Local finiteness and bisimulation games

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Terms and identities

An abstract algebra is a set with operations of arities ≥0:

$$\mathbf{A}=(A, (f_i)_{i\in I}),$$

0-ary operations are elements of A.

The operation (function) symbols make the *signature* of **A**.

E.g. $\mathbf{Z}=(Z, +, -, \cdot, 0, 1)$.

A term of a given signature is an expression built from variables (x,y,...) and operation symbols.

E.g. $(x + y)\cdot(x - y)$.

An identity is an equality of terms.

Varieties of algebras-1

A $\models t(x_1,\ldots,x_n)=r(x_1,\ldots,x_n)$ (an identity holds in **A**),

if $t(a_1,\ldots,a_n)=r(a_1,\ldots,a_n)$ for any $a_1,\ldots,a_n \in A$.

A variety of algebras is a class of algebras of the same signature satisfying a certain system of identities Σ.

$V(\Sigma) = \{\mathbf{A} \mid \mathbf{A} \models t=r, \text{ for all } (t=r)\in\Sigma\}$.

Example In the signature $(\cdot, {}^{-1}, 1)$ the variety of groups is axiomatized by identities

$$(x\cdot y)\cdot z= x\cdot(y\cdot z),\ 1\cdot x = x,\ x^{-1}\cdot x=1.$$

Varieties of algebras-2

Birkhoff theorem A non-empty class $\mathscr{C}$ of algebras of the same signature is a variety

$\Leftrightarrow$ $\mathscr{C}$ is closed under taking subalgebras, homomorphic images, and direct products.

- A free algebra of rank k of a variety $\mathscr{M}$

  $\mathscr{F}_{\mathscr{M}}(k)$ consists of terms in variables $x_1,\ldots,x_k$, up to equivalence in $\mathscr{M}$ ($\mathscr{M} \vDash t=r$).

  Every k-generated algebra in $\mathscr{M}$ is a homomorphic image of $\mathscr{F}_{\mathscr{M}}(k)$.

Finite (finitely generated) varieties

A finite variety is generated by a single finite algebra. I.e., this is the least variety containing a given finite algebra **A**:

$$V(\{ t=r \mid \mathbf{A} \models t=r\}).$$

Example 1 The variety of Boolean rings is finite.

Boolean rings are defined by axioms of commutative rings with unit in the signature $(+, \cdot, 0, 1)$, plus idempotence: $x \cdot x = x$. This variety is generated by the field $\mathbf{F}_2$.

Example 1' A Boolean algebra is an equivalent of a Boolean ring in the signature $(\cup, \cap, -, 0, 1)$. The axioms are well-known.

Every Boolean ring can be made a Boolean algebra, and vice versa.

The variety of Boolean algebras is generated by the two-element algebra **2**.

Local finiteness-1

A variety is locally finite, if all its finitely generated algebras are finite.

Or: all free algebras of finite rank are finite.

Finiteness ⇒ Local finiteness

Examples The variety of Abelian groups of period n

(axioms: Groups ∪ {xy=yx, x^n=1}) is finite: it is generated by $\mathbf{Z}_n$.

The variety of groups of period 3 is locally finite, but not finite.

Local finiteness-2

The *bounded Burnside problem*: is the variety of groups of period n locally finite for any finite n?

Counterexamples (Adian – Novikov, reworked by Adian): any odd n>665.

The problem is open for small odd n.

Finite approximability-1

A finitely approximable variety is generated by finite algebras.

Local finiteness ⇒ finite approximability

This is because $\mathscr{M}$ is generated by algebras $\mathscr{F}_{\mathscr{M}}(k)$ for finite k.

<u>Example</u> The variety of groups $\mathscr{G}$ is finitely approximable, but not locally finite. It is generated by symmetric groups $\mathbf{S}_n$, but $\mathscr{F}_{\mathscr{G}}(2)$ is infinite.

