct system of abelian o-groups equipped with a little extra structure. From a general viewpoint our representation theorem contributes to the structural

Key words and phrases. Involutive residuated lattices; lack of postulating idempotency, divisibility, and integrality; construction; representation; direct system; abelian o-groups.

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description of residuated lattices which are *neither integral nor naturally ordered nor idempotent*. Our result is reminiscent to the representation theorem of a subclass of inverse semigroups, called Clifford semigroups. Every Clifford semigroup is isomorphic to a strong semilattice of groups [13, 20], and the starting point of the strong semilattice construction is a family of groups indexed by a semilattice and equipped with certain homomorphisms. In our case the starting point is a family of abelian o-groups indexed by a chain and equipped with certain homomorphisms. The definition of the product in the strong semilattice construction is of the same Płonka-fashion [18] as in our construction. However, in our case we first need to deform the abelian o-groups and we also need to handle the extra structure: we have to properly define the ordering etc.

The basic insight in obtaining our representation theorem is the decomposition of the algebra with respect to its so-called layers determined by the so-called local unit function of the algebra (a concept that has been introduced in [9]), and observing that thus defined layer algebras are either cancellative or there exists a canonical homomorphism with a cancelative image from which the layer algebra can be recovered. The interested reader might want to consult [10] or https://sites.google.com/view/nonclassicallogicwebinar/talks (Jan 8, 2021) for a more detailed account.

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An extended class of non-classical models of ZF

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Abstract

In this talk, we explore the construction of models for non-classical set theories. The methodology we use follows the presentation of [1] and its extension to non-classical contexts as in [3], [2] and [4]. In particular, we build algebra-valued models on top of linear Heyting algebras expanded with a unary operator which we will be using to interpret negation in the language of set theory. We show that given a suitable choice of this unary operator, we can build paraconsistent, paracomplete and paradefinite models of ZF. Moreover, these models seem particularly interesting due to their stable ontology and their proximity with their classical cousins, i.e., Boolean-valued models.

More specifically, we go on to show that each of these models validates a different set of sentences in the language of set theory (they are non-$\in$-elementarily equivalent with each other) and that the internal logic of these models matches actually the logic of the underlying algebras. We show, as well, that a certain class of these models validates Leibniz's law of indiscernibility of identicals, thus, opening up the possibility of constructing quotient models out of these algebra-valued models. Finally, we conclude by talking about the prospects and applications of these non-classical models of ZF.

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Unitary Menger Systems of Idempotent Cyclic Multiplace Functions

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Abstract

Multiplace functions and their algebras called Menger algebras have been widely studying in various fields of universal algebra. Based on the theory of many-sorted algebra, the primary aim of this paper is to present the ideas of Menger systems and Menger systems of full multiplace functions which are natural generalizations of Menger algebras and Menger algebras of n-ary operations, respectively. A specific type of n-ary operations, which is called idempotent cyclic generated by cyclic terms, are provided. The Menger algebra of such n-ary operation defined on a fixed set is constructed and some algebraic properties are investigated. Particularly, we provide the necessary and sufficient conditions in which the abstract Menger algebra and the Menger algebras of such n-ary operations can be isomorphic.

Keywords: Menger algebra, multiplace function, cyclic term.

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Associative posets

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For every poset $\mathbf{P} := \langle P, \leqslant \rangle$ one can define a product $\cdot$ such that

$$a \leqslant b \iff a \cdot b = a. \tag{1}$$

holds. This $\cdot$ can be chosen to be commutative if $\mathbf{P}$ is not the two-element antichain. The prime examples of such posets are meet-semilattices $\mathbf{P}$, characterized by the fact that we can choose $\cdot$ to be commutative and associative.

A poset is said to be *associative* if it admits an associative operation satisfying (1). It is easy to see that the resulting algebra is a right regular band [1], *posemigroups* for short.

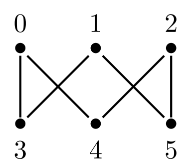

Figure 1: A non associative poset.

The general problem of characterizing associative posets is not trivial (in a personal communication, B. Steinberg informed us that determining if a finite poset is associative is NP-hard). We show some results in this direction.

Theorem 1. *The class of associative posets in the signature* $\{\leqslant\}$ *is not a first order class (although it is closed under ultraproducts).*

A (*downwards-growing*) *tree* $\mathbf{T} = (T, \leqslant)$ is a poset with a top element such that $\{y \in T : x \leqslant y\}$ is linearly ordered for each $x \in T$. A tree is *well-founded* if every (maximal) chain is finite.

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Theorem 2. *The following are equivalent:*

- *Every well-founded tree is associative.*
- *The axiom of choice.*

The previous equivalence is a consequence of the following result:

Theorem 3. *Let* $\mathbf{P}$ *be a poset such that there exists a well-founded tree* $\mathbf{T}$ *and a surjective homomorphism* $f : \mathbf{P} \to \mathbf{T}$ *which satisfies:*

- $f(x) < f(y) \Longrightarrow x < y$
- *For every* $x \in \mathbf{T}$, $f^{-1}(x)$ *is an associative poset.*
- *If* x *is a minimal element of* $\mathbf{T}$ *then* $f^{-1}(x)$ *has a minimum.*

then $\mathbf{P}$ *is associative.*

Moreover, the results above work more generally for downwards-growing trees all of whose (possibly infinite) chains are well-ordered by $\leqslant$.

On another tack, we generalized the results on semilattices contained in [2] obtaining "inner" direct product representations for posemigroups having a central (commuting) element. In order to do this, we define generalized notions of order filters and of direct sum of filters of posemigroups.

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A CHAIN OF ADJUNCTIONS FOR BA AND ANY VARIETY GENERATED BY A SEMI-PRIMAL BOUNDED LATTICE EXPANSION.

ALEXANDER KURZ, WOLFGANG POIGER, AND BRUNO TEHEUX

Recall that a finite algebra $\mathbf{L}$ is called *semi-primal* if every operation $f\colon L^n \to L$ which preserves subalgebras is a term-function on $\mathbf{L}$. Let $\mathbf{L}$ be a semi-primal bounded lattice expansion and $\mathcal{A}$ be the variety generated by $\mathbf{L}$. We denote by $\mathsf{Stone}_{\mathbf{L}}$ the category whose objects (X, v) are given by a Stone space X and a map $v\colon X \to \mathbb{S}(\mathbf{L})$ such that $v^{-1}(\mathbf{S})$ is a closed subspace of X for every $\mathbf{S} \in \mathbb{S}(\mathbf{L})$, and whose arrows $f\colon (X, v) \to (Y, v')$ are the continuous maps $f\colon X \to Y$ that satisfy $v'(f(x)) \subseteq v(x)$ for every $x \in X$.

The theory of natural duality [1] provides a dual equivalence

$$\begin{tikzcd} \mathsf{Stone}_{\mathbf{L}} \arrow[r, bend left, "P"] & \mathcal{A} \arrow[l, bend left, "S"] \end{tikzcd},$$

where P and S are given by homming into $\mathbf{L}$ (we consider L as an element of $\mathsf{Stone}_{\mathbf{L}}$ by equipping it with the map $v\colon L \to \mathbb{S}(\mathbf{L})$ where $v(x) = \langle x \rangle$ for every $x \in L$). In the particular case where $\mathbf{L}$ is the 2-element Boolean algebra, we recover Stone duality for the category BA of Boolean algebas. In that case, we denote P by Π and S by Σ.

Let $\mathcal{U} : \mathsf{Stone}_{\mathbf{L}} \to \mathsf{Stone}$ be the forgetful functor. Then we can define two adjoints $V^{\top} \dashv \mathcal{U} \dashv V^{\perp}$ by

$$V^{\top}(X) = (X, v^{\top}) \text{ where } v^{\top}(x) = \mathbf{L} \text{ for all } x \in X,$$
$$V^{\perp}(X) = (X, v^{\perp}) \text{ where } v^{\perp}(x) = \langle \varnothing \rangle \text{ for all } x \in X.$$

The functor $V^{\perp}$ has a right-adjoint $C : \mathsf{Stone}_{\mathbf{L}} \to \mathsf{Stone}$ defined by

$$C(X, v) = \{x \in X \mid v(x) = \langle \varnothing \rangle\}.$$

Now, walking through the equivalences, we can find the dual of $\mathcal{U}$ which is a functor $\mathfrak{B} : \mathcal{A} \to \mathsf{BA}$ and the corresponding chain of adjunctions. The whole situation can be summarized in the diagram of Fig. 1.

In the case where $\mathbf{L}$ is a finite MV-chain Ł_n (and thus $\mathcal{A} = \mathsf{MV_n}$) the functor $\mathfrak{B}$ is the Boolean skeleton [3], a concept which can be generalized [2] for $\mathcal{A}$ as follows. First, note that for all $\ell \in \mathbf{L}$ the map $T_\ell : \mathbf{L} \to \mathbf{L}$ defined by

$$T_\ell(x) = \begin{cases} 1 & \text{if } x = \ell \\ 0 & \text{if } x \neq \ell \end{cases}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathsf{Stone}_{\mathbf{L}} \arrow[r, bend left=8, "P"] \arrow[d, bend right=8, "\mathcal{U}" description] \arrow[d, bend left=30, "C"] & \mathcal{A} \arrow[l, bend left=8, "S"] \arrow[d, bend right=8, "\mathfrak{B}" description] \arrow[d, bend left=30, "Q"] \\
\mathsf{Stone} \arrow[r, bend left=8, "\Pi"] \arrow[u, bend left=30, "V^{\top}"] \arrow[u, bend right=8, "V^{\perp}" description] & \mathsf{BA} \arrow[l, bend left=8, "\Sigma"] \arrow[u, bend left=30, "\mathfrak{M}"] \arrow[u, bend right=8, "J" description]
\end{tikzcd}$$

$V^{\top} \dashv \mathcal{U} \dashv V^{\perp} \dashv C \qquad \mathfrak{M} \vdash \mathfrak{B} \vdash J \vdash Q$

FIGURE 1. Chain of adjunctions

is term-definable since it preserves subalgebras. Then, for every $\mathbf{A} \in \mathcal{A}$, the set $\mathfrak{B}(\mathbf{A}) := \{a \in A \mid T_1(a) = a\}$ forms a Boolean algebra with the lattice operations inherited from $\mathbf{L}$ and negation T_0.

One crucial fact about $\mathfrak{B}$ is that for every $\mathbf{A} \in \mathcal{A}$, the map that restricts any $u \in \mathcal{A}(\mathbf{A}, \mathbf{L})$ to $\mathfrak{B}(\mathbf{A})$ is a homeomorphism between $\mathcal{A}(\mathbf{A}, \mathbf{L})$ (the underlying topological space of $S(\mathbf{A})$) and $\mathsf{BA}(\mathfrak{B}(\mathbf{A}), \mathbf{2})$.

During the talk we will show how to construct and interpret $\mathfrak{M}$, J and Q in the finite case, and how to extend them to $\mathcal{A}$ and BA.

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Modal Logics of Locally Compact Ordered Spaces

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Joint work with: Guram Bezhanishvili, Nick Bezhanishvili, and Jan van Mill

Abstract: The classic McKinsey-Tarski theorem states that when the modal box is interpreted as topological interior, the logic of a crowded (dense-in-itself) metrizable space is Lewis's well-known modal system $\mathsf{S4}$. The McKinsey-Tarski theorem, in conjunction with Telgarsky's theorem, yields a characterization of exactly which modal logics arise as the logic of an arbitrary metrizable space. A metrizable space can be thought of as a generalization of the real line $\mathbb{R}$ equipped with usual topology which is induced by the usual metric.

Alternatively, the usual topology on $\mathbb{R}$ also coincides with the interval topology induced by the usual linear ordering of $\mathbb{R}$. Recall that the topology τ of a linearly ordered topological space (X, τ), or simply a LOTS, is the interval topology induced by some linear ordering of X. Hence, a LOTS generalizes the space $\mathbb{R}$ in a different direction. In contrast to the class of metrizable spaces, the class of LOTS is not closed under taking subspaces. Closing this class under subspaces leads to the concept of a generalized ordered space, or simply a GO-space. The Sorgenfrey line and the long line are typical examples of non-metrizable GO-spaces. Whereas, any Euclidean space of dimension ≥ 2 is a typical example of a metrizable space that is not a GO-space.

We present an analogue of the McKinsey-Tarski theorem by showing that the logic of a crowded locally compact GO-space is $\mathsf{S4}$. Our result, in conjunction with a generalization of Herrlich's result, yields a characterization of those modal logics that arise as the logic of an arbitrary locally compact GO-space. In fact, such logics coincide with those aforementioned logics arising from metrizable spaces. Thus, despite the class of metrizable spaces and the class of locally compact GO-spaces being incomparable, these two classes lead to the same logical invariants.

Canonical formulas for IK4

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The method of canonical formulas is a uniform axiomatization method applied predominately in the context of intuitionistic and transitive modal logic. The method axiomatizes logics by encoding the structure of finite algebras or frames. Algebraically, the method relies on a suitable locally finite reduct. Traditionally, the type of the reduct is encoded fully for each algebra and the remaining operators of the full type depend on parameters fed to the formula. We will extend the method to the domain of transitive intuitionistic modal logic (IK4) by introducing some asymmetry to this standard approach. The difficulty in this setting is that there is no established locally finite reduct that can be extended back into the full type of modal Heyting algebras. However, we are able to use the variety of bounded distributive lattices to obtain finite transitive modal Heyting algebras. By generalizing the method of canonical formulas to encode box halfway we can use finite transitive algebras as refutation patterns for all transitive intuitionistic modal logics.

Moreover, by mirroring stable canonical formulas along the axis of the introduced asymmetry we obtain canonical formulas for the lax logic – an intuitionistic modal logic that formalizes nuclei of pointless topology and has applications in formal hardware verification. We will investigate the *steady canonical formulas* that axiomatize lax logics more thoroughly and use them to characterize a class of lax logics that is structurally very similar to the class of subframe logics in the intuitionistic setting. For example, all *steady logics* have the finite model property and are generated by a class closed under (steady) subframes. Furthermore, we look at translations of intermediate logics into lax logics and show a number of preservation results using steady canonical formulas. In particular, we prove a lax analog for the Dummett-Lemmon conjecture that the least modal companion of each Kripke-complete intermediate logic is Kripke-complete.

Deciding dependence

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In [6] we consider a generalized algebraic version of the notion of dependence introduced in [1] by De Jongh and Chagrova for intuitionistic propositional logic IPC, where formulas $\varphi_1, \dots, \varphi_n$ are called IPC-*dependent* if there exists a formula $\psi(p_1, \dots, p_n)$ such that $\vdash_{\mathsf{IPC}} \psi(\varphi_1, \dots, \varphi_n)$ and $\nvdash_{\mathsf{IPC}} \psi(p_1, \dots, p_n)$, otherwise IPC-*independent*.

Let $\mathcal{L}$ be an algebraic language and let $\mathcal{V}$ be any variety (equational class) of $\mathcal{L}$-algebras. We denote by $\mathrm{Tm}(\bar{x})$ the set of $\mathcal{L}$-terms over a set $\bar{x}$ and call $t_1, \dots, t_n \in \mathrm{Tm}(\bar{x})$ $\mathcal{V}$-*dependent* if there exists an equation $\varepsilon(y_1, \dots, y_n)$ such that $\mathcal{V} \vDash \varepsilon(t_1, \dots, t_n)$ and $\mathcal{V} \nvDash \varepsilon$, otherwise $\mathcal{V}$-*independent*.

For example, if $\mathcal{V}$ is the variety of vector spaces over some field $\mathbf{K}$, then the notion of $\mathcal{V}$-independence coincides with the usual notion of independence in linear algebra. Just observe that terms $t_1, \dots, t_n$ in the language with the usual group operations and scalar multiplication are $\mathcal{V}$-independent if, and only if, for any equation $\lambda_1 y_1 + \cdots + \lambda_n y_n \approx 0$ with $\lambda_1, \dots, \lambda_n \in K$,

$$\mathcal{V} \vDash \lambda_1 t_1 + \cdots + \lambda_n t_n \approx 0 \implies \mathcal{V} \vDash \lambda_1 y_1 + \cdots + \lambda_n y_n \approx 0,$$

and that $\mathcal{V} \vDash \lambda_1 y_1 + \cdots + \lambda_n y_n \approx 0$ if, and only if, $\lambda_1, \dots, \lambda_n = 0$.

We denote by $\mathbf{F}_{\mathcal{V}}(\bar{x})$ the $\mathcal{V}$-free algebra over a set $\bar{x}$ and deliberately confuse terms in $\mathrm{Tm}(\bar{x})$ with the respective elements in $\mathrm{F}_{\mathcal{V}}(\bar{x})$. We also denote by $[n]$ the set $\{1, \dots, n\}$ for any $n \in \mathbb{N}$. Then for any terms $t_1, \dots, t_n \in \mathrm{Tm}(\bar{x})$ and set $\bar{y} = \{y_1, \dots, y_n\}$, the following are equivalent:

(1) $t_1, \dots, t_n$ are $\mathcal{V}$-independent.

(2) The homomorphism $h\colon \mathbf{F}_{\mathcal{V}}(\bar{y}) \to \mathbf{F}_{\mathcal{V}}(\bar{x})$ defined by mapping y_i to t_i for $i \in [n]$ is injective.

$\mathcal{V}$-dependence may also be viewed as a special case of an algebraic notion of dependence introduced by Marczewski in [5] and extensively studied in universal algebra (see [3]) and later in semigroup theory (see [2]). We say that elements $a_1, \dots, a_n$ of an $\mathcal{L}$-algebra $\mathbf{A}$ are *Marczewski-dependent* in $\mathbf{A}$ if there exist $\mathcal{L}$-terms $u(y_1, \dots, y_n), v(y_1, \dots, y_n)$ satisfying

$$u^{\mathbf{A}}(a_1, \dots, a_n) = v^{\mathbf{A}}(a_1, \dots, a_n) \quad \text{and} \quad u^{\mathbf{A}} \neq v^{\mathbf{A}},$$

otherwise *Marczewski-independent* in $\mathbf{A}$. It is not hard to prove that terms $t_1, \dots, t_n \in \mathrm{Tm}(\bar{x})$ are $\mathcal{V}$-dependent if, and only if, $t_1, \dots, t_n$ are Marczewski-dependent in $\mathbf{F}_{\mathcal{V}}(\bar{x})$.

Let us call the problem of determining whether finitely many terms are $\mathcal{V}$-dependent, the *dependence problem for* $\mathcal{V}$. De Jongh and Chagrova proved the decidability of the dependence problem for IPC in [1] using Pitts' constructive proof of uniform interpolation for IPC [7]. More generally, the decidability of the dependence problem for a variety $\mathcal{V}$ with a decidable equational theory can be obtained by a constructive proof of a weaker property than uniform interpolation. That is, the dependence problem for $\mathcal{V}$ is decidable if there exists an algorithm to check for $t_1, \dots, t_n \in \mathrm{Tm}(\bar{x})$ and $\bar{y} = \{y_1, \dots, y_n\}$ whether for any equation $\varepsilon(\bar{y})$,

$$\mathcal{V} \vDash (y_1 \approx t_1 \,\&\, \cdots \,\&\, y_n \approx t_n) \to \varepsilon(t_1, \dots, t_n) \iff \mathcal{V} \vDash \varepsilon.$$

In particular, if the finite set of equations $\{y_1 \approx t_1, \dots, y_n \approx t_n\}$ has a right deductive uniform interpolant $\Sigma(\bar{y})$ (also a finite set of equations), then $t_1, \dots, t_n$ are $\mathcal{V}$-independent if, and only if,

$\mathcal{V} \models \Sigma$. Constructive proofs of deductive uniform interpolation have been obtained for several intermediate, modal, and substructural logics, as well as varieties such as lattice-ordered abelian groups (see [4, 9] for details and references).

Let us also remark that the dependence problem for the variety of semigroups corresponds to the problem of checking whether a given variable-length code is uniquely decodable and can be decided by the Sardinas-Patterson algorithm [8]. Moreover, by the Nielsen-Schreier theorem, every finitely generated subgroup of a finitely generated free group is again a finitely generated free group, and the dependence problem for the variety of groups can be decided by determining the rank of a finitely generated subgroup of a finitely generated free group.

The dependence problem for a variety $\mathcal{V}$ can also be decided if the equational theory of $\mathcal{V}$ is decidable and we can find a finite $\mathcal{V}$*-refuting* set for each $n \in \mathbb{N}$: that is, a finite set Δ_n of equations with variables in $\bar{y} = \{y_1, \ldots, y_n\}$ such that for any equation $\varepsilon(\bar{y})$, if $\mathcal{V} \not\models \varepsilon$ and $\mathcal{V} \models \sigma(\varepsilon)$ for some substitution $\sigma\colon \mathrm{Tm}(\bar{y}) \to \mathrm{Tm}(\omega)$, then also $\mathcal{V} \models \sigma(\delta)$ for some $\delta \in \Delta_n$. If this is the case for a variety $\mathcal{V}$, then the terms $t_1, \ldots, t_n \in \mathrm{Tm}(\bar{x})$ are $\mathcal{V}$-dependent if, and only if, there is an equation δ in a finite $\mathcal{V}$-refuting set Δ_n for n such that $\mathcal{V} \models \delta(t_1, \ldots, t_n)$.

For the (locally finite) variety $\mathcal{DL}at$ of distributive lattices, it is easy to show that a $\mathcal{DL}at$-refuting set of equations is given for each $n \in \mathbb{N}$ by

$$\Delta_n := \Big\{ \bigwedge_{i \in I} y_i \;\leq\; \bigvee_{j \in [n] \setminus I} y_j \mid \emptyset \neq I \subsetneq [n] \Big\}.$$

Similarly, it can be shown that for the (non-locally finite) variety $\mathcal{L}at$ of lattices, a $\mathcal{L}at$-refuting set of equations is given for each $n \in \mathbb{N}$ by

$$\Delta_n := \Big\{ y_i \leq \bigvee_{j \in [n] \setminus \{i\}} y_j \mid i \in [n] \Big\} \;\cup\; \Big\{ \bigwedge_{j \in [n] \setminus \{i\}} y_j \leq y_i \mid i \in [n] \Big\}.$$

Hence the dependence problems for $\mathcal{DL}at$ and $\mathcal{L}at$ are decidable.

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The univeral completion of C(L) and a localic representation of Riesz spaces

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In [1] and [3] we presented the construction of the Dedekind completion of the lattice C(L) of continuous real functions on a completely regular frame L. Our approach not only turned out to be useful to generalize the classical constructions to the pointfree setting, but it actually provided simpler and more direct methods that can be restricted to the classical case of topological spaces in a conservative manner.

The aim of this talk is to present the construction another order completion of C(L), namely the *universal completion*, from a localic perspective. We will describe this completion as the space of continuous real functions on the Booleanization of L. This construction has no counterpart in the classical settings, as the Booleanization of a frame L is not spatial in general. Not even when L is spatial. As a corollary we show that C(L) is universally complete if and only if L is almost Boolean. The restriction of this result to the classical case provides a characterization of topological spaces X such that their corresponding space C(X) of continuous real functions is universally complete. It is worth to be noted that, although there is a long history of deep results which relate metric, algebraic and order theoretic properties of spaces of continuous real functions to properties of the underlying topological spaces, to the best of the authors' knowledge, this spatial theorem is new.

Finally we will give account of a localic version of the classical Maeda-Ogasawara-Vulikh representation theorem for Archimedean Riesz spaces with weak unit and use it to show that the universal completion of an Archimedean Riesz spaces with weak unit can be described as the space of continuous real functions on a Boolean frame.

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VARIETIES OF PBZ*–LATTICES

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PBZ–lattices* are bounded lattices endowed with two unary operations:

- an involution $\cdot'$, called *Kleene complement*, which satisfies the *Kleene condition*: $x \wedge x' \leq y \vee y'$ (and thus the bounded involution lattice reduct of any PBZ*–lattice is a pseudo–Kleene algebra), as well as paraorthomodularity (which becomes an equational condition under the other axioms of PBZ*–lattices),
- and the *Brouwer complement*, which reverses order, is smaller than the Kleene complement, and satisfies only one of the De Morgan laws, along with condition $(*)$, which is a weakening of the other De Morgan law (called *Strong De Morgan*), obtained from it by replacing one of the variables with the Kleene complement of the other.

These algebras arise in the study of Quantum Logics and they include orthomodular lattices with an extended signature (with the two complements coinciding), as well as antiortholattices (whose Brouwer complements are trivial). From the other algebras of Quantum Logics, PBZ*–lattices present the advantage of forming a variety, that we denote by $\mathbb{PBZL}^*$.

The lattice of subvarieties of the variety of pseudo–Kleene algebras turns out to be embedded as a poset into the lattice of subvarieties of the variety $\mathbb{SAOL}$ of PBZ*–lattices generated by antiortholattices with the Strong De Morgan property and this poset embedding also reflects order and it allows all members of its image to be axiomatized with respect to $\mathbb{SAOL}$ and thus to $\mathbb{PBZL}^*$ based on the axiomatizations of their preimages with respect to the variety of pseudo–Kleene algebras.

Since orthomodular lattices become PBZ*–lattices with the extended signature, the well–known infinite ascending chain of varieties of modular ortholattices is embedded in the lattice of subvarieties of $\mathbb{PBZL}^*$, but so is the image of this infinite ascending chain through the poset embedding mentioned above. These two infinite ascending chains are disjoint.

Another infinite ascending chain can be found in the lattice of subvarieties of the variety of the distributive PBZ*–lattices, and this chain is disjoint from the previous two.

Moreover, a direct decomposition of an interval in the lattice of subvarieties of $\mathbb{PBZL}^*$ (with the lower bound given by its only atom, namely the variety of Boolean algebras with the same extended signature as its supervariety of orthomodular lattices and the upper bound given by the subvariety of $\mathbb{PBZL}^*$ generated by orthomodular lattices and antiortholattices) turns the previous three pairwise disjoint infinite ascending chains into an infinity of pairwise disjoint infinite ascending chains of varieties of PBZ*–lattices.

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ORDERINGS OF ULTRAFILTERS ON COMPLETE BOOLEAN ALGEBRAS

FRANCESCO PARENTE

Abstract. In the literature, there is currently no agreement on what constitutes an adequate generalization of the Rudin-Keisler ordering to ultrafilters on Boolean algebras. Indeed, two papers [3, 2] independently introduced two candidate notions, whose relation is still totally unexplored.

In this talk, I will compare the two notions and show they need not be equivalent under CH. Furthermore, I will highlight their connection with cofinal types of ultrafilters, as recently investigated in [1].

This is joint work with Jörg Brendle.

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Algebraic Semantics for the Logic of Proofs

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Justification logics are modal-like logics that provide a framework for reasoning about epistemic justifications (see [2] for a survey). The language of justification logics extends the language of propositional logic by justification terms and expressions of the form $t : A$, with the intended meaning "t is a justification for A" or "t is a proof for A." The *Logic of Proofs* LP was the first logic in the family of justification logics, introduced by Artemov in [1]. In this paper we impose a Boolean structure on justification terms of LP. This also gives a Boolean structure on mathematical proofs. The resulting logic is an extension of LP and is denoted by LP^B. Let Var be a set of justification variables, Const be a set of justification constants, and Prop be a set of propositional variables. Justification terms and formulas of LP^B are constructed by the following mutual grammars:

$$t, s ::= c \mid x \mid 0 \mid -t \mid t + s \mid t \cdot s \mid\, !t\,,$$
$$\varphi, \psi ::= p \mid t \approx s \mid \bot \mid \neg\varphi \mid \varphi \vee \psi \mid t : \varphi\,.$$

where $c \in \mathsf{Const}$, $x \in \mathsf{Var}$, $p \in \mathsf{Prop}$, and 0 is a distinguished constant. Define $1 := -0$ and $s \odot t := -(-s + -t)$. The set of axioms of LP^B is an extension of that of LP:

PC. All tautologies of classical propositional logic,
Appl. $s : (\varphi \to \psi) \to (t : \varphi \to (s \cdot t) : \psi)$,
Sum. $s : \varphi \to (s + t) : \varphi\ ,\ s : \varphi \to (t + s) : \varphi$,
jT. $t : \varphi \to \varphi$,
j4. $t : \varphi \to !t : t : \varphi$,

with the following axioms of the Boolean algebra for terms:

B1. $s + t \approx t + s$ $\quad$ $s \odot t \approx t \odot s$
B2. $s + (t + u) \approx (s + t) + u$ $\quad$ $s \odot (t \odot u) \approx (s \odot t) \odot u$
B3. $s + 0 \approx s$ $\quad$ $s \odot 1 \approx s$
B4. $s + (-s) \approx 1$ $\quad$ $s \odot (-s) \approx 0$
B5. $s + (t \odot u) \approx (s + t) \odot (s + u)$ $\quad$ $s \odot (t + u) \approx (s \odot t) + (s \odot u)$

and the following axioms for equality:

Eq1. $t \approx t$
Eq2. $s \approx t \wedge \varphi[x/s] \to \varphi[x/t]$, where $\varphi[x/s]$ denotes the result of substitution of s for x in φ.

The rules of LP^B are Modus ponens MP and the following rule:

Int.
$$\frac{\varphi}{1 : \varphi}$$

Next we introduce an algebraic semantics for LP^B.

Definition 0.1. *Let $\mathcal{T} = (T, 0_\mathcal{T}, -_\mathcal{T}, +_\mathcal{T}, \cdot_\mathcal{T}, !_\mathcal{T})$, where $(T, 0_\mathcal{T}, -_\mathcal{T}, +_\mathcal{T})$ is a Boolean algebra, and $!_\mathcal{T} : T \to T$ and $\cdot_\mathcal{T} : T \times T \to T$ are respectively unary and binary operators on T. An LP^B algebra $\mathcal{A} = (A, \mathrm{o}, \ominus, \oplus, \Box_\alpha, I)_{\alpha \in T}$ over $\mathcal{T}$ consists of a Boolean algebra $(A, \mathrm{o}, \ominus, \oplus)$, and an interpretation*

$$I : \mathit{Const} \cup \{0\} \to T$$

such that $I(0) = 0_\mathcal{T}$, and relations $\Box_\alpha \subseteq A \times A$, for $\alpha \in T$, satisfy the following conditions. For all $X, Y \in A$, and all $\alpha, \beta \in T$:

WO-B. *$\Box_\alpha(X)$ is a well-ordered set, where $\Box_\alpha(X) := \{Y \in A \mid (X, Y) \in \Box_\alpha\}$.*
Al-Appl-B. $\min \Box_\alpha(X \Rightarrow Y) \otimes \min \Box_\beta(X) \leq \min \Box_{\alpha \cdot_{\mathcal{T}} \beta}(Y)$,
Al-Sum-B. $\min \Box_\alpha(X) \oplus \min \Box_\beta(X) \leq \min \Box_{\alpha +_{\mathcal{T}} \beta}(X)$,
Al-jT-B. $X \in \Box_\alpha(X)$,
Al-j4-B. $\min \Box_\alpha(X) \leq \min \Box_{!_{\mathcal{T}}\alpha}(\min \Box_\alpha(X))$,
Al-1-B $\Box_{1_{\mathcal{T}}}(\mathrm{1}) = \{\mathrm{1}\}$*, where* $\mathrm{1} := \ominus \mathrm{0}$.

The class of all LP^B algebras with singleton $\nabla = \{\mathrm{1}\}$ is denoted by $\mathcal{A}_{\mathsf{LP}^B}$.

Definition 0.2. *Let $\mathcal{A} = (A, \mathrm{0}, \ominus, \oplus, \Box_\alpha, I)_{\alpha \in T}$ be an LP^B algebra over $\mathcal{T} = (T, 0_{\mathcal{T}}, -_{\mathcal{T}}, +_{\mathcal{T}}, \cdot_{\mathcal{T}}, !_{\mathcal{T}})$. Any valuation $\theta : \mathsf{Prop} \to A$ can be extended to the assignment $\tilde{\theta}$ on all formulas in the standard way with the following conditions on justification assertions and equalities:*

$$\tilde{\theta}(t : \varphi) = \min \Box_{t_I^v}(\tilde{\theta}(\varphi)),$$
$$\tilde{\theta}(s \approx t) = \begin{cases} \mathrm{1} & \text{if } s_I^v = t_I^v \\ \mathrm{0} & \text{if } s_I^v \neq t_I^v, \end{cases}$$

where t_I^v is the value of the term t with respect to the interpretation I and function $v : \mathsf{Var} \to T$. Validity is defined as truth at all algebras under every valuation and every function $v : \mathsf{Var} \to T$.

In order to prove the completeness of LP^B, we use the Tarski-Lindenbaum algebra construction. In fact, we construct two Tarski-Lindenbaum algebras; one out of formulas and the other out of terms.

Theorem 0.3 (Soundness and completeness). *φ is provable in LP^B iff φ is valid in $\mathcal{A}_{\mathsf{LP}^B}$.*

We then present a counterpart of the Stone's representation theorem in this setting. In contrast with the standard representation theorems we construct two isomorphisms for a given LP^B algebra $\mathcal{A}$ over $\mathcal{T}$: One for the algebra $\mathcal{A}$ and one for the algebra $\mathcal{T}$. This leads to a bi-isomorphism of LP^B algebras.

Definition 0.4. *Let $\mathcal{A} = (A, \mathrm{0}, \ominus, \oplus, \Box_\alpha, I)_{\alpha \in T}$ and $\mathcal{A}' = (A', \mathrm{0}', \ominus', \oplus', \Box_\alpha, I')_{\alpha \in T'}$ be LP^B algebras over $\mathcal{T} = (T, 0_{\mathcal{T}}, -_{\mathcal{T}}, +_{\mathcal{T}}, \cdot_{\mathcal{T}}, !_{\mathcal{T}})$ and $\mathcal{T}' = (T', 0_{\mathcal{T}'}, -_{\mathcal{T}'}, +_{\mathcal{T}'}, \cdot_{\mathcal{T}'}, !_{\mathcal{T}'})$ respectively. A bi-isomorphism between $\mathcal{A}$ and $\mathcal{A}'$ is a pair of Boolean isomorphisms (f, g) such that $f : (A, \mathrm{0}, \ominus, \oplus) \to (A', \mathrm{0}', \ominus', \oplus')$ and $g : (T, 0_{\mathcal{T}}, -_{\mathcal{T}}, +_{\mathcal{T}}) \to (T', 0_{\mathcal{T}'}, -_{\mathcal{T}'}, +_{\mathcal{T}'})$ satisfying the following conditions:*

$$(X, Y) \in \Box_\alpha \text{ iff } (f(X), f(Y)) \in \Box_{g(\alpha)},$$
$$g(I(c)) = I'(c), \qquad \text{for } c \in \mathsf{Const} \cup \{0\},$$
$$g(\alpha \cdot_{\mathcal{T}} \beta) = \alpha' \cdot_{\mathcal{T}'} \beta',$$
$$g(!_{\mathcal{T}}\alpha) = !_{\mathcal{T}'}\alpha'.$$

Definition 0.5. *Given two set algebras $(A, \emptyset, -, \cup)$ and $(B, \emptyset, -, \cup)$, an LP^B set algebra is a structure*

$$\mathcal{A} = (A, \emptyset, -, \cup, \Box_\alpha, I)_{\alpha \in B}$$

over $\mathcal{B} = (B, \emptyset, -, \cup, \cdot_{\mathcal{B}}, !_{\mathcal{B}})$ satisfying the conditions of Definition 0.1.