Finite approximability-2

- Post–Harrop theorem If a variriety $\mathscr{M}$ is axiomatized by a finite set of identities (finitely axiomatizable) and finitely approximable, then the set of identities true in $\mathscr{M}$ (the equational theory) is algorithmically decidable.
- Remark For any variety

$$\mathscr{M} \vDash t(x_1,\dots,x_n)=r(x_1,\dots,x_n) \Leftrightarrow$$

$$\mathscr{F}_{\mathscr{M}}(n) \vDash t(x_1,\dots,x_n)=r(x_1,\dots,x_n).$$

Thus decidability of identities in $\mathscr{M}$ is equivalent to decidablity of identities in all free algebras of finite ranks $\mathscr{F}_{\mathscr{M}}(k)$ ("word problems" - for the case of groups).

Modal algebras-1

The variety of normal modal algebras.

Signature: Boolean with an extra unary operation:

$(\cup, \cap, -, 0, 1, \Box)$

Axioms:

- Boolean
- $\Box(x \cap y) = \Box x \cap \Box y$
- $\Box 1 = 1$

Similarly *polymodal algebras* (with several extra $\Box$s) can be defined.

Modal algebras-2

Example 1 Let S be a topological space, then there is a modal algebra

$MA(S)=(2^S, I)$, where

2^S is the Boolean algebra of all subsets of S,

IX is the interior of X.

There are additional identities in this algebra:

$$\Box x \cap x = \Box x, \Box\Box x = \Box x.$$

Remark In a modal algebra there is an opeation dual to $\Box$:

$\Diamond x = -\Box -x.$

In MA(S) this is the topological closure operation.

Modal algebras-3

<u>Example 2</u> Let F=(W,R) be a non-empty set with a binary relation (a Kripke frame). Then there is a modal algebra

MA(F)=(2^W, $\Box$), where

- $\Box X = \{u \in W \mid R(u) \subseteq X\}$. In this algebra

$\Diamond X = R^{-1}(X)$.

The well-known Stone representation theorem for Boolean algebras extends to modal algebras as follows.

<u>Theorem (Tarski & Jonsson)</u> Every modal algebra $\mathfrak{A}$ is embeddable in the algebra of its *canonical Kripke frame*:

$$\mathfrak{A} \rightarrowtail \mathfrak{A}_+ = MA(Uf(\mathfrak{A}), R),$$

where Uf($\mathfrak{A}$) is the set of ultrafilters of $\mathfrak{A}$,

$$uRv \Leftrightarrow \forall x\,(\Box x \in u \Rightarrow x \in v).$$

Modal formulas-1

Modal formulas are terms in the signature of modal algebras in a countable set of variables (proposition letters) $PL=\{p_1,p_2,...\}$. Boolean operations in formulas are denoted by *logical connectives:* $\vee$ (disjunction), $\wedge$ (conjunction), $\neg$ (negation), $\Box$ (necessity), $\bot$ (falsity), $\top$ (truth).

Derived connectives are used for abbreviations:

$A \to B = \neg A \vee B$ (implication),

$A \leftrightarrow B = (A \to B) \wedge (B \to A)$ (equivalence),

$\Diamond = \neg \Box \neg$ (possibility).

Modal formulas-2

Identities in Boolean and modal algebras can be rewritten:

$\mathfrak{A} \models A = B \Leftrightarrow \mathfrak{A} \models (A \leftrightarrow B)=T$

So we can consider only identitites of the form A = T.

$\mathfrak{A} \models A = T$ is usually written as $\mathfrak{A} \models A$ and read as a formula

A is true in an algebra $\mathfrak{A}$.