Theorem 0.6 (Bi-Representation Theorem). *Every LP^B algebra is bi-isomorphic to an LP^B set algebra.*

The above theorem shows that LP^B is complete with respect to set algebras.

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Forcing with $\aleph_1$-Free Groups

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$\aleph_1$-free groups, abelian groups whose countable subgroups are free, are objects of both algebraic and set-theoretic interest. Illustrating this, we note that $\aleph_1$-free groups, and in particular the question of when $\aleph_1$-free groups are free, were central to the resolution of the Whitehead problem as undecidable. In elucidating the relationship between $\aleph_1$-freeness and freeness, we prove the following result: an abelian group G is $\aleph_1$-free in a countable transitive model of ZFC (and thus by absoluteness, in every transitive model of ZFC) if and only if it is free in some generic model extension. We would like to answer the more specific question of when an $\aleph_1$-free group can be forced to be free while preserving the cardinality of the group. For groups of size $\aleph_1$, we establish a necessary and sufficient condition for when such forcings are possible. We also identify both existing and novel forcings which force such $\aleph_1$-free groups of size $\aleph_1$ to become free with cardinal preservation. These forcings lay the groundwork for a larger project which uses forcing to explore various algebraic properties of $\aleph_1$-free groups and develops new set-theoretical tools for working with them.

Reducts of representable relation algebras: finite representability and axiomatisability

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Relation algebras are the kind of Boolean algebras with operators [9]. The additional operators are ;, a binary operator that reflects relational composition, $\smile$, an involution, and $\mathbf{1}$, the identity relation. Relation algebras provide algebraisation of binary relations. One needs to distinguish abstract relation algebras and relation set algebras. By a relation set algebra, we mean a relation algebra on some set of binary relations defined of some non-empty base set.

A relation algebra is called *representable* if it is isomorphic to a subalgebra of some relation set algebra. There are plenty of examples of non-representable relation algebras, so we do not have a uniform representation theorem in contrast to Boolean algebras. Moreover, the class of all representable relation algebras (written as **RRA**) is not finitely axiomatisable. The representability problem for finite algebras is undecidable [7].

For this reason, reducts of relation algebras are of interest, see, e.g., [5]. In particular, if a class of reducts of representable relation algebras has a recursively enumerable axiomatisation and it has the finite representation property, then the representability problem for finite algebras is decidable. Recall that the class of representable reducts of **RRA** has the finite representation property (FRP) if every finite member is representable over a finite base.

In this abstract, we consider two kinds of reducts, residuated semigroups and upper semilattice-ordered semigroups.

A residuated semigroup is an algebra $\langle A, ;, \backslash, /, \leq \rangle$, where $\langle A, ;, \leq \rangle$ is a partially ordered semigroup and $\backslash, /$ are binary operations that obey the residuation property, that is, for all $a, b, c \in A$:

$$a \leq c/b \Leftrightarrow a; b \leq c \Leftrightarrow b \leq a \setminus c \tag{1}$$

A representation of a residuated semigroup $\mathcal{A}$ is an embedding $h : \mathcal{A} \hookrightarrow 2^{X \times X}$ (where $X \neq \emptyset$ is called the base set) such that, for all $a, b \in A$:

1. $a \leq b$ iff $h(a) \subseteq h(b)$
2. $h(a; b) = \{(x, y) \in X^2 \mid \exists z \in X \ (x, z) \in h(a) \ \& \ (z, y) \in h(b)\} = h(a); h(b)$
3. $h(a \setminus b) = \{(x, y) \in X^2 \mid \forall z \in X \ ((z, x) \in h(a) \Rightarrow (z, y) \in h(b))\} = h(a) \setminus h(b)$
4. $h(a/b) = \{(x, y) \in X^2 \mid \forall z \in X \ ((y, z) \in h(b) \Rightarrow (x, z) \in h(a))\} = h(a)/h(b)$

According to Andréka and Mikulás, every residuated semigroup is representable, see [1].

We show that the class of representable residuated semigroup has the FRP using the relational representation of quantales [3] and Dedekind-MacNeille completions for residuated semigroups [4]. As a result, we have the following theorem that provides a positive solution to [8, Problem 19.17].

Theorem 1. *Let $\mathcal{A}$ be a finite representable residuated semigroup, then $\mathcal{A}$ is representable over a finite base.*

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As a corollary, the representability is decidable for finite residuated semigroups since the corresponding theory of representable structures is finitely axiomatisable.

An upper semilattice-ordered semigroup is an algebra $\langle A, ;, + \rangle$, where $\langle A, + \rangle$ is an upper semilattice, $\langle A, ; \rangle$ is a semigroup, and for all $a, b, c \in A$, the following hold:

1. $a; (b + c) = a; b + a; c$
2. $(a + b); c = a; c + b; c$

A representation of an upper semilattice-ordered semigroup $\mathcal{A}$ over the base set X is an embedding $h : \mathcal{A} \hookrightarrow 2^{X \times X}$ such that, for all $a, b \in \mathcal{A}$:

1. $h(a; b) = h(a); h(b)$ (as previously)
2. $h(a + b) = h(a) \cup h(b)$.

There exist non-representable upper semilattice-ordered semigroups, and, moreover, the class of all representable upper semilattice-ordered semigroups is not finitely axiomatisable [2]. We show that the class of all representable upper semilattice-ordered semigroups has a recursively enumerable universal axiomatisation.

Theorem 2. *Let $\mathcal{A}$ be an upper semilattice-ordered semigroup, then there exists an inductively defined set of universal formulas $\Sigma = (\sigma_m)_{m<\omega}$ such that $\mathcal{A} \models \Sigma$ iff $\mathcal{A}$ is representable.*

We characterise representable upper semilattice-ordered semigroups by means of back-and-forth games on networks (see the version of this technique for relation and cylindric algebras here [6]), that is, directed graphs whose edges are labelled with upper cones of a given upper semilattice-ordered semigroup.

We do not know whether the class of upper semilattice-ordered semigroups has the FRP at the time of writing. The author's conjecture is that this problem is likely to have a positive solution. If so, then the problem of representability for finite algebras is also decidable.

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Directed games

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Ehrenfeucht-Fraïssé games is a commonly used technique for establishing decidability. We present a refinement of this technique for many-sorted structures. Let $\mathcal{L}$ be some many-sorted language with sorts indexed by a non-empty set I, and let $\mathscr{A}$ and $\mathscr{B}$ be two $\mathcal{L}$-structures. Let $\bar{s} = (s_1, \ldots, s_n) \in I^n$ for some $n > 0$. We define a two-person game $\mathrm{G}_n^{\bar{s}}(\mathscr{A}, \mathscr{B})$ with n rounds (we call such games directed). Suppose that we are in the i-th round of the game. Player I plays first, and either chooses an element a_i of sort s_i from $\mathscr{A}$ in which case player II must respond by choosing b_i of sort s_i from $\mathscr{B}$, or an element b_i of sort s_i from $\mathscr{B}$ in which case player II must respond by choosing a_i of sort s_i from $\mathscr{A}$. The game stops after round n. Player II wins the game if the partial mapping corresponding to the pairs of elements $\{(a_i, b_i) : 1 \leq i \leq n\}$ is a partial isomorphism from $\mathscr{A}$ to $\mathscr{B}$, otherwise he loses. It is easy to show that if Player II has a winning strategy in $\mathrm{G}_n^{\bar{s}}(\mathscr{A}, \mathscr{B})$, then for all quantifiers $\mathrm{Q}_1, \ldots, \mathrm{Q}_n$ and quantifier-free $\mathcal{L}$-formulas ϕ, the sentence $\mathrm{Q}_1\, x_1^{s_1} \ldots \mathrm{Q}_n\, x_n^{s_n}\, \phi(x_1^{s_1}, \ldots, x_n^{s_n})$ is true in $\mathscr{A}$ if and only if it is true in $\mathscr{B}$ (where each variable $x_i^{s_i}$ is of sort s_i). We describe how directed games can be used to establish the decidability of many interesting classes of sentences in Simple Type Theory and Quine's New Foundations (e.g., increasing sentences, existential strictly-decreasing sentences, three-quantifier sentences).

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Quantale-theoretic tools for the working logician

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An approach to abstract deductive systems by means of quantale modules was proposed by N. Galatos and C. Tsinakis in [3] and eventually developed by the present author [5,6,8]. However, apart from occasional references to logic appeared, for example, in [1, 2, 4], many concrete applications of the order-theoretic apparatus are still lacking.

More precisely, in [3] the authors showed that propositional deductive systems can be fully described by means of their lattices of theories which are modules over the powerset quantales of the monoids of substitutions of their languages. Among other things, they proved that, given two deductive systems over the same language, the existence of a quantale module homomorphism between the corresponding modules of theories is a sufficient condition for the existence of an interpretation of one system into the other. The present author extended that result in two directions [6], namely, to the case of systems with different underlying languages and to both weaker and stronger forms of interpretations.

Further possibile applications were pointed out by all of us in those papers, but most of them have not been concretely explored in details. In the present talk, we shall precisely address this issue by proving the results hereafter briefly summarized.

Concerning the algebraic tools, we shall only extend to quantale modules the approach to congruences by means of the so-called saturated elements, already known for quantale congruences.

Definition 1. Let Q be a quantale, M a Q-module, and ϑ be a binary relation on M. An element s of M is called *ϑ-saturated* if, for all $(v, w) \in \vartheta$ and $a \in Q$, $av \leq s \iff aw \leq s$. Let M_ϑ be the set of ϑ-saturated elements of M.

Theorem 2. *Let M be a Q-module, $\vartheta \subseteq M^2$, and $\rho_\vartheta : v \in M \mapsto \bigwedge\{s \in M_\vartheta \mid v \leq s\} \in M$.*
Then ρ_ϑ is a Q-module nucleus whose image is M_ϑ. Moreover, M_ϑ, with the structure induced by ρ_ϑ, is isomorphic to the quotient of M w.r.t. the congruence generated by ϑ.

According to [6, Theorem 6.7], each quantale morphism $h : Q \to R$ induces an adjoint and co-adjoint functor $(\ )_h : R\text{-}\mathcal{M}od \to Q\text{-}\mathcal{M}od$, whose left adjoint is $R \otimes_Q \_$. Such a result suggested the possibility of expanding the language of a deductive system, but the main problem was to maintain the possibility of recovering the original lattice of theories within the new one. The expansion is an obvious application of the tensor products, while the preservation of the original system is the content of the following result.

Theorem 3. *Let $\mathcal{L}_1$ be an expansion of $\mathcal{L}$, $i : \mathscr{P}(\Sigma_\mathcal{L}) \to \mathscr{P}(\Sigma_{\mathcal{L}_1})$ the natural embedding, and $(D, \vdash)$ a deductive system in the language $\mathcal{L}$ whose module of theories shall be denoted by Th. Then Th is isomorphic to a $\mathscr{P}(\Sigma_\mathcal{L})$-submodule of $(\mathscr{P}(\Sigma_{\mathcal{L}_1}) \otimes_{\mathscr{P}(\Sigma_\mathcal{L})} Th)_i$.*

The failure of the amalgamation property for quantales is an obvious consequence of the one for semigroups and monoids. However, we prove that quantales of substitutions of propositional languages can actually be amalgamated, even in a very easy way.

Proposition 4. *Let $\mathcal{L}_1 = (L_1, \nu_1)$ and $\mathcal{L}_2 = (L_2, \nu_2)$ be propositional languages with a common fragment $\mathcal{L} = (L_1 \cap L_2, \nu)$, where $\nu = \nu_1 \restriction_{L_1 \cap L_2} = \nu_2 \restriction_{L_1 \cap L_2}$. Hence we have an amalgam of monoids $\Sigma_{\mathcal{L}_1} \overset{i_1}{\leftarrow} \Sigma_{\mathcal{L}} \overset{i_2}{\rightarrow} \Sigma_{\mathcal{L}_2}$ which is strongly embeddable in $\Sigma_{\mathcal{L}_1 \cup \mathcal{L}_2}$.*

Consequently, the quantale $\mathscr{P}(\Sigma_{\mathcal{L}_1 \cup \mathcal{L}_2})$ is the strong amalgamated coproduct of $\mathscr{P}(\Sigma_{\mathcal{L}_1})$ and $\mathscr{P}(\Sigma_{\mathcal{L}_2})$ w.r.t. $\mathscr{P}(\Sigma_{\mathcal{L}})$.

Last, we will show various techniques for combining different deductive systems thus obtaining a new one which encompasses the consequence relations of all of the initial systems. In particular, we will prove that the whole machinery is flexible enough not only to handle different situations but also to propose alternative constructions for each single case. We can briefly resume such constructions as follows

- Two systems with the same underlying language can always be merged either using $\mathcal{Q}$-module (possibly amalgamated) coproduct or by "doubling" the language. Module coproduct gives a system on some kind of two-sorted syntactic constructs and has the advantage of working in a pretty standard way, in the sense that its concrete implementation does not depend on the particular systems at hand. On the other hand, the construction based on the coproduct of quantales is definitely more effective when dealing with systems of the same type (both on formulas, equations, or sequents).
- Merging systems over different languages is obviously more laborious using module (possibly amalgamated) coproduct. On the contrary, the construction with quantale coproducts is essentially the same as the single-language case, but the capacity of taking into account possible isomorphic pieces of the modules of theories depends on the existence or not of some (possibly partial) translation and interpretation.

Besides the obvious interest for algebraic logicians, a standard method for merging different deductive systems may be useful in various applications in automated reasoning, such as automated theorem provers or decision-making processes.

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Resolving finite indeterminacy
A definitive constructive universal prime ideal theorem

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Abstract algebra abounds with ideal objects and the invocations of transfinite methods, typically Zorn's Lemma, that grant those object's existence. Put under logical scrutiny, ideal objects often serve for proving the semantic conservation of additional non-deterministic sequents, that is, with finite but not necessarily singleton succedents. By design, dynamical methods in algebra [2, 4, 9] allow to eliminate the use of ideal methods by shifting focus from semantic model extension principles to syntactical conservation theorems, which move has enabled Hilbert's Programme for modern algebra.

A paradigmatic case, which to a certain extent has been neglected in dynamical algebra proper, is Krull's Lemma for prime ideals. A particular form of this asserts that a multiplicative subset of a commutative ring contains the zero element if and only if the set at hand meets every prime ideal. Prompted by Kemper and Yengui's novel treatment of valuative dimension [3], the authors of the present note together with Yengui have recently put Krull's Lemma under constructive scrutiny [7]. This development has eventually helped to unearth the underlying general phenomenon [8]: Whenever a certificate is obtained by the semantic conservation of certain additional non-deterministic axioms, there is a finite labelled tree belonging to a suitable inductively generated class which tree encodes the desired computation.

Recall that a *consequence relation* on a set S is a relation $\rhd$ between finite subsets[1] and elements of S, which is *reflexive, monotone* and *transitive*:

$$\frac{U \ni a}{U \rhd a}\ (\mathrm{R}) \qquad\qquad \frac{U \rhd a}{U, V \rhd a}\ (\mathrm{M}) \qquad\qquad \frac{U \rhd b \quad U, b \rhd a}{U \rhd a}\ (\mathrm{T})$$

where the usual shorthand notations are in place. The *ideals* of a consequence relation are the subsets $\mathfrak{a}$ of S *closed* under $\rhd$ in the sense that if $\mathfrak{a} \supseteq U$ and $U \rhd a$, then $a \in \mathfrak{a}$. If U is a finite subset of S, then its *closure* is an ideal:

$$\langle U \rangle = \{\, a \in S \mid U \rhd a \,\}$$

A decisive aspect of our approach is the notion of a regular set for certain non-deterministic axioms over a fixed consequence relation, where by a *non-deterministic axiom* on S we understand a pair (A, B) of finite subsets of S. A subset $\mathfrak{p}$ of S is *closed* under (A, B) if $A \subseteq \mathfrak{p}$ implies $\mathfrak{p} \between B$, where the latter is to say that $\mathfrak{p}$ and B have an element in common.

Let $\mathcal{E}$ be a set of non-deterministic axioms over $\rhd$. A *prime ideal* is an ideal of $\rhd$ that is closed under every element of $\mathcal{E}$. For instance, if $\rhd$ denotes deduction, and $\mathcal{E}$ consists of all pairs $(\emptyset, \{\, \varphi, \neg\varphi \,\})$ for sentences φ, then the (prime) ideals are exactly the (complete) theories.

A subset R of S is *regular* with respect to $\mathcal{E}$ if, for all finite subsets U of S and all $(A, B) \in \mathcal{E}$,

$$\frac{(\forall b \in B)\, \langle U, b \rangle \between R}{\langle U, A \rangle \between R}$$

Abstracted from the multiplicative subsets occurring in Krull's Lemma, regular sets haved proved the right concept for our *Universal Prime Ideal Theorem*:

[1] We understand a set to be *finite* if it can be written as $\{\, a_1, \ldots, a_n \,\}$ for some $n \geqslant 0$.

Proposition 1 ($\mathbf{ZFC}$)**.** *A subset R of S is regular if and only if for every ideal $\mathfrak{a}$ we have $R \between \mathfrak{a}$ precisely when $R \between \mathfrak{p}$ for all prime ideals $\mathfrak{p} \supseteq \mathfrak{a}$.*

Regular sets further account for the constructive version of Proposition 1. To this end, given an ideal $\mathfrak{a}$, we next define a collection $T_\mathfrak{a}$ of *finite* labelled trees such that the root of every $t \in T_\mathfrak{a}$ be labelled with a finite subset U of $\mathfrak{a}$, and the non-root nodes with elements of S. The latter will be determined successively by consequences of U along the elements of $\mathcal{E}$.

We understand paths, which necessarily are finite, to lead from the root of a tree to one of its leaves. Given a path π of $t \in T_\mathfrak{a}$, we write $\pi \rhd a$ whenever $U, b_1, \ldots, b_n \rhd a$ where U labels the root of t and $b_1, \ldots, b_n$ are the labels occurring at the non-root nodes of π.

Definition. Let $\mathfrak{a}$ be an ideal. We generate $T_\mathfrak{a}$ inductively according to the following rules:

1. For every finite $U \subseteq \mathfrak{a}$, the trivial tree (i.e., the root-only tree) labelled with U belongs to $T_\mathfrak{a}$.

2. If $(A, B) \in \mathcal{E}$ and if $t \in T_\mathfrak{a}$ has a path π such that $\pi \rhd a$ for every $a \in A$, then add, for every $b \in B$, a child labelled with b at the leaf of π.

We say that $t \in T_\mathfrak{a}$ *terminates* in $R \subseteq S$ if for every path π of t there is $r \in R$ such that $\pi \rhd r$.

Our *Constructive Universal Prime Ideal Theorem* works in (a fragment of) Constructive Zermelo–Fraenkel set theory $\mathbf{CZF}$:

Proposition 2 ($\mathbf{CZF}$)**.** *A subset R of S is regular if and only if for every ideal $\mathfrak{a}$ we have $R \between \mathfrak{a}$ precisely when there is a tree $t \in T_\mathfrak{a}$ which terminates in R.*

We thus uniformise many instances of the dynamical method and generalise the universal proof-theoretic conservation criterion offered before [6], which by Scott–style entailment relations [1] unifies numerous phenomena, e.g. [5].

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Lattice-ordered Effect Algebras with $\sqrt{\prime}$*

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Quantum logic originally started as an intricate algebraic structure of a set of projective operators. Such algebraic structures were later called orthomodular lattices, which were actively investigated. However, most of the attempts to build any adequate logic on the basis of this lattice for various reasons were unsuccessful. By the end of the 20th century, algebras of effects appeared [4], which turned out to be a generalization of many important algebraic structures, including orthomodular lattices. Research in the field of residual algebras of affect appeared to be the most attractive area(?) for modern logicians. This research would (will (?)) make possible to connect quantum logic with other substructural logics.

Research in the field of residual effect algebras appeared to be the most attractive area for modern quantum logicians. This research would will make possible to connect quantum logic with other substructural logics.

Logic of Lattice Effect algebras was introduced in [2] and [5] as logic of residuated lattice effect algebras. The main feature of these logics is the presence of a logical connective of implication, which is unambiguously can be expressed as Sasaki arrow (or Sasaki projection) with good properties. The introduction of an implication with deductive properties and good axiomatizability of Lattice Effect algebras Logic became possible only thanks to a well-constructed residual algebra of effects. But does this approach open up all the possibilities of quantum logic?

The main idea of this work is to extend the residueted effect algebra by the unary operation $\sqrt{\prime}$, double application of which corresponds to involution operation $\prime$. The possibility of using of such operation is actively discussed in [3] and [1]. This report proposes to discuss possible options for the extension of residuated lattice effect algebras by $\sqrt{\prime}$ operation and properties of this extension.

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ON GOLDIE ABSOLUTE DIRECT SUMMANDS IN MODULAR LATTICES

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Abstract. In this paper absolute direct summand(ADS) in lattices is defined and some of its properties in modular lattices are studied. It is shown that in a certain class of modular lattices, the direct sum of two elements has ADS if and only if the elements are relatively injective. As a generalization of ADS the concept of Goldie absolute direct summand in lattices is introduced and studied. It is shown that Goldie ADS property is inherited by direct summands. A necessary and sufficient condition is given for an element of modular lattice to have Goldie ADS.

Keywords: Extending elements, Goldie extending element, Absolute direct summands, Goldie absolute direct summands.

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Modal Logics of Cayley Graphs

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Groups are an example of highly regular mathematical structures. Though some of them may be surprisingly complicated – like the famous Fischer–Griess monster group [2] – in general, their simple yet powerful axioms ensure remarkable fine structure. Caley graphs expressively demonstrate this good behavior of groups. In terms of graph theory, they are vertex-transitive, which means that for any pair of vertices v and u there exists an automorphism f such that $f(v) = u$ [6].

Graphs now are studied in a tremendous number of ways, and one of them is modal logic [1]. With Kripke semantics, one may express graph properties with modal formulas. The concept of frame validity allows modal expression of properties that are true for every vertex of a graph. For example, it is well-known that the formula $AT = \Box A \to A$ is valid in graphs where every vertex has a loop. We thus expect that graphs that have many vertex-invariant features allow for expressive modal characterization.

In this study we figure out if for any family of Cayley graphs there exists an efficiently enumerable modal logic that is complete with respect to them.

Suppose that G is a group and S is a generating set of G. The Cayley graph $\Gamma = \Gamma(G, S)$ is a directed colored graph where vertices represent the elements of G and for each $g \in G$, $s \in S$ there is an edge of color c_s between g and gs [3]. We interpret a Cayley graph as a Kripke frame with the two following approaches:

1. A *simple Cayley frame* for a Cayley graph $\Gamma(G, S)$ is a Kripke frame $F = (G, \mathcal{R})$ where the relation is identified with the edge set of $\Gamma(G, S)$. The modal language for simple Cayley frames has one modality $\Box$ that is associated with $\mathcal{R}$.

2. A *multimodal Cayley frame* for a Cayley graph $\Gamma(G, S)$ is a frame $F = (G, \{\mathcal{R}_s\}_{s \in S})$ where each relation $\mathcal{R}_s$ corresponds to the set of edges of color c_s in $\Gamma(G, S)$. In the modal language for multimodal Cayley frames, for each $s \in S$ there is a modality $\Box_s$ that is associated with $\mathcal{R}_s$.

Our goal is to describe the modal logics of Cayley graphs with a given signature, namely a fixed generator set S, $|S| = n$. This task breaks down into the following problems:

1. What is the logic of the class $Cayley_S$ of all simple Cayley frames of this signature?

2. For a given class of Cayley graphs of the same signature, what is the logic of their simple Cayley frame?

3. What is the logic of the class $CayleySimple_n$ of all multimodal Cayley frames of this signature?

4. For a given class of Cayley graphs of the same signature, what is the logic of their multimodal Cayley frame?

The task of describing logics of an arbitrary class of Cayley graphs appears too general. To approach this problem we use group presentations [4]. We study if given a group presentation one can find the logic of its Cayley graph. Having this problem solved, we may find the logic of some simple family of Cayley graphs as the intersection of logics of all members of this family.

The following theorems present our findings.

Theorem 1. $\mathrm{Log}(Cayley_S) = SL_S = K + \bigwedge_{s\in S} \Box_s p \leftrightarrow \Diamond_s p$.

This logic is known as the logic of all Kripke frames where each modality represents a function on the set of possible worlds. It was also studied as 'the logic of tomorrow' [5] [7].

Theorem 2. *Let $G = \langle S \mid R\rangle$, where S is finite and $S = S^{-1}$. Let $F = (G, \{\mathcal{R}_s\}_{s\in S})$ be the multimodal Cayley frame for G. Then*

$$\mathrm{Log}\ F = SL_S + \bigwedge_{r\in R} \phi_r,$$

where ϕ_r is a translation of the relation $r \in R$, constructed as follows:

1. *$e^{\#}$ is an empty word;*
2. *$s^{\#} := \Diamond_s$ for any $s \in S$;*
3. *$(As)^{\#} := A^{\#}s^{\#}$ for any word A in S and any $s \in S$;*
4. *$(A = B)^{\#} := (A^{\#}p \leftrightarrow B^{\#}p)$ for any words A, B in S;*
5. *$\phi_r := r^{\#}$.*

Theorem 3. $\mathrm{Log}(CayleySimple_n) = K + AD + Alt_n$.

This is the logic that defines the frame condition $\forall x\ 1 \leq |\mathcal{R}(x)| \leq n$.

Theorem 4. *There exists no modal logic L and translation function τ that maps group relations to modal formulas, such that for any group $G = \langle S \mid R\rangle$ and its simple Cayley frame $F = (G, \mathcal{R})$,*

$$Log(F) = L + \bigwedge_{r\in R} \tau(r).$$

Despite the general negative result of Theorem 4, we discovered a construction for a family of formulas that are nontrivially valid a simple Cayley frame. The formulas use an infinite number of propositional letters that cannot be reduced.

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A new perspective on quantum substructural logics

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Joint work with Wesley Fussner

Orthomodular lattices have been studied extensively as an algebraic foundation for reasoning in quantum mechanics (see, e.g., [2]), and their assertional logic is among the most prominent quantum logics (see, e.g., [4]). However, the algebraic theory of orthomodular lattices suffers from several obstacles that have inhibited its study. For instance, the variety of orthomodular lattices is not closed under canonical [6] or even MacNeille completions [5]. This makes it more difficult to tackle decidability questions given that, for example, existing methods showing the finite model property for ortholattices use MacNeille completions [1].

One plausible approach to address these questions is to embed orthomodular lattices in an environment that is more amenable from the perspective of completions, decidability, proof theory, and related issues. In particular, efforts have been made (e.g. [3]) to import the techniques and tools proper of substructural logics.

We contribute to the research in this direction with the introduction of *residuated ortholattices*. We provide several characterizations of orthomodular lattices within this environment. One key feature, in contrast to ortholattices, is that residuated ortholattices are the equivalent algebraic semantics of their assertional logic. Furthermore, we exhibit a relative decidability result for (the equational theory of) orthmodular lattices through a double-negation translation into residuated ortholattices.

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Monadic Residuated Lattices

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One-variable fragments of first-order logics can be studied under a modal guise: any (unary) predicate $P(x)$ can be identified with a propositional variable p, and quantifiers $(\forall x)$ and $(\exists x)$ with unary modalities $\Box$ and $\Diamond$, respectively. This comes with a number of advantages. Not only can we apply the well-developed theory of modal logic, we can also study such fragments algebraically. Such an algebraic study of one-variable fragments of first-order logics was initiated by Halmos in the 1950s [10]. He defined monadic Boolean algebras as the algebraic semantics for the one-variable fragment of first-order classical logic. Since then, various generalizations have appeared in the literature, typically to study other one-variable fragments. Examples include monadic Heyting algebras as algebraic semantics to the one-variable fragment of first-order intuitionistic logic [12], monadic MV-algebras to do the same for first-order Łukasiewicz logic [13], monadic Gödel algebras for first-order Gödel logic [9, 4], and monadic abelian ℓ-groups for first-order Abelian logic [11].

All such algebras have more in common than the adjective "monadic". This work is a first general algebraic investigation into such commonalities. In order to do so, we work in the framework of (first-order) substructural logics whose Gentzen-style proof system admits the rule of exchange. Algebraically, we work with *commmutative pointed residuated lattices* (or FL_e*-algebras*), that is, with algebras $\langle A, \wedge, \vee, \cdot, \rightarrow, f, e\rangle$ such that $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, e\rangle$ is a commutative monoid, and $\cdot$ is residuated in the underlying lattice order with residual $\rightarrow$, i.e., for all $a, b, c \in A$, $a \cdot b \leq c \iff a \leq b \rightarrow c$ (see [8] for details).

We define a monadic version of FL_e-algebras, both to study all monadic algebras in a more general framework, and as a first attempt at a general approach to one-variable fragments of first-order substructural logics.

Definition 1. A *monadic* FL_e*-algebra* is an algebra $\langle A, \wedge, \vee, \cdot, \rightarrow, f, e, \Box, \Diamond\rangle$ (also written $\langle \mathbf{A}; \Box, \Diamond\rangle$) such that $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, f, e\rangle$ is an FL_e-algebra and for all $a, b, c \in A$,

(L1)	$\Box a \leq a$	(L5)	$\Box e = e$
(L2)	$\Box\Diamond a = \Diamond a$	(L6)	$\Box(a \rightarrow \Box b) = \Diamond a \rightarrow \Box b$
(L3)	$\Box(a \wedge b) = \Box a \wedge \Box b$	(L7)	$\Box(\Box a \rightarrow b) = \Box a \rightarrow \Box b$.
(L4)	$\Box f = f$		

These monadic FL_e-algebras indeed generalize all mentioned monadic algebras. Additionally, they are sound with respect to the (one-variable fragment of the) minimal first-order substructural logic that admits exchange QFL_e, defined proof-theoretically in, e.g., [1, 7].

Moreover, monadic FL_e-algebras admit an elegant characterization. Let $\mathbf{A}$ be any FL_e-algebra. A subalgebra $\mathbf{A}_0$ of $\mathbf{A}$ is called *relatively complete* if for all $a \in A$, the sets $\{c \in A_0 \mid c \leq a\}$ and $\{c \in A_0 \mid a \leq c\}$ contain a greatest and least element, respectively. Such relatively complete subalgebras were already considered by Halmos, and appeared in works on, e.g., monadic Heyting algebras [2] and monadic MV-algebras [6]. For a relatively complete subalgebra $\mathbf{A}_0$ of $\mathbf{A}$, we can define modalities $\Box_0 a \coloneqq \max\{c \in A_0 \mid c \leq a\}$ and $\Diamond_0 a \coloneqq \min\{c \in A_0 \mid a \leq c\}$. Then $\langle \mathbf{A}; \Box_0, \Diamond_0\rangle$ is a monadic FL_e-algebra. Conversely, let $\langle \mathbf{A}; \Box, \Diamond\rangle$ be a monadic FL_e-algebra and define $\Box A \coloneqq \{\Box a \mid a \in A\}$. Then $\Box A$ is in fact a relatively complete subuniverse of $\mathbf{A}$, and we write $\Box\mathbf{A}$ to denote the corresponding algebra. We obtain the following correspondence.

Theorem 2. *There exists a one-to-one correspondence between monadic FL_e-algebras and pairs $\langle \mathbf{A}, \mathbf{A}_0 \rangle$ of FL_e-algebras such that $\mathbf{A}_0$ is a relatively complete subalgebra of $\mathbf{A}$.*

Monadic FL_e-algebras hence exactly capture the notion of relative completeness in the context of FL_e-algebras. Moreover, for a monadic FL_e-algebra $\langle \mathbf{A}; \Box, \Diamond \rangle$, the subalgebra $\Box\mathbf{A}$ determines the lattice of congruences of $\langle \mathbf{A}; \Box, \Diamond \rangle$.

Theorem 3. *Let $\langle \mathbf{A}; \Box, \Diamond \rangle$ be a monadic FL_e-algebra. Then the lattice of congruences of $\langle \mathbf{A}; \Box, \Diamond \rangle$ is isomorphic to the lattice of congruences of $\Box\mathbf{A}$.*

This reduces the study of congruences of a monadic FL_e-algebra $\langle \mathbf{A}; \Box, \Diamond \rangle$ to the study of congruences of the FL_e-algebra $\Box\mathbf{A}$. The latter are in many cases easier to study, or well-studied already. Moreover, we obtain corollaries like the following.

Corollary 4. *A monadic FL_e-algebra $\langle \mathbf{A}; \Box, \Diamond \rangle$ is subdirectly irreducible (simple) if and only if $\Box\mathbf{A}$ is subdirectly irreducible (simple).*

These characterizations and reductions can be further exploited. For example, one can generalize the functional representation results obtained in [3] for monadic Heyting algebras and in [5] for monadic BL-, Łukasiewicz and Gödel algebras to a much more general setting.

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The computation of the 768-th Laver table

Joseph Van Name

The n-th Laver table is the unique algebraic structure

$$A_n = (\{1, 2 \ldots, 2^n - 1, 2^n\}, *_n)$$

such that

$$x *_n (y *_n z) = (x *_n y) *_n (x *_n z)$$

and

$$x *_n 1 = x + 1 \mod 2^n$$

whenever

$$x, y, z \in \{1, 2, \ldots, 2^n - 1, 2^n\}.$$

The Laver tables arise in set theory from rank-into-rank embeddings. In the 1990's, Randall Dougherty computed the 48-th Laver table. By enhancing Dougherty's algorithm, we have computed an operation that is presumably the 768-th Laver table. However, since the data file for A_{768} was obtained by repeatedly searching for and correcting instances of non-distributivity, it is an open problem as to whether this computation of the 768-th Laver table is completely correct.

THE AUTOMORPHISM GROUP OF THE FRAÏSSÉ LIMIT OF FINITE HEYTING ALGEBRAS

KENTARÔ YAMAMOTO

We present various results on the automorphism group of the Fraïssé limit of finite Heyting algebras. This extends a long line of research on the automorphism groups of homogeneous structures into a setting where the structure is not uniformly locally finite. The highlights include the non-amenability and the simplicity of the said automorphism group. The latter is proved from the superamalgamation property of the class of finite Heyting algebras and is applicable to other homogeneous lattices.

Institute of Informatics, the Czech Academy of Sciences.

HOLLAND-MARTINEZ SESSION

Invited talks

κ-HOLLOW FRAMES AND κ-REPLETE VECTOR LATTICES

R. N. BALL, A. W. HAGER, AND J. WALTERS-WAYLAND

A completely regular frame is a (canonical) union of an ascending cardinally indexed sequence of κ-frames, starting with the cozero part. In addition, every such frame has a (canonical) descending cardinally indexed sequence of minimal dense κ-Lindelöf sublocales, ending with its booleanization. The connection between the two is that the minimal dense κ-Lindelöf sublocale is the intersection of the dense open κ-cozero sublocales.

This focuses attention on the κ-hollow frames, i.e., those without dense κ-cozero elements. These frames have a coarse but transparent structure: if κ-Lindelöf, they are free over their skeletons, and their skeletons are simply sub-κ-frames of a complete boolean algebra. Characterizing these frames in various ways, and explaining how they fit together to determine the structure of the overlying frame, are the topics of the first part of this talk.