Modal logics-1

A modal logic is a set L of modal formulas such that

- L contains all Boolean tautologies
- L is closed under Modus Ponens: $A,\ A \to B \in L \Rightarrow B \in L$
- L is closed under adding $\Box$: $A \in L \Rightarrow \Box A \in L$
- L is closed under substitutions:

  $A(p_1,\ldots,p_n) \in L \Rightarrow A(B_1,\ldots,B_n) \in L$ (for any formulas $B_1,\ldots,B_n$)
- $\Box(p_1 \to p_2) \to (\Box p_1 \to \Box p_2) \in L$

  K is the minimal modal logic; **K**+Γ – is the minimal modal logic containing a set of formulas Γ.
- Polymodal logics are defined similarly.

Modal logics-2

Soundness theorem The formulas true in a certain modal algebra, constitute a modal logic:

$$L(\mathfrak{A}) = \{A \mid \mathfrak{A} \models A\}$$

Completeness theorem Every modal logic Λ corresponds to a modal algebra, the free algebra of its variety:

$$\Lambda = L(\mathscr{F}_{V(\Lambda)}(\omega))$$

Also

$$\Lambda = L(\{\mathscr{F}_{V(\Lambda)}(n) \mid n \text{ is finite}\})$$

Hence we have a 1-1 correspondence

Modal logics ⇹ Varieties of modal algebras

Kripke models-1

A formula A is *valid* on a Kripke frame F ($F \models A$),

if $MA(F) \models A$.

An equivalent definition is given in terms of Kripke models.

A *Kripke model* on F is (F,θ), where θ is a *valuation*

$\theta: PL \to 2^W$ (i.e., $\theta(p_i) \subseteq W$).

In *k-weak Kripke models* a valuation is defined on $p_1,\ldots,p_k$.

Kripke models-2

<u>Def</u> A formula A is true at a point x of a Kripke model M ($M,x \vDash A$):

- $M,x \vDash p_i \Leftrightarrow x \in \theta(p_i)$
- $M,x \vDash A \vee B \Leftrightarrow (M,x \vDash A$ or $M,x \vDash B)$
- $M,x \vDash A \wedge B \Leftrightarrow (M,x \vDash A$ and $M,x \vDash B)$
- $M,x \vDash \neg A \Leftrightarrow M,x \nvDash A$
- $M,x \vDash \Box A \Leftrightarrow \forall y(xRy \Rightarrow M,y \vDash A)$

Then

- $M,x \vDash A \to B \Leftrightarrow (M,x \Vdash A \Rightarrow M,x \Vdash B)$
- $M,x \vDash \Diamond A \Leftrightarrow \exists y(xRy \;\&\; M,y \vDash A)$

<u>Claim</u> $F \vDash A$ iff A is true at all points of Kripke models on F.

Kripke models-3

Canonical model

This model M_L is obtained from the proof of Tarski–Jonsson theorem

- For any formula A

$M_L \vDash A$ (at all points) $\Leftrightarrow A \in L$

Also in M_L

- x,y verify the same formulas $\Leftrightarrow$ x=y.

Kripke completeness, tabularity, finite model property (FMP)

Kripke complete logics

L(F) := { A | F ⊨ A}=L(MA(F)) (the logic of a frame F),

L($\mathcal{C}$) := $\bigcap$ {**L**(F)|F ∈ $\mathcal{C}$} (the logic of a class of frames $\mathcal{C}$).

If F is finite, **L**(F) is called *tabular (finite)*

$$\Lambda \text{ is tabular} \Leftrightarrow \mathrm{V}(\Lambda) \text{ is finite}$$

- *If* $\mathcal{C}$ consists of finite frames, then **L**($\mathcal{C}$) *has the FMP*. This is an intersection of tabular logics.

$$\Lambda \text{ has the FMP} \Leftrightarrow \mathrm{V}(\Lambda) \text{ is finitely approximable}$$

('Harrop theorem') If a logic has the FMP and is finitely axiomatizable, then it is decidable.

Local tabularity

<u>Def</u> A logic L is locally tabular *(locally finite)*, if for any k there exist finitely many formulas in $p_1,\ldots,p_k$, up to equivalence in L.