Application of the by-now-classical Madden representation, the pointfree version of the very classical Yosida representation, leads to the concept of repletion in **W**, the category of unital archimedean vector lattices. The frame maps appropriate for κ-hollow frames are those that are κ-*skeletal*, i.e., those that take dense κ-cozeros to dense κ-cozeros. These dualize to κ-complete **W**-homomorphisms, i.e., those that preserve κ-joins. In the second part of the talk, we will take the first steps towards characterizing the κ-replete objects and articulating their structural and universal properties.

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(Hager) Department of Mathematics, Wesleyan University, Middletown, CT 06457, U.S. A.

(Walters-Wayland) J. Walters-Wayland, CECAT, Chapman University, Orange CA 92866

The Conrad-Harvey-Holland Theorem: a review of its impact on the study of lattice-ordered groups

Michael Darnel

In 1963, Paul Conrad and his first two doctoral students John Harvey and W. Charles Holland successfully generalized the Hahn Embedding Theorem for totally ordered abelian groups to abelian lattice-ordered groups, providing a standard representation for such structures. This article will review some of the consequences of this theorem, ending with a discussion of later successful efforts to further generalize it to normal-valued lattice-ordered groups.

W. Charles Holland and Automorphism Groups of Ordered Sets

Manfred Droste (Leipzig University)

This talk will describe some of Charles Holland's research achievements at Bowling Green State University, in particular for automorphism groups of ordered sets.

Fraction-dense algebraic frames

Themba Dube
University of South Africa

The notion of fraction-density in the category of *f*-rings was introduced by Tony Hager and Jorge Martinez. I will show how to abstract this to algebraic frames in such a way that a reduced *f*-ring is fraction-dense precisely when its frame of radical ideals is fraction-dense. [This is based on joint work with Papiya Bhattacharjee, one of numerous mathematical descendants of Jorge Martinez].

WHAT ARE WEAK PSEUDO EMV-ALGEBRAS?

ANATOLIJ DVUREČENSKIJ, OMID ZAHIRI

1. Weak Pseudo EMV-algebras

In [DvZa, DvZa1, DvZa2], we have introduced a commutative and non-commutative generalization of MV-algebras and generalized Boolean algebras as algebras where the top element is not assumed and every element is majorized by some idempotent element, moreover, every interval $[0, a]$, where a is an idempotent, is an MV-algebra or a pseudo MV-algebra. These algebras are said to be *EMV-algebras* or *pseudo EMV-algebras* (EMV stands for extended MV-algebras). If such an algebra possesses a top element, then it is equivalent to an MV-algebra or to a pseudo MV-algebra. The principal representing result says that every EMV-algebra (pseudo EMV-algebra) M without top element can be embedded into an EMV-algebra (pseudo EMV-algebra) N with top element as a maximal and normal ideal of N and every element not belonging to the image of M is a complement of the image of some element from M.

These algebras do not form a variety because they are not closed under subalgebras, but they are closed to varieties, special classes called q-variety. Consequently, there are countably many q-subvarieties of EMV-algebras and uncountably many q-subvarieties of pseudo EMV-algebras.

Therefore, in [DvZa3], we have found classes of algebras called *weak EMV-algebras* (wEMV-algebras) which form a variety and this class contains also all EMV-algebras and this variety is the least variety containing all wEMV-algebras. In this contribution, we present a non-commutative generalization of algebras called weak pseudo EMV-algebras:

Definition 1.1. An algebra $(M; \vee, \wedge, \oplus, \ominus, \circleddash, 0)$ of type (2,2,2,2,2,0) is called a *wPEMV-algebra* (w means weak) if it satisfies the following conditions:

- (W1) $(M, \vee, \wedge, 0)$ is a distributive lattice with the least element 0;
- (W2) $(M; \oplus, 0)$ is a monoid;
- (W3) $(y \oplus x) \ominus x \leq y$ and $x \circleddash (x \oplus y) \leq y$;
- (W4) $(y \ominus x) \oplus x = x \vee y = x \oplus (x \circleddash y)$;
- (W5) $x \ominus (x \wedge y) = x \ominus y$ and $(x \wedge y) \circleddash y = x \circleddash y$;
- (W6) $y \ominus (x \circleddash y) = x \wedge y = (y \ominus x) \circleddash y$;
- (W7) $z \ominus (x \vee y) = (z \ominus x) \wedge (z \ominus y)$ and $(x \vee y) \circleddash z = (x \circleddash y) \wedge (y \circleddash z)$;
- (W8) $(x \wedge y) \ominus z = (x \ominus z) \wedge (y \ominus z)$ and $z \circleddash (x \wedge y) = (z \circleddash x) \wedge (z \circleddash y)$;
- (W9) $x \ominus (y \oplus z) = (x \ominus z) \ominus y$ and $(y \oplus z) \circleddash x = z \circleddash (y \circleddash x)$;
- (W10) $x \oplus (y \vee z) = (x \oplus y) \vee (x \oplus z)$ and $(y \vee z) \oplus x = (y \oplus x) \vee (z \oplus x)$.

Examples of wPEMV-algebras are pseudo MV-algebras or pseudo EMV-algebras. If G is an ℓ-group, then the positive cone G^+ endowed with the standard $\vee$, $\wedge$ and $x \oplus y = z + y$, $x \ominus y = (x - y) \vee 0$, $x \circleddash y = (-x + y) \vee 0$, is a wPEMV-algebra.

These algebras form a variety and in the contribution, we present basic properties of these algebras. We concentrate to a question when a wPEMV-algebra M without top element can be represented by a wPEMV-algebra N with top element as its maximal and normal ideal. The wPEMV-algebra N is said

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to be a *representing wEMV-algebra*. We show that the variety of representable wEPV-algebras has this property. We present an equational base for this subvariety.

We show that a necessary and sufficient condition for a wPEMV-algebra M in order to admit a representing wPEMV-algebra with top element is an existence of a left (and right) unitizing automorphism. For example weakly commutative wPEMV-algebras or cancellative wPEMVa-algebras have a left unitizing automorphism. We prove that every wPEMV-algebra can be embedded into a positive cone of an ℓ-group preserving $0, \vee, \wedge, \ominus, \oslash$.

Finally, we show that the class of cancellative, or the class of weakly commutative or the class of normal-valued wPEMV-algebras forms a variety.

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Some research directions in the theory of lattice-ordered groups

In memoriam W. Charles Holland (1935-2020) and J. Martinez (1945-2020)

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In this talk I would like to discuss a number of research directions in which lattice-ordered groups are either featured as the main object of study, or else play a central rôle. The choice of topics is guided by my personal interests. In the talk I emphasise the major influence that both Charles Holland and Jorge Martinez had on the entire field of lattice-ordered groups.

THE MATHEMATICAL LIFE OF JORGE MARTINEZ

WARREN WM. MCGOVERN[2]

ABSTRACT. Jorge Martinez passed away on August 19, 2020. Jorge was my Ph.D. advisor, but most importantly he was a dear friend. In this talk I would like to celebrate his life and his mathematical work.

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Key words and phrases. Jorge Martinez mathematical life.

The topology of closure systems in algebraic lattices

Niels Schwartz

Algebraic lattices are spectral spaces with respect to the coarse lower topology. Closure systems in algebraic lattices are viewed as subspaces. There are close connections between algebraic properties of a closure system and topological properties of the subspace. A closure system is algebraic if and only if the subspace is patch closed in the ambient algebraic lattice. Let X be a subset of an algebraic lattice P and L the closure system generated by X. Then the patch closure of X generates a closure system M, which is the patch closure of L. Assume that X is contained in the set of nontrivial prime elements of P. If X is patch closed in P then L is a coherent algebraic frame. Conversely, if P is a coherent algebraic frame then its set of nontrivial prime elements generates P and is patch closed.

The Archimedean Property: New Perspectives

In memoriam: W. C. Holland and Jorge Martinez

Constantine Tsinakis
Vanderbilt University, USA

[This talk is based on article [11] in the references.] There have been a number of attempts to define the concept of an Archimedean algebra for individual classes of residuated lattices, but there is not a general definition that subsumes the existing special cases. In this talk we propose such a definition and single out a sufficiently large class $\mathcal{K}$ in which the Archimedean property implies commutativity.

Our approach is inspired by the work of P.F. Conrad, who, in the 1960s, launched a general program for the investigation of ℓ-groups [4, 5, 6, 7], aimed at capturing relevant information about these algebras by studying the structure of their lattices of convex ℓ-subgroups, as well as at showing that many significant properties of ℓ-groups have a dominant lattice-theoretic component. A natural continuation of Conrad's original program consists in extending it from ℓ-groups to the more comprehensive domain of residuated lattices. This *extended Conrad program* has fueled some of the recent developments in the theory of residuated structures, leading to promising results e.g. in the study of semilinear and Hamiltonian varieties [2], in the investigation of normal-valued residuated lattices [3], in the study of hulls of residuated lattices [9], and in the description of projectable objects [8, 10].

The most far-reaching attempt to capture the Archimedean property so far, due to Jorge Martinez [12], is consistent with Conrad's approach. An algebraic distributive lattice $\mathbf{L}$, where the set of compact elements is closed under finite meets, has the *zero radical compact property*[1] if for each compact element $c \in L$, the meet of all the maximal elements in the interval $[\bot, c]$ in $\mathbf{L}$ is $\bot$. Martinez observed that an Abelian ℓ-group is Archimedean if and only if its lattice of convex subuniverses has the zero radical compact property [12, p. 249]. In this special case, therefore, the Archimedean property is fully captured in the lattices of convex subuniverses of the ℓ-groups in question. However, this is no longer true in general, even in the ℓ-group setting. For example, there exist Archimedean ℓ-groups and non-normal-valued ℓ-groups whose lattices of convex subuniverses are isomorphic [1, Ex. E53].

We advance hereby a suggestion to the effect that a residuated lattice is *Archimedean* provided (1) it is cyclic, satisfies the equation $e/x \approx x\backslash e$; (2) it is normal

[1] Martinez calls such lattices *Archimedean*. We prefer to use a brand new label in order to avoid conflicts with the notion of an Archimedean residuated lattice.

valued, that is, every completely meet-irreducible convex subuniverse is normal in its cover; and (3) its lattice of convex subuniverses has the zero radical compact property. The restriction to e-cyclic residuated lattices is motivated by the fact that only in the e-cyclic case lattices of convex subuniverses are known to behave in the appropriate way – for example, they are distributive lattices [2]. The *desideratum* (3) is clearly inspired by Martinez's characterization of Archimedean Abelian ℓ-groups. Finally, (2) supplements the lattice-theoretic toolbox with the needed extra information for fully describing the property.

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Contributed talks

SUBLOCALES OF THE ASSEMBLY OF A FRAME AND THE T_D-DUALITY

IGOR ARRIETA

University of Coimbra and University of the Basque Country EHU/UPV

If L is a locale, the ordered collection $\mathcal{S}(L)$ of all its sublocales is a coframe. In this talk we will introduce and discuss the relation between several subcolocales (or, more generally, subsets) of the coframe of sublocales of a locale. Some examples are the Booleanization of $\mathcal{S}(L)$ (see e.g. [1]) or the system $\mathcal{S}_D(L)$ of D-sublocales from [2].

We will show properties of these subsets sometimes encode interesting properties of the locale itself. Using this approach, we will revisit some aspects of the important T_D axiom of Aull and Thron and in particular the connection with T_D-spatiality and the T_D-duality of Banaschewski and Pultr.

A part of the talk is joint work with Anna Laura Suarez (University of Birmingham).

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LOCALIC MAPS AND z-EMBEDDED SUBLOCALES

ANA BELÉN AVILEZ

Abstract. Let $\mathsf{S}(L)$ denote the coframe of sublocales of a locale L. Given a localic map $f\colon L \to M$ (a morphism in the category of locales), consider the familiar adjunction between localic images and preimages

$$\begin{tikzcd} \mathsf{S}(L) \arrow[r, bend left, "f[-]"] \arrow[r, phantom, "\perp"] & \mathsf{S}(M) \arrow[l, bend left, "f_{-1}[-]"] \end{tikzcd}$$

where $f[-](S) = f[S]$ is just the set theoretical image of S in M and $f_{-1}[-][T]$ is the *largest* sublocale of L contained in the set theoretical preimage $f^{-1}[T]$. The preimage functor $f_{-1}[-]$ is a coframe homomorphism that preserves complements while the left adjoint $f[-]$ is a colocalic map.

In this talk we will use this adjunction and the notion of a *zero sublocale* to motivate and discuss the definition of z-embeddings. Unlike the usual approach in [2] and [3], where the authors consider coz-onto quotients, we look at z-embeddings from the intuitive point of view of sublocales. We will illustrate this with some results from [1] that are nicely formulated in the localic language.

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Reflective and coreflective subcategories of bounded archimedean ℓ-algebras

G. Bezhanishvili, P. J. Morandi, B. Olberding

New Mexico State University

In Memory of W. C. Holland and Jorge Martínez

Jorge Martínez and Tony Hager each visited us at NMSU back in the early 2010's. They, along with many others including Banaschewski, Ball, and Madden, have done considerable work with the category $\mathbf{W}$ of archimedean lattice-ordered groups with a weak-order unit and the subcategory $\mathbf{W}^*$ where the weak order-unit is a strong-order unit. The category $\mathbf{W}^*$ is the group analogue of the category $\boldsymbol{ba\ell}$ of bounded archimedean lattice-ordered $\mathbb{R}$-algebras that we study. After visiting us and learning about our first paper [1] in the subject, they were particularly interested in the following result of ours:

Lemma. Every nontrivial full reflective subcategory of $\boldsymbol{ba\ell}$ is bireflective.

Thinking about this result led to their paper [2]. In this talk we will discuss in detail this result, how we proved it, and its consequences.

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Maximal d-Elements of M-Frames

Papiya Bhattacharjee

The space of maximal d-ideals of $C(X)$ is homeomorphic to the $Z^\sharp$-ultrafilters, and this space is the minimal quasi-F cover of a compact Tychonoff space X. In this talk the notions of maximal d-ideals and $Z^\sharp$-ultrafilters will be generalized for M-frames (algebraic frames with the FIP). For an M-frame L, it is straight forward that the maximal d-elements of the frame is the counterpart of the maximal d-ideals. A curious fact is, the counterpart of the $Z^\sharp$-ultrafilters is the ultrafilters on $\mathfrak{K}L^\perp$, where $\mathfrak{K}L$ is the lattice of compact elements of an M-frame L. The speaker will discuss the spaces of minimal prime elements $Min(L)$, maximal d-elements $Max(dL)$, and their properties. Furthermore, the relation between $Max(dL)$ and $Min(L)$ will be established.

COUNTABLE METRIC SPACES WITHOUT ISOLATED POINTS

FREDERICK K. DASHIELL, JR.

Challenge: Prove directly that the space $\mathbb{Q}$ of rational numbers is homeomorphic to the space $\mathbb{Q} \times \mathbb{Q}$ of rational points in the Euclidean plane. Or that $\mathbb{Q} \cap (0, 1)$ is homeomorphic to $\mathbb{Q} \cap (0, 1]$.

This talk describes a short, self-contained proof of the famous fact that any countable metric space without isolated points is homeomorphic to $\mathbb{Q}$. The novel feature is that the proof is carried out entirely in the language of countable metric spaces. Unlike all other other proofs (!), there is no dependence on the notion of a basis for a topology or on the complete metrizability of any space (such as $\mathbb{R}$, $2^{\mathbb{N}}$, or $\mathbb{N}^{\mathbb{N}}$). No "analysis" or convergent sequences are used. A direct corollary is the fact that any countable metric space is homeomorphic to a subspace of the rationals. This can be presented in a first course in real analysis.

Center of Excellence for Computation, Algebra, and Topology (CECAT), Chapman University, Orange, CA

Anthony Hager and Brian Wynne

Title: Atoms in the lattice of hull operators on $\mathbf{W}$ and in related lattices

Abstract: Let $\mathbf{W}$ be the category of Archimedean lattice-ordered groups with distinguished weak unit and unit-preserving ℓ-homomorphisms. The class $\mathbf{hoW}$ of all hull operators on $\mathbf{W}$, which includes (just to name a few) the divisible hull, the projectable hull, and the Dedekind-MacNeille completion operators, has the structure of a complete lattice with top the maximum essential extension operator and bottom the identity operator. I will discuss what we know so far about the atoms in $\mathbf{hoW}$, including some specific examples. While the full story in $\mathbf{W}$ remains a mystery, I will present complete descriptions of the atoms in several related lattices, such as the lattice of covering operators on compact Hausdorff spaces.

On o-Automorphisms of o-Groups of Finite Archimedean Rank

Ramiro H. Lafuente-Rodriguez
University of South Dakota

Abstract

We provide a full description of o-groups of finite archimedean rank and their groups of o-automorphisms. We also determine divisibility criteria for elements in the group of o-automorphisms. Finally we study embeddings of such groups in divisible o-groups of higher archimedean ranks.

BLAST 2021 Abstract
Paper: ON THE RIESZ STRUCTURES OF A LATTICE ORDERED ABELIAN GROUP

G. Lenzi

ABSTRACT. A Riesz structure on a lattice ordered abelian group G is a real vector space structure where the product of a positive element of G and a positive real is positive. In this paper we show that for every cardinal k there is a totally ordered abelian group with at least k Riesz structures, all of them isomorphic. Moreover two Riesz structures on the same totally ordered group are partially isomorphic in the sense of model theory. Further, as a main result, we build two nonisomorphic Riesz structures on the same l-group with strong unit. This gives a solution to a problem posed by Conrad in 1975. Finally we apply the main result to MV-algebras and Riesz MV-algebras.

Conjunctive Join-Semilattices

James J. Madden, LSU, Baton Rouge LA USA (presenter)
Charles N. Delzell, LSU, Baton Rouge LA USA
Oghenetega Ighedo, UNISA, Johannesburg SA

Abstract. A join-semilattice L is said to be *conjunctive* if it has a top element 1 and it satisfies the following first-order condition: for any two distinct $a, b \in L$, there is $c \in L$ such that either $a \vee c \neq 1 = b \vee c$ or $a \vee c = 1 \neq b \vee c$. It is known that this is equivalent to the condition that every principal ideal is an intersection of maximal ideals. All the maximal ideals of a join-semilattice L with 1 (conjunctive or not) are prime if and only if L is 1-$\vee$-distributive, but a conjunctive join-semilattice may fail to have any prime ideals, and a 1-$\vee$-distributive, conjunctive join semilattice may fail to be distributive. **Theorem.** Every conjunctive join-semilattice is isomorphic to a join-closed subbase for a compact T_1-topology on $\max L$, the set of maximal ideals of L. A join-semilattice morphism $\phi : L \to M$ is conjunctive if $\phi^{-1}(\mathfrak{m})$ is an intersection of maximal ideals of L whenever $\mathfrak{m}$ is a maximal ideal of M. **Theorem.** Every conjunctive morphism between conjunctive join-semilattices is induced by a multi-valued function from $\max M$ to $\max L$. A base for a topological space is said to be *annular* if it is a lattice, and *Wallman* if it is annular and for any point u in any basic open U, there is a basic open V that misses u and together with U covers X. It is easy to show that every Wallman base is conjunctive. We give an example of a conjunctive annular base that is not Wallman. Finally, we examine the free distributive lattice over a conjunctive join semilattice. In general, it is not conjunctive, but we show that a certain canonical, algebraically-defined quotient of it is isomorphic to the sublattice of the topology of the representation space. We mention numerous applications of the theory presented here, including new proofs and generalizations of results of Martinez and Zenk concerning Yosida Frames.

Mathematics Subject Classification 2020: Primary 06B15; Secondary 06B75, 54H10.

A Grothendieck problem for finitely presented MV-algebras

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1. Introduction

A well-known problem in algebra related to profinite completion is a problem posed by Grothendieck in 1970 [2] that asks whether or not the profinite completion functor from the category of residually finite and finitely presented groups into the category of profinite groups reflects isomorphisms. Having gathered all the related ingredients for MV-algebras[6, 7], it seemed very natural to consider the analog of that problem for MV-algebras. We use the representation of finitely presented MV-algebras by means of rational polyhedra [5, Thm. 6.3] and obtain a positive answer to the problem. More precisely, we obtain that if $\varphi : A \to B$ is an MV-algebra homomorphism where A and B are finitely presented such that $\widehat{\varphi} : \widehat{A} \to \widehat{B}$ is an isomorphism, then φ is an isomorphism.

2. Profinite completions of residually finite MV-algebras

We wish to consider the problem of whether or not the profinite completion functor from the category of residually finite MV-algebras to that of Stone MV-algebras reflects isomorphisms.

First, one can show that an MV-algebra is residually finite if and only if it is a sub-MV-algebra of an MV-algebra of the form $\prod_{x\in X} \text{Ł}_{n_x}$ for some nonempty set X.
The next result shows that finitely presented MV-algebras are residually finite.

Proposition 2.1. *Finitely presented MV-algebras are subdirect products of finite Łukasiewicz chains. In particular, finitely presented MV-algebras are residually finite.*

Proof. Let A be a finitely presented MV-algebra. By the duality between finitely presented MV-algebras and rational polyhedra with $\mathbb{Z}$-maps, there exists a rational polyhedron $P \subseteq [0,1]^n$ such that $A = \mathcal{M}(P)$, the MV-algebra of all $\mathbb{Z}$-maps from $P \to [0,1]$, where $[0,1]$ is regarded as a rational polyhedron. Since P is a rational polyhedron, the subset $R(P)$ of points of P having rational coordinates is dense in P. For each such point $x \in R(P)$, the evaluation of members of A at x yields a function $A \to [0,1]$ which can be easily proved to be a homomorphism of MV-algebras. Because the point x is rational, it can also be proved that the image of this homomorphism is a finite subalgebra of $[0,1]$, which we denote by $L_A(x)$. One can show that the induced homomorphism $\mu_A : A \to \prod_{x\in R(P)} L_A(x)$, is a subdirect embedding. $\square$

One observes from Proposition 2.1 that finitely presented MV-algebras are automatically residually finite. Therefore the MV-algebraic version of the Grothendieck problem as described above can simply be formulated about finitely presented MV-algebras. In the following results and observations, we gather some of the main ingredients needed to offer a positive answer to the problem. One key such ingredient is a concrete description of the profinite completion of an MV-algebra.

Corollary 2.2. *Let $A := \mathcal{M}(P)$ be a finitely presented MV-algebra. With the notations of Proposition 2.1 and its proof;*

1. *The map $\sigma_A : P \to \mathrm{Max}A$ defined by $x \mapsto \mathfrak{h}_x := \{f \in A : f(x) = 0\}$ is a homeomorphism.*
2. *For every $x \in P$, $\sigma_A(x)$ has finite rank in A if and only if $x \in R(P)$. In addition, $A/\sigma_A(x) \cong L_A(x)$ for all $x \in R(P)$.*
3. *$\widehat{A} \cong \prod_{x\in R(P)} L_A(x)$.*

Remark 2.3. *Let $\varphi : \mathcal{M}(P) \to \mathcal{M}(Q)$ be a homomorphism, where $P \subseteq [0,1]^n$ and $Q \subseteq [0,1]^m$ are rational polyhedra. Let, $\pi_i : [0,1]^n \to [0,1]$ be the ith projection ($1 \leq i \leq n$). For each i, let $f_i = \varphi(\pi_i \restriction P)$, where $\pi_i \restriction P$ denotes the restriction of π_i to P. One has a map $\sigma_\varphi : Q \to P$ defined by $\sigma_\varphi(y) = (f_1(y), f_2(y), \dots, f_n(y))$ which is indeed a $\mathbb{Z}$-map* [5, Lem. 3.8(ii)].

Lemma 2.4. *Let $\varphi : A := \mathcal{M}(P) \to B := \mathcal{M}(Q)$ be a homomorphism as in the previous Remark. Then, $\sigma_\varphi(R(Q)) \subseteq R(P)$ and for every $y \in R(Q)$, and every $f \in \mathcal{M}(P)$, $f(\sigma_\varphi(y)) = \varphi(f)(y)$. In particular, $L_A(\sigma_\varphi(y)) \subseteq L_B(y)$ for all $y \in R(Q)$.*

The next result offers a positive answer to the MV-algebraic version of the Grothendieck's problem as discussed in the introductory paragraph of this section.

Theorem 2.5. *Let A, B be two finitely presented MV-algebras and $\varphi : A \to B$ be a homomorphism. Then φ is an isomorphism if and only if $\widehat{\varphi}$ is an isomorphism.*

Proof. For the sufficiency, we shall rely heavily on the fact that A and B are finitely presented. We write $A := \mathcal{M}(P)$ and $B := \mathcal{M}(Q)$, where $P \subseteq [0,1]^n$ and $Q \subseteq [0,1]^m$ are rational polyhedra. In this context, we have from Corollary 2.2(3) that $\widehat{A} = \prod_{x\in R(P)} L_A(x)$ and $\widehat{B} = \prod_{y\in R(Q)} L_B(y)$. In addition, $\widehat{\varphi} : \prod_{x\in R(P)} L_A(x) \to \prod_{y\in R(Q)} L_B(y)$ is defined by $\widehat{\varphi}(\alpha)(y) = \alpha(\sigma_\varphi(y))$ for all $\alpha \in \prod_{x\in R(P)} L_A(x)$ and $y \in R(Q)$. Note that $\widehat{\varphi}$ is well-defined thanks to Lemma 2.4.

One can now assume that $\widehat{\varphi}$ is an isomorphism use the previous results and ingredients above to show that φ is a bijection. □

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The Relationship between Partial Metric Varieties and Commuting Powers Varieties

Homeira Pajoohesh

Let $\mathcal{L}_n$ be the variety of ℓ-groups satisfying $x^n y^n = y^n x^n$ and $\mathcal{E}_n$ be the variety of n-partial metrics, ℓ-groups satisfying $(x \vee z)^n y^n \leq (x \vee y)^n (y \vee z)^n$. In this talk first we show that $\mathcal{L}_n \cap \mathcal{A}^2 = \mathcal{E}_n$ for every n, where $\mathcal{A}^2$ is the metabelian variety.

HAUSDORFF REFLECTION OF INTERNAL PRENEIGHBOURHOOD SPACES

PARTHA PRATIM GHOSH

The notion of an *internal preneighbourhood space* was initiated in [Gho20]. The notion of *Hausdorff preneighbourhood space* is developed in [Gho21]. The purpose of the present paper is to provide the existence and a construction of the Hausdorff reflection of an internal preneighbourhood space.

A *context* is a triple $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$, where $\mathbb{A}$ is a finitely complete category with finite coproducts such that (E, M) is a proper factorisation structure on $\mathbb{A}$ and the $\mathtt{Sub}_{\mathsf{M}}(X)$ of all M-subobjects of X (also called *admissible subobjects*) is a complete lattice. A *preneighbourhood system* on X is an order preserving map $\mathtt{Sub}_{\mathsf{M}}(X)^{\mathrm{op}} \xrightarrow{\mu} \mathtt{Fil}X$ ($\mathtt{Fil}X$ is the lattice of all filters in $\mathtt{Sub}_{\mathsf{M}}(X)$) such that $p \in \mu(m)$ implies $p \geq m$. An *internal preneighbourhood space* is (X, μ) where X is an object and μ is a preneighbourhood system on X. The function $\mathtt{Sub}_{\mathsf{M}}(X) \xrightarrow{\mathrm{cl}_\mu} \mathtt{Sub}_{\mathsf{M}}(X)$ defined by $\mathrm{cl}_\mu p = \bigvee\{x \in \mathtt{Sub}_{\mathsf{M}}(X) : x \neq \mathbf{1}_X \text{ and } u \in \mu(x) \Rightarrow u \wedge p \neq \sigma_X\}$ (where σ_X is the smallest admissible subobject of X) is a grounded, transitive and hereditary closure operator. If (X, μ) and (Y, ϕ) are internal preneighbourhood spaces then a morphism $X \xrightarrow{f} Y$ is a *preneighbourhood morphism* if $p \in \phi(m)$ implies $f^{-\mathbf{1}}p \in \mu(f^{-\mathbf{1}}m)$; the morphism f is *closed* if $\mathrm{cl}_\phi \exists_f p \leq \exists_f \mathrm{cl}_\mu p$ (where $\exists_f p$ is the subobject obtained from the (E, M)-factorisation of $f \circ p$, the *image* of p under f). The category $\mathtt{pNbd}[\mathbb{A}]$ is topological over $\mathbb{A}$. Hence every limit (respectively, colimit) object of a diagram of internal preneighbourhood spaces is usually endowed with the smallest (respectively, largest) preneighbourhood system which makes each component of the limiting (respectively, colimiting) cone a preneighbourhood morphism. A preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ is *proper* if its stably closed. A preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ is *separated* if the *diagonal morphism* $(X, \mu) \xrightarrow{d_f = (\mathbf{1}_X, \mathbf{1}_X)} (\mathtt{kerp}\, f, \mu \times_\phi \mu)$ (where $\mu \times_\phi \mu$ is the smallest preneighbourhood system on $\mathtt{kerp}\, f$ which make the kernel pair projections preneighbourhood morphisms) is proper. An internal preneighbourhood space (X, μ) is *Hausdorff* if the preneighbourhood morphism $(X, \mu) \xrightarrow{f} (\mathtt{1}, \nabla_{\mathtt{1}})$ (where ∇_X is the smallest preneighbourhood system on X) is separated.

In this talk it shall be shown:

(a) In a reflecting zero context (i.e., for every morphism $X \xrightarrow{f} Y$, $f^{-\mathbf{1}}\sigma_Y = \sigma_X$) with finite product projections in E the full subcategory $\mathtt{Haus}[\mathtt{pNbd}[\mathbb{A}]]$ of internal Hausdorff preneighbourhood spaces is a regular epimorphic reflective subcategory of $\mathtt{pNbd}[\mathbb{A}]$.
(b) In a reflecting zero M-well powered context with finite product projections in E there exists a transfinite construction of the Hausdorff reflection.

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Heyting congruences and Booleanness conditions for partial frames

Anneliese Schauerte

I would like to offer this talk as part of the special session in memory of W. CHARLES HOLLAND (1935—2020) and JORGE MARTINEZ (1945—2020).

This is joint work with John Frith, also of the University of Cape Town, South Africa.

Congruences have a long history as a useful tool in universal algebra, but also in point-free topology. Indeed, the collection of congruences of a frame forms a frame itself, and the embedding of a frame into its congruence frame has useful universal properties. It has been considered by many authors and has appeared in the literature in many guises (being variously called: assembly, modal extension, parts of a frame, dissolution locale, splitting locale).

The context of this work is more general than frame theory, encompassing as it does bounded distributive lattices, sigma-frames, kappa-frames and (full) frames. Partial frames are meet-semilattices where, in contrast with frames, not all subsets need have joins. A selection function, $\mathcal{S}$, specifies, for all meet-semilattices, certain subsets under consideration, which we call the "designated" ones; an $\mathcal{S}$-frame then must have joins of (at least) all such subsets and binary meet must distribute over these. For details of our previous work on the congruence frame in this context, we refer the reader to: J. Frith and A. Schauerte, "The congruence frame and the Madden quotient for partial frames," Algebra Univers. 79 (2018) Article 73.

In this talk we introduce a new class of congruences, which we call Heyting congruences. Their definition involves the Heyting arrow in the free frame over a partial frame, hence the name. The free frame over a partial frame provides a useful tool in this regard, with unexpected connections arising. We characterize those Heyting congruences which are co-atoms of the congruence frame, in terms of meet-irreducibility. We also characterize those co-atoms of the congruence frame which are Heyting congruences. Examples illustrate how the full and partial cases differ; for instance, any frame congruence is an intersection of Heyting congruences, but this need not be the case for, say, sigma-frames. We conclude with a discussion of several (provably different) conditions akin to Booleanness for partial frames, in which free frames, congruence frames and Heyting congruences all play a role. We justify regarding them as "Booleanness conditions" since, in the case of full frames, they do indeed restrict to the simple statement that each element has a complement.

A hyperarchimedean ℓ-group not embeddable into any hyperarchimedean ℓ-group with strong unit

Philip Scowcroft

Wesleyan University

In "Settling a number of questions about hyper-Archimedean lattice-ordered groups" Conrad and Martinez were the first to describe a hyper-archimedean (h.a.) ℓ-group not embeddable into any Archimedean ℓ-group with strong unit. This work was refined and extended by Hager and Johnson in "Some comments and examples on generation of (hyper-) archimedean ℓ-groups and f-rings." Both Conrad-Martinez and Hager-Johnson build their h.a. ℓ-groups with the help of uncountable families of almost-disjoint sets of natural numbers. Without relying on this machinery, this talk will present another example of an h.a. ℓ-group not embeddable into any h.a. ℓ-group with strong unit. If time permits I will describe the Boolean-algebraic source of the new example.

DIRECTED PARTIAL ORDERS AND RIEMANN HYPOTHESIS

ZHIPENG XU, YUEHUI ZHANG
SHANGHAI JIAO TONG UNIVERSITY

This talk is about the classification of directed partial orders over the imaginary quadratic extension $F(i) = F+Fi$ of a non-archimedean o-field F. This is uniformly done by a new concept *doubly convex set* without dividing into two cases $1 > 0$ and $1 \not> 0$ as in the literature. This new concept works also very well when applied to the classification of the quaternions $H = F + Fi + Fj + Fk$. This classification of H is a negative answer to the Fuch's problem. We also mention the relationship of directed partial orders on complex numbers and Riemann Hypothesis.

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STONE-PRIESTLEY SESSION

Invited talks

Extending Stone-Priestley duality along full embeddings

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Stone and Priestley dualities for bounded distributive lattices establish dual equivalences with the categories of coherent (or spectral) spaces and of Priestley spaces, respectively (and these two categories are isomorphic). Since bounded distributive lattices are, in turn, equivalent to coherent frames, these dualities may be seen as suitable restrictions and co-restrictions of the spatial-sober duality between spatial frames and sober spaces. Notice however that neither coherent nor Priestley spaces are fully embedded in the category of sober spaces: morphisms of coherent spaces are required to be well-behaved with respect to compactness, while Priestley spaces come equipped with an order, and their morphisms are required to be order-preserving. Similarly, coherent frames do not form a full subcategory of frames as the morphisms are required to preserve compact elements. In the past few years, several authors suggested it could be more natural to understand duality for bounded distributive lattices bitopologically [3, 1]. Unlike in the monotopological setting, Stone-Priestley duality may be seen as a restriction and co-restriction of the bitopological version of the spatial-sober duality along full subcategory embeddings.

In this talk we will see an alternative extension of Stone-Priestley duality along full embeddings to a spatial-sober-like adjunction. The categories at play are those of *Pervin spaces* and of *Frith frames.* Pervin spaces, introduced in [2], are known to capture totally bounded and transitive quasi-uniform spaces. Frith frames are its pointfree version, and we show that they capture totally bounded and transitive quasi-uniform frames. In particular, there exist natural notions of *completion of a Pervin space* and of *completion of a Frith frame* available. It may be derived from [4] that the category of complete T_0 Pervin spaces is equivalent to the category of coherent topological spaces, and we show that complete Frith frames are equivalent to coherent frames. We will also see how this relates to the bitopological point of view mentioned above.