$L\lceil k$ denotes the restriction of L to formulas in $p_1,\ldots,p_k$. $L\lceil k$ are called weak logics.

Local tabular is equivalent to each of the following conditions:

- All the $L\lceil k$ are tabular.
- The variety V(L) is locally finite.
- All weak canonical models $M_{L\lceil k}$ are finite.

Tabularity criterion-1

Theorem (Chagrov 1994)

L is tabular iff L ⊢ $\alpha_n \wedge Alt_n$ for some n.

Validity conditions for α_n, Alt_n on Kripke frames

- α_n forbids simple paths of length n+1:

  $x_0Rx_2...Rx_n$, where all x_i are different.

- Alt_n forbids (n+1)-branching: $xRx_0,..., xRx_n$, where x_i are different.

Tabularity criterion-2

$$\alpha_n = \neg\Diamond(P_0 \wedge \Diamond(P_2 \wedge \ldots \Diamond(P_{n-1} \wedge \Diamond P_n)\ldots)),$$

$$Alt_n = \neg(\Diamond P_0 \wedge \Diamond P_2 \wedge \ldots \wedge \Diamond P_n),$$

where

$$P_i = \neg p_i \wedge \bigwedge\{p_j \mid 0 \leq j \leq n,\ j \neq i\}.$$

Bisimulations-1

Bisimulations appeared in the 1970s in the works as a method for Kripke models comparison in the works by Johan Van Benthem (the original name was 'zigzag morphism').

They were later re-invented in Theoretical CS (David Park, Robin Milner) and applied to study of concurrency.

Bounded bisimulations were also introduced by Van Benthem (1979) for the modal logic purposes. Bisimulation games give a more visual description of bounded bisimulations; they first appeared in Theoretical CS (Colin Stirling, 1995).

Bisimulations-2

Def For Kripke models $M=(W,R,\theta)$, $M'=(W',R',\theta')$ (with the same proposition letters) there is the *0-equivalence* relation between points

$$M,x \equiv_0 M',x' := \text{for any } q \in PL \text{ (or } PL\lceil k) \ (M,x \vDash q \Leftrightarrow M',x' \vDash q).$$

Def (Van Benthem, 1974) A *bisimulation* between Kripke models $M=(W,R,\theta)$, $M'=(W',R',\theta')$ is a binary relation between their points

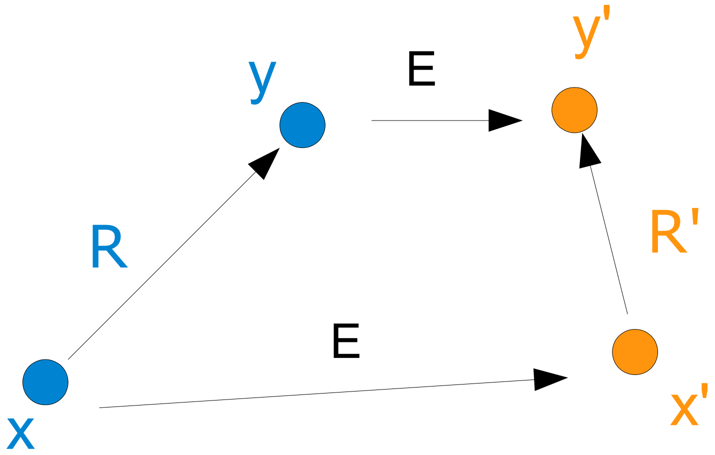
$E \subseteq W \times W'$ with the following properties.

(0) $xEx' \Rightarrow x \equiv_0 x'$

(zig) $xEx' \ \& \ xRy \Rightarrow \exists y' \ (x'R'y' \ \& \ yEy')$

(zag) $xEx' \ \& \ x'R'y' \Rightarrow \exists y \ (xRy \ \& \ yEy')$

Bisimulations-3



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Bisimulations-4

Def Kripke models with designated points M,x and M',x' are called *bisimilar* if there exists a bisimulation E between M and M' such xEx'.