This is based on ongoing joint work with Anna Laura Suarez.

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Toward choice-free Esakia duality

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In a recent paper with Nick Bezhanishvili, "Choice-free Stone duality" (*Journal of Symbolic Logic*, 2020), we developed a choice-free topological duality theory for Boolean algebras using special spectral spaces, called upper Vietoris spaces. In this talk, I will report on our progress toward generalizing the ideas of that paper to give a choice-free duality theory for Heyting algebras, inspired by Esakia duality (Leo Esakia, *Heyting Algebras: Duality Theory*, Springer, 2019).

A VARIETY OF BI-HEYTING ALGEBRAS NOT GENERATED BY COMPLETE ALGEBRAS

MAMUKA JIBLADZE
(JOINT WORK WITH GURAM BEZHANISHVILI AND DAVID GABELAIA)

In 1974 Fine constructed a Kripke incomplete logic above S4.In 1977 Shehtman found a Kripke incomplete intuitionistic counterpart of Fine's logic.We use Esakia duality to produce an incompleteness argument that allows a generalizationfrom Kripke semantics to complete bi-Heyting algebra semantics.This provides an example of a variety of bi-Heyting algebras not generated by its complete members.Thus there exists a topologically incomplete extension of the Heyting-Brouwer logic.

Lovász-type theorems and polyadic spaces*

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A classical result by Lovász [8] states that two finite relational structures A and B are isomorphic if, and only if, for all finite relational structures C, the sets of homomorphisms $\hom(C,A)$ and $\hom(C,B)$ have the same cardinality. This theorem has inspired several other homomorphism-counting results (see e.g. [4, 5, 9]), which nowadays play an important role in finite model theory and graph theory.

I will describe a systematic approach to homomorphism-counting results based on the framework of *game comonads*, which have been recently introduced by Abramsky, Dawar et al. [1, 2]. This approach relies on a special instance of Joyal's notion of *polyadic spaces* [6] — certain functors into the category of Stone spaces and continuous maps that are dual to Boolean hyperdoctrines [7]. I shall discuss some ideas to further develop this connection between homomorphism-counting results in finite model theory and Stone duality.

This talk is in large part based on joint work with Anuj Dawar and Tomáš Jakl [3].

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Title: dcpo completions and Scott topologies of partially ordered sets.
Speaker: Marcus Tressl

Abstract:

https://personalpages.manchester.ac.uk/staff/Marcus.Tressl/dcpo.pdf

Priestley duality for MV-algebras and beyond

Sam van Gool

We provide a new perspective on extended Priestley duality for a large class of distributive lattices equipped with binary double quasioperators. Under this approach, non-lattice binary operations are each presented as a pair of partial binary operations on dual spaces. In this enriched environment, equational conditions on the algebraic side of the duality may more often be rendered as first-order conditions on dual spaces. In particular, we specialize our general results to the variety of MV-algebras, obtaining a duality for these in which the equations axiomatizing MV-algebras are dualized as first-order conditions.

This talk will be based on joint work with Wesley Fussner, Mai Gehrke, and Vincenzo Marra, accepted for publication in Forum Mathematicum, preprint available as arXiv:2002.12715.

Contributed talks

The opposite of the category of compact ordered spaces as an infinitary variety

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1 Background and main question

In 1936, M. H. Stone described what is nowadays known as Stone duality for Boolean algebras [12]. In modern terms, the result states that the category of Boolean algebras and homomorphisms is dually equivalent to the category of totally disconnected compact Hausdorff spaces and continuous maps, now known as Stone or Boolean spaces.

If we drop the assumption of total disconnectedness, we are left with the category CH of compact Hausdorff spaces and continuous maps. J. Duskin observed in 1969 that the opposite category $\mathsf{CH}^{\mathsf{op}}$ is monadic over the category Set of sets and functions [5, 5.15.3]. In fact, $\mathsf{CH}^{\mathsf{op}}$ is equivalent to a variety of algebras with primitive operations of at most countable arity: a finite generating set of operations was exhibited by J. Isbell [7], while a finite equational axiomatisation was provided by V. Marra and L. Reggio [9]. Therefore, if we allow for infinitary operations, Stone duality can be lifted to compact Hausdorff spaces, retaining the algebraic nature of the category involved.

In 1970, H. A. Priestley introduced what are now known as Priestley spaces, i.e. compact spaces equipped with a partial order satisfying a condition called total order-disconnectedness, and showed that the category of bounded distributive lattices and homomorphisms is dually equivalent to the category of Priestley spaces and order-preserving continuous maps [11].

The category of Priestley spaces is a full subcategory of the category $\mathsf{CompOrd}$ of compact ordered spaces and order-preserving continuous maps: here, by a *compact ordered space*, we mean a compact space equipped with a partial order which is closed in the product topology—a partially ordered version of compact Hausdorff spaces introduced by L. Nachbin in 1948 [10].

Similarly to the case of Boolean algebras, one may ask if Priestley duality can be lifted to $\mathsf{CompOrd}$ retaining the algebraic nature of the opposite category. In other words:

> Is the category $\mathsf{CompOrd}$ of compact ordered spaces dually equivalent to a variety of (possibly infinitary) algebras?

This appeared as an open question in a paper by D. Hofmann, R. Neves and P. Nora [6].

2 Results

The following is our main result.

Theorem 1. *The category* $\mathsf{CompOrd}$ *of compact ordered spaces and order-preserving continuous maps is dually equivalent to a variety of algebras, with operations of at most countable arity.*

This gives a positive answer to the open question in [6]. This result was first proved by the author in [1], but a shorter proof was obtained in a joint work with L. Reggio [4]. A natural way to describe a variety dual to $\mathsf{CompOrd}$ uses the signature Σ of all the order-preserving continuous maps from finite or countably infinite powers of $[0,1]$ to $[0,1]$ itself: it was already known that $\mathsf{CompOrd}^{\mathsf{op}}$ is equivalent to the class $\mathbb{SP}([0,1])$ of subalgebras of powers of $[0,1]$ (with obvious interpretation of the function symbols in Σ), and via categorical means we prove that this class is closed under homomorphic images and thus it is a variety of (infinitary) algebras.

The countable bound on the arity is the best possible, since CompOrd is not dually equivalent to any variety of finitary algebras. Indeed, the following stronger results hold:

1. CompOrd is not dually equivalent to any finitely accessible category;
2. CompOrd is not dually equivalent to any first-order definable class (as suggested by S. Vasey as an application of a result by M. Lieberman, J. Rosický and S. Vasey [8]);
3. CompOrd is not dually equivalent to any class of finitary algebras closed under products and subalgebras.

Manageable sets of primitive operations and equational axioms for $\mathsf{CompOrd}^{\mathsf{op}}$ exist: we exhibit a *finite* equational axiomatisation of $\mathsf{CompOrd}^{\mathsf{op}}$, meaning that we use only finitely many function symbols (of at most countable arity) and finitely many equational axioms to present the variety.

To conclude, we recall that D. Hofmann, R. Neves and P. Nora proved that the opposite of the category of coalgebras for the Vietoris endofunctor on CompOrd is equivalent to an $\aleph_1$-ary quasivariety [6]. Our results can be used to show that this is actually a variety.

The results described in this abstract are part of the author's doctoral thesis [3], supervised by V. Marra at the University of Milan. Some of them are covered in [1, 2, 4].

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PROFINITENESS AND SPECTRA OF HEYTING ALGEBRAS

G. BEZHANISHVILI, N. BEZHANISHVILI, T. MORASCHINI, AND M. STRONKOWSKI

An algebra is said to be *profinite* if it is isomorphic to the inverse limit of an inverse system of finite algebras. Similarly, the *profinite completion* of an algebra $\boldsymbol{A}$ is the inverse limit of the inverse system consisting of the finite algebras of the form $\boldsymbol{A}/\theta$, where θ is a congruence of $\boldsymbol{A}$. It follows that every profinite completion is a profinite algebra, while the converse need not be true in general.

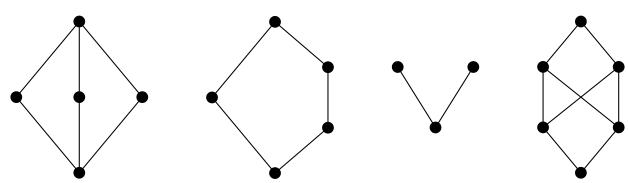
Even though the study of profinite Heyting algebras and completions has recently gained attention [1, 2, 3, 4], the problem of determining whether all profinite Heyting algebras are profinite completions remained open [4]. In this talk we will resolve it, by characterizing the varieties of Heyting algebras whose profinite members are profinite completions. As a consequence, we will be able to exhibit an array of profinite Heyting algebras that cannot be obtained as profinite completions of any Heyting algebra.

To this end, we rely on the following description of profinite Heyting algebras and completions [1, 2]. A poset is said to be *image finite* when its principal upsets are finite. Accordingly, the *image finite part* X_{fin} of a poset X is the subposet of X with universe

$$X_{\text{fin}} := \{x \in X : \uparrow x \text{ is finite}\}.$$

It follows immediately that X_{fin} is an image finite poset. For the present purpose, the interest of image finite posets is that a Heyting algebra is profinite precisely when it is isomorphic to the algebra of upsets $\mathsf{Up}(X)$ of an image finite poset X. Furthermore, the profinite completion of a Heyting algebra $\boldsymbol{A}$ is isomorphic to $\mathsf{Up}(X)$ where X is the image finite part of the Esakia space that dualizes $\boldsymbol{A}$ in the sense of [7, 9].

Consequently, in order to construct a profinite Heyting algebra that is not a profinite completion, it suffices to exhibit an image finite poset X that is not the image finite part of any Esakia space (in which case, $\mathsf{Up}(X)$ is a profinite Heyting algebra that is not a profinite completion). We shall do this by constructing, for each poset P depicted below, a p-morphic image X of a disjoint union of copies of P that is image finite, but is not the image finite part of any Esakia space.



P_1 P_2 P_3 P_4

Since p-morphic images and disjoint unions preserve the validity of formulas, this implies that if the profinite members of a variety K of Heyting algebras are profinite completions, then K must omit the Heyting algebra $\mathsf{Up}(P)$, for every poset P in the above picture.

The converse of this result is also true and constitutes the main result of the talk. To prove it observe that, in view of Jankov's Lemma, there exists a largest variety of Heyting algebras that omits $\mathsf{Up}(P_i)$ for $i = 1, \ldots, 4$. We call its members *diamond Heyting algebras*, because of the shape of their Esakia duals. More precisely, we say that a poset X is a *diamond system* if it satisfies the following conditions:

(i) $\uparrow x$ satisfies Esakia's three point rule for each $x \in X$;
(ii) X has width at most two;
(iii) Principal upsets are upward directed in X;

(iv) For every $\bot, x, y, z, v, \in X$, if $\bot \leqslant x, y \leqslant z, v$, there is $w \in X$ such that

$$x, y \leqslant w \leqslant z, v.$$

Image finite downward directed diamond systems are linear sums of diamonds and lines, whencefrom this terminology.

Diamond algebras and systems are related as follows.

Theorem 1. *The following conditions hold:*

(i) *A Heyting algebra is diamond if and only if the order reduct of its Esakia dual is a diamond system;*
(ii) *If X is a poset and $\mathsf{Up}(X)$ a diamond Heyting algebra, then X is a diamond system;*
(iii) *Every image finite diamond system is the image finie part of the Esakia dual of a diamond Heyting algebra.*

Bearing this in mind, let K be a variety of diamond Heyting algebras and $\boldsymbol{A}$ a profinite member of K. Since $\boldsymbol{A}$ is profinite, it has the form $\mathsf{Up}(X)$ for an image finite poset X that, moreover, is a diamond system in view of Condition (ii) of the above theorem. By Condition (iii) of the same theorem, X is the image finite part of some Esakia space, whence $\mathsf{Up}(X)$ (and, therefore, $\boldsymbol{A}$) is a profinite completion. We conclude that all the profinite members of K are profinite completions. This establishes the remaining part of the main result of the talk.

Theorem 2. *Let K be a variety of Heyting algebras. The profinite members of K are profinite completions if and only if K is a variety of diamond Heyting algebras.*

Corollary 3. *The problem of determining whether the profinite members of a variety of Heyting algebras (which can be presented either by a finite set of equations of by a finite set of algebras) are profinite completions is decidable.*

We closed this talk by observing that intermediate logics algebraized by varieties of diamond Heyting algebras form a denumerable set and are both locally tabular [8] and hereditarily structurally complete [6]. Furthermore, they have the infinite Beth definability property [10]. These results have been collected in [5].

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BI-GÖDEL ALGEBRAS AND CO-TREES

NICK BEZHANISHVILI, MIGUEL MARTINS, AND TOMMASO MORASCHINI

A Heyting algebra whose order dual is also a Heyting algebra is called a *bi-Heyting algebra*. From a logical standpoint, the interest for bi-Heyting algebras stems from the fact that they algebraize the bi-intuitionistic logic **Bi-IPC**. Unlike the thoroughly investigated lattice of superintuitionistic logics $\Lambda(\mathbf{IPC})$ (see, e.g., [3]), the lattice $\Lambda(\mathbf{Bi\text{-}IPC})$ of axiomatic extensions of **Bi-IPC** lacks such an in-depth investigation. Accordingly, in this talk, we study a simpler sublattice of $\Lambda(\mathbf{Bi\text{-}IPC})$: the lattice of axiomatic extensions of the bi-intuitionistic linear calculus

$$\mathbf{Bi\text{-}LC} \coloneqq \mathbf{Bi\text{-}IPC} + (p \to q) \vee (q \to p),$$

which we call *bi-Gödel-Dummett logic*.

It turns out that the lattice of axiomatic extensions of **Bi-LC** has a rich and complex, yet understandable structure. Moreover, the logic **Bi-LC** has some characteristics that make it an interesting object of study by itself, as we explain below.

First, the bi-Heyting algebras (respectively, the subdirectly irreducible bi-Heyting algebras) which satisfy the Gödel-Dummett prelinearity axiom

$$(p \to q) \vee (q \to p)$$

are exactly those whose bi-Esakia duals are co-forests (respectively, co-trees). Consequently, **Bi-LC** is the bi-intuitionistic logic of co-trees. Moreover, we show that the bi-intuitionistic logic of chains is a proper extension of **Bi-LC**. This contrasts with the intuitionistic case, where the logic of chains, **LC**, coincides with that of co-trees. Thus, from the point of view of **IPC**, the class of co-trees is indistinguishable from that of chains. This can be viewed as an indication that the language of **Bi-IPC** is more appropriate to study tree-like structures.

We also prove that, unlike many axiomatic extensions of **Bi-IPC**, **Bi-LC** admits a form of a classical inconsistency lemma in the sense of [4], that is, for every set of formulas $\Sigma \cup \{\alpha\}$, we have that

$$\Sigma, \sim\neg\sim\alpha \vdash_{\mathbf{Bi\text{-}LC}} \bot \iff \Sigma \vdash_{\mathbf{Bi\text{-}LC}} \alpha.$$

Note that the above equivalence entails that in the deductive system $\vdash_{\mathbf{Bi\text{-}LC}}$, one can reason by *reductio ad absurdum*.

In this work, we investigate the structure of the lattice $\Lambda(\mathbf{Bi\text{-}LC})$ of axiomatic extensions of **Bi-LC**. Since **Bi-LC** is algebraized by the class **bG** of *bi-Gödel algebras*, the lattice $\Lambda(\mathbf{Bi\text{-}LC})$ is dually isomorphic to that of varieties of bi-Gödel algebras. Consequently, we can study the former through the lenses of the latter. Some of the fundamental properties of the variety of bi-Gödel algebras are collected in the following theorem, which highlights a sharp contrast with the **HA**-reduct of **bG**, which is well-known to be locally finite [3], hence all of its subvarieties enjoy the finite model property (FMP for short).

Theorem 1. **bG** *is a discriminator variety, is not locally finite, and while it has the FMP, there are subvarieties of* **bG** *which lack the FMP.*

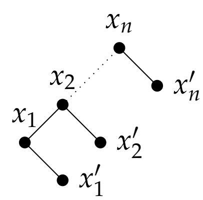
We also develop the theories of Jankov-style formulas for bi-Gödel algebras. In particular, we define the Jankov, stable canonical, and subframe formulas for these algebras. We also construct an infinite antichain of finite subdirectly irreducible bi-Gödel algebras with respect to the standard order of being (isomorphic to) a subalgebra. By adjusting the standard methods to our setting (as in [1] and [3]), we obtain:

Theorem 2. $|\Lambda(\mathbf{Bi\text{-}LC})| = 2^{\aleph_0}$, *and if* $L \in \Lambda(\mathbf{Bi\text{-}LC})$, *then:*

(i) *L is a splitting logic iff L is the logic of a finite co-tree;*
(ii) *L is axiomatizable by stable canonical formulas.*

Using the defining properties of the subframe formulas and the frame-theoretic properties of a particular class of co-trees, the finite combs, we derive a characterization of the locally finite varieties of bi-Gödel algebras, as stated in the theorem below.

Definition 3. For each positive $n \in \omega$, we define the *n-comb* $\mathfrak{C}_n := (C_n, R_n)$ as the co-tree depicted below. We also let $\beta(\mathfrak{C}_n)$ denote the subframe formula of the algebraic dual of $\mathfrak{C}_n$.



Theorem 4. *Let* $\mathbf{V}$ *be a variety of bi-Gödel algebras. Then* $\mathbf{V}$ *is locally finite iff* $\mathbf{V} \vDash \beta(\mathfrak{C}_n)$, *for some* $n \in \omega$.

This work is based on [2].

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Interpolation and Beth definability in implicative fragments of IPC

Intermediate logics are axiomatic extensions of the intuitionistic propositional calculus IPC that are contained in the classical propositional calculus CPC. There is a dual isomorphism from the lattice of extensions of IPC to the lattice of all varieties of Heyting algebras. Under this correspondence, logical properties can be translated into algebraic terms that, in turn, are amenable to the powerful methods of universal algebra. A classic example of this is Maksimova's characterization of intermediate logics with the interpolation and the projective Beth properties. More precisely, in [Mak79] it is proved that there are exactly eight intermediate logics with the interpolation property, and in [Mak03] it is shown that there are exactly sixteen intermediate logics with the projective Beth property.

Interpolation and Beth definability are important logical properties (see, e.g., [GM05]). For example, if we prove that there is a unique mathematical object X satisfying certain properties, we might like to find an 'explicit' formulation or definition of X. If we are working in a logic satisfying the relevant variant of Beth definability, we know that an explicit definition of X indeed can be found (see, e.g., [GM05, Chapter 1]). In this talk, we consider three kinds of Beth definability, whose strengths are related as follows:

$$\text{projective Beth property} \Rightarrow \text{infinite Beth property} \Rightarrow \text{finite Beth property.}$$

Interpolation and Beth properties are very sensitive to changes in the signature and, in particular, need not persist in fragments of a given logic. For instance, let $\mathrm{IPC}_{\to}$ be the fragment of IPC that employs only the connective $\to$; define $\mathrm{IPC}_{\wedge,\to}$ and $\mathrm{IPC}_{\wedge,\to,\bot}$ similarly. Blok and Hoogland [BH06, Corollary 4.6] proved that no consistent axiomatic extension of $\mathrm{IPC}_{\to}$ has the infinite Beth property. This contrasts with the full signature case, as there are continuum many intermediate logics with the infinite Beth property, which was proved in [BMR17].

In this talk, we investigate interpolation and the three variants of the Beth property in axiomatic extensions of $\mathrm{IPC}_{\wedge,\to}$ and $\mathrm{IPC}_{\wedge,\to,\bot}$. These logics are algebraized, respectively, by the varieties of implicative semilattices (ISL's) and bounded ISL's. An axiomatic extension of these logics has the interpolation property if and only if the variety of (bounded) ISL's associated with it has the amalgamation property. Similarly the various Beth properties correspond to algebraic properties that concern the surjectivity of epimorphisms.

Theorem **1**. There are exactly four varieties of ISL's and nine varieties of bounded ISL's with the amalgamation property.

Corollary **1**. There are exactly four logics above $\mathrm{IPC}_{\wedge,\to}$ and nine logics above $\mathrm{IPC}_{\wedge,\to,\bot}$ with the interpolation property.

For example, those four logics are IPC, the Gödel logic, CPC and the inconsistent logic.

Theorem **2**. There are exactly eight varieties of ISL's and thirty varieties of bounded ISL's with strong epimorphism surjectivity.

Corollary **2**. There are exactly eight logics above $\mathrm{IPC}_{\wedge,\to}$ and thirty logics above $\mathrm{IPC}_{\wedge,\to,\bot}$ with the projective Beth property.

Theorem **3**. Continuum many (but not all) varieties of (bounded) ISL's have epimorphism surjectivity.

Corollary **3**. Continuum many (but not all) axiomatic extensions of $\mathrm{IPC}_{\wedge,\to(,\bot)}$ have the infinite Beth property.

Our main tool is Köhler's duality from [Köh81] between the categories of finite implicative semilattices and finite posets with certain partial functions as arrows.

The following table summarizes the main results, notably the amounts of varieties that have the properties stated in the left column. Every algebraic property is associated with a logical property; the numbers in the table also indicate how many axiomatic extensions of IPC restricted to the indicated signature, have the concerning logical property. The entries in blue are the results of this research; references to the essential discoveries of the other results are included in the table.

Signature	$\wedge,\to$	$\wedge,\to,0$	$\wedge,\to,\vee$	$\wedge,\to,0,\vee$
Structure	ISL	bounded ISL	IL	Heyting algebra
Amalgamation / interpolation	4	9	4 [Mak03]	8 [Mak79]
Strong ES / projective Beth	8	30	7 [Mak03]	16 [Mak00]
ES / infinite Beth	$2^{\aleph_0}$	idem	?	$2^{\aleph_0}$ [BMR17]
Weak ES / finite Beth	all [Kre60]	idem	idem	idem

This work is based on P. M. Dekker's Bachelor's thesis [Dek20].

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A topological duality for monotone expansions of semilattices

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Abstract

In this paper, we provide a Stone-type duality for monotone meet semilattices by combining the topological duality developed in [3] for meet semilattices with top element together with the approach of [2] used to obtain a duality for monotonic distributive semilattices.

In [1] Birkhoff proved that every finite distributive lattice is represented by the lower sets of a finite partial order. In [6] Stone generalized the results given by Birkhoff showing a categorical duality (known as Stone duality) for bounded distributive lattices through spectral spaces with continuous functions. Along the years the Stone duality has shown to be a powerful tool not only for the study of bounded distributive lattices, but also for the study of many ordered algebraic structures associated with non-classical logics. A particular case of interest raises when combining this duality with some other tools coming from algebra. This is the case of canonical extensions. The study of canonical extensions started with Jónsson and Tarski's works [4],[5] for Boolean algebras with operators. The most remarkable contribution of these papers was that they provided a procedure to transfer the benefits and the working methodology from the duality for Boolean algebras to several classes of algebras with additional operators. Later on, these results were generalized to several kinds of lattice expansions. In particular, in [2] Celani and Menchón showed that the employment of canonical extensions can be especially useful at the moment of establishing Stone-type dualities for distributive semilattices with suitable relational topological spaces.

Let X be a topological space, $\mathcal{K}$ be a subbasis for X and $Y \subseteq X$. Now consider the sets $S(X) = \{U^c \colon U \in \mathcal{K}\}$ and $C_{\mathcal{K}}(X) = \{\bigcap \mathcal{A} \colon \mathcal{A} \subseteq \mathcal{K}\}$. In [3] an Y-family was defined as a family $\mathcal{F} \subseteq S(X)$ in which for all $A, B \in \mathcal{F}$, there exists $H, C \in S(X)$ such that $Y \subseteq H$, $A \cap H \subseteq C$ and $B \cap H \subseteq C$. In the same paper *S-spaces* were defined as pairs $\langle X, \mathcal{K}\rangle$ satisfying the following conditions:

1. X is a T_0-space.
2. $\mathcal{K}$ is a subbase of compact open subsets, which is closed under finite unions and $\emptyset \in \mathcal{K}$.
3. For every $U, V \in \mathcal{K}$, if $x \in U \cap V$, then there exist $W, D \in \mathcal{K}$ such that $x \notin W$, $x \in D$ and $D \subseteq (U \cap V) \cup W$.
4. If $Y \in C_{\mathcal{K}}(X)$ and $\mathcal{F} \subseteq S(X)$ is a Y-family such that $Y \cap A^c \neq \emptyset$, for all $A \in \mathcal{F}$, then $Y \cap \bigcap\{A^c \colon A \in \mathcal{F}\} \neq \emptyset$.

Let $\langle X_1, \mathcal{K}_1\rangle$, $\langle X_2, \mathcal{K}_2\rangle$ be two S-spaces, $T \subseteq X_1 \times X_2$ be a relation and let $\Box_T : \mathcal{P}(X_2) \to \mathcal{P}(X_1)$ be the mapping defined by $\Box_T(U) = \{x \in X_1 \colon T(x) \subseteq U\}$. A *meet-relation* between two S-spaces $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ was defined in [3] as a subset $T \subseteq X_1 \times X_2$ satisfying the following conditions:

1. For every $U \in S(X_2)$, $\Box_T(U) \in S(X_1)$, and
2. $T(x) = \bigcap\{U \in S(X_2) : T(x) \subseteq U\}$ for all $x \in X_1$.

Let $\langle X_i, \mathcal{K}_i \rangle$, with $i = 1, 2, 3$ be S-spaces and $R \subseteq X_1 \times X_2$ and $T \subseteq X_2 \times X_3$ be meet-relations. Consider now the following relation

$$T * R = \{(x, z) \in X_1 \times X_3 \colon (\forall U \in S(X_3))((T \circ R)(x) \subseteq U \Rightarrow z \in U)\}.$$

In [3] it was proved that S-spaces and meet-relations form a category in which the composition of maps is given by $*$ and the identity arrow is given by the dual of the specialization order.

Let $\langle X, \mathcal{K} \rangle$ be an S-space. We say that $Z \subseteq X$ is a *subbasic saturated subset* if there exists a dually directed family $\mathcal{L} \subseteq \mathcal{K}$ such that $Z = \bigcap\{W \colon W \in \mathcal{L}\}$. We denote by $\mathcal{Z}(X)$ the set of all subbasic saturated subsets of an S-space $\langle X, \mathcal{K} \rangle$. For each $U \in S(X)$ we define the subset L_U of $\mathcal{Z}(X)$ as follows:

$$L_U = \{Z \in \mathcal{Z}(X) \colon Z \cap U \neq \emptyset\}.$$

Definition 1. A *monotonic S-space* (mS-space, for short) is a structure $\langle X, \mathcal{K}, R \rangle$, such that $\langle X, \mathcal{K} \rangle$ is a S-space, and $R \subseteq X \times \mathcal{Z}(X)$ is a multirelation such that:

1. $m_R(U) = \{x \in X : \forall Z \in R(x)[Z \cap U \neq \emptyset]\} \in S(X)$, for every $U \in S(X)$.
2. $R(x) = \bigcap\{L_U : U \in S(X) \text{ and } x \in m_R(U)\}$, for every $x \in X$.

Definition 2. Let $\langle X_1, \mathcal{K}_1 \rangle$ and $\langle X_2, \mathcal{K}_2 \rangle$ be two S-spaces. A meet-relation $T \subseteq X_1 \times X_2$ is a *monotonic meet-relation* if $(m_{R_1} \circ \Box_T)(U) = (\Box_T \circ m_{R_2})(U)$, for every $U \in S(X_2)$.

Let $\mathbf{A} = \langle A, \wedge, 1 \rangle$ be a semilattice. An operator $m : A \to A$ is said to be monotone if it is an order preserving map. By a *monotone semilattice* we mean a pair $\langle \mathbf{A}, m \rangle$ where $\mathbf{A}$ is a semilattice and m is a monotone operator on A. We write mMS for the algebraic category of monotonic semilattices and homomorphisms.

In this paper we prove that mS-spaces and monotonic meet-relations form a category called mSsp in which the identity map is given by the dual of the specialization order. In addition, we prove that this category is indeed dually equivalent to mMS. Moreover, as an application of this duality we provide a characterization of congruences of monotone semilattices by means of lower Vietoris type topologies. It is worth mentioning that a key tool for developing this duality is the topological description of canonical extension of semilattices that we provide, in terms of the *subbasic saturated subsets* of the associated S-space.

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Representable forests and diamond systems.

A poset is said to be *Priestley* (resp. *Esakia*) *representable* if it is isomorphic to the prime spectrum of a bounded distributive lattice (resp. Heyting algebra). In view of Priestley [11, 12] and Esakia [4, 5] dualities, a poset $(X, \leq)$ is Priestley (resp. Esakia) representable if and only if it can be endowed with a topology τ turning $(X, \leq, \tau)$ into a Priestley (resp. Esakia) space. The problem of describing the structure of Priestley representable posets was raised in [2, 7] and it is equivalent to the classical problem of characterizing the posets isomorphic to the spectrum of a commutative ring with unit [3, 8, 9]. The problem of characterizing Esakia representable posets was raised in [5]. Both Priestley and Esakia representability problems remain open.

We study the representability problems by restricting the attention to two classes of posets: **forests**, i.e., posets whose principal downsets are chains, and **diamond systems**, a class that includes the order duals of forests. Diamond systems have been introduced recently in [1] in order to characterize the varieties of Heyting algebras whose profinite members are profinite completions.

First, we provide a characterization of Priestley and Esakia representable diamond systems. Specifically, we prove that a diamond system $(X, \leq)$ is Priestley (resp. Esakia) representable if and only if its nonempty chains have infima and suprema in $(X, \leq)$ and, for every $x, y \in X$,

if $x < y$, there are $x', y' \in X$ such that $x \leq x' < y' \leq y$ and $[x', y'] = \{x', y'\}$.

These two conditions have already been introduced in [7, 9, 10]. As Priestley representable posets are closed under order duals, this yields a new proof of Lewis' description of Priestley representable forests [10]. Moreover, the representation of diamond systems can be used to simplify the main proof of [1]. While a classification of arbitrary Esakia representable forests remains open, the main result of this talk is a characterization of Esakia representable well-ordered forests. In particular, we will prove that a well-ordered forest is Esakia representable if and only if its nonempty chains have suprema. We will also suggest how to proceed in the case of Esakia representable countable forests, by providing two new classes of countable forests which are not Esakia representable.

These results appear in [6]. The work contained therein has been done in collaboration with Nick Bezhanishvili and Tommaso Moraschini.

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Choice-free duality for orthocomplemented lattices by means of spectral spaces

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Abstract

The existing topological representation of an orthocomplemented lattice via the clopen orthoregular subsets of a Stone space depends upon Alexander's Subbase Theorem, which asserts that a topological space X is compact if every subbasic open cover of X admits of a finite subcover. This is an easy consequence of the Ultrafilter Theorem—whose proof depends upon Zorn's Lemma, which is well known to be equivalent to the Axiom of Choice. Within this work, we give a choice-free topological representation of orthocomplemented lattices by means of a special subclass of spectral spaces; choice-free in the sense that our representation avoids use of Alexander's Subbase Theorem, along with its associated nonconstructive choice principles. We then introduce a new subclass of spectral spaces which we call *upper Vietoris orthospaces* in order to characterize up to homeomorphism (and isomorphism with respect to their orthospace reducts) the spectral spaces of proper lattice filters used in our representation. It is then shown how our constructions give rise to a choice-free dual equivalence of categories between the category of orthocomplemented lattices and the category of upper Vietoris orthospaces. Lastly, in light of our newly established duality, we develop a duality dictionary, which explicitly demonstrates how various lattice-theoretic concepts (as applied to orthocomplemented lattices and their homomorphisms) can be translated into their corresponding dual topological counterparts (as applied to upper Vietoris orthospaces and their spectral p-morphisms). Our duality combines Bezhanishvili and Holliday's choice-free spectral space approach to Stone duality for Boolean algebras with Goldblatt and Bimbó's choice-dependent orthospace approach to Stone duality for orthocomplemented lattices.

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Generalized Heyting Algebras and Duality

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Let $\mathcal{A} = (A, \leq, \wedge, \vee, 0, 1)$ be a bounded (distributive) lattice. A tuple $(\mathcal{A}, \nabla, \to)$ is called a (distributive) ∇-algebra if $\nabla c \wedge a \leq b$ is equivalent to $c \leq a \to b$, for any $a, b, c \in A$, or in a more abstract term $\nabla(-) \wedge a \dashv a \to (-)$, for any $a \in A$. A ∇-algebra is called normal if ∇ commutes with all finite meets. ∇-algebras generalize both bounded lattices ($\nabla c = 0$ and $a \to b = 1$) and Heyting algebras ($\nabla c = c$). More interestingly, their generalization of Heyting implication provides a representation for abstract implications as introduced in [1] in which any implication becomes a part of an adjunction presenting a full pair of introduction and elimination rules. They also provide a point-free presentation for dynamic systems [4] and algebraic models for basic intuitionistic tense logics, as explained in the following two examples.

Example 0.1. *(Dynamic systems)* Let X be a topological space and $f : X \to X$ be a continuous function. Define $\to_f$ over $\mathcal{O}(X)$ by $U \to_f V = f_*(U^c \cup V)$, where $f_* : \mathcal{O}(X) \to \mathcal{O}(X)$ is the right adjoint of f^{-1}. Then, the structure $(\mathcal{O}(X), f^{-1}, \to_f)$ is a normal distributive ∇-algebra. This ∇-algebra is the point-free version of the dynamic system (X, f), using the adjunction $f^{-1} \dashv f_*$ to encode the map f.

Example 0.2. *(Intuitionistic Kripke Frames)* Let $(W, \leq)$ be a poset. By an *intuitionistic Kripke frame* we mean a tuple $\mathcal{K} = (W, \leq, R)$, where R is a binary relation over W, compatible with the partial order, i.e., if $k' \leq k$, $l \leq l'$ and $(k, l) \in R$ then $(k', l') \in R$, for any $k, k', l, l' \in W$. To any intuitionistic Kripke frame, we can assign a canonical ∇-algebra, encoding its structure via topology. Set $\mathfrak{X}$ as the locale of all upsets of $(W, \leq)$ and define $\nabla : \mathfrak{X} \to \mathfrak{X}$ as $\nabla_{\mathcal{K}} U = \{x \in W \mid \exists y \in U \; R(y, x)\}$ and $U \to_{\mathcal{K}} V = \{x \in W \mid \forall y \in W [R(x, y) \wedge y \in U \Rightarrow y \in V]\}$. It is easy to see that $(\mathfrak{X}, \nabla_{\mathcal{K}}, \to_{\mathcal{K}})$ is a distributive ∇-algebra. This ∇-algebra presents an algebraic model for intuitionistic temporal logic in which ∇ refers to a $\Diamond$-style modality for the past while $1 \to (-)$ refers to a $\Box$-style modality for the future.

In this talk we will present the algebraic properties of different varieties of ∇-algebras inspired by either their temporal interpretation or their reading as dynamical systems. These properties include Dedekind-MacNeille completion, canonical extension, congruence extension property, amalgamation property and the characterization of subdirectly irreducible and simple ∇-algebras. We will also present Priestley-Esakia and spectral dualities for distributive ∇-algebras, following the approaches in [3] and [2].

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