Lemma Bisimilarity is an equivalence relation.

Bisimilarity is denoted by M,x $\underline{\leftrightarrow}$ M',x' (or just by x $\underline{\leftrightarrow}$ x').

Theorem (bisimulation invariance). Bisimilar points satisfy the same formulas:

if M,x $\underline{\leftrightarrow}$ M',x', then M,x $\equiv$ M',x' (where $\equiv$ means equivalence modulo all formulas of the corresponding language).

Proof. By induction on the length of formulas. This fact also follows from further considerations.

Bisimulations-5

A well-known example of a bisimulation.

A *morphism* M ↠ M' is a map $f: W \to W'$

such that

(0) $x \equiv_0 f(x)$

(monotonicity) $xR_i y \Rightarrow f(x)R'_i f(y')$

(lift property) $f(x)R'_i y' \Rightarrow \exists y\ (xR_i y\ \&\ f(y)=y')$

The latter two properties are exactly (zig) and (zag) if we put

$xEy \Leftrightarrow y=f(x)$. So a Kripke model morphism is nothing but a total functional bisimulation.

If M is a *generated submodel* of M', we have a paricular case of a morphism: f is the inclusion map, $f(x)=x$.

Monotonicity comes from “submodel”, the lift property from “generated”.

Bisimulation games-1

Origin: Colin Stirling (1995) << n-bisimulations by Johan Van Benthem (1989) << n-equivalence by Kit Fine (1974)

As above, we have two Kripke models with designated points (M,x_0), (M',y_0).

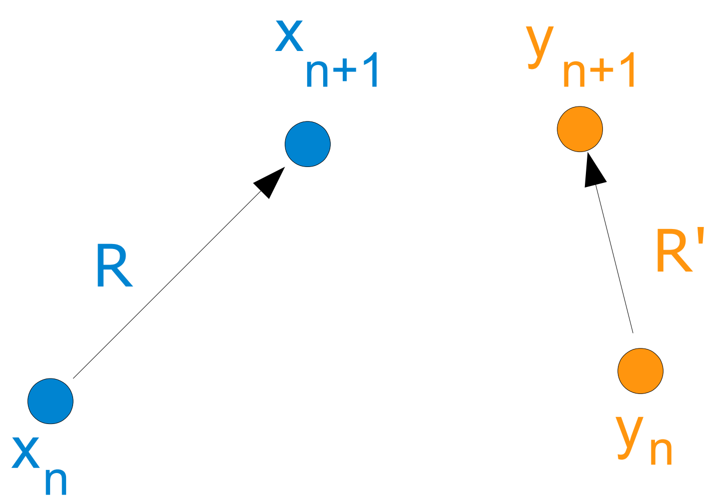
The *r-round bisimulation game* $BG_r(M,M',x_0,y_0)$ is played by two players: Spoiler (Abelard, **A**) and Duplicator (Éloïse, **E**).

Rules of the game

Round 0 If $x_0 \not\equiv_0 y_0$, **A** wins.

Bisimulation games-2

The n-th position (after n rounds) in $BG_r(M, x_0, y_0)$ is (x_n, y_n).



Round (n+1)

- **A** chooses x_{n+1} [or y_{n+1}] such that $x_n \, R \, x_{n+1}$ [$y_n R' y_{n+1}$]
- **E** chooses y_{n+1} [x_{n+1}] such that $y_n \, R' \, y_{n+1}$ [$x_n R x_{n+1}$] and $x_{n+1} \equiv_0 y_{n+1}$ '
- A player loses if he/she cannot move.
- **E** wins after r rounds.

Bisimulation games-3

md(A), the *modal depth* of a formula A, is defined by induction:

$md(p_i)=0$,

$md(A*B)= \max(md(A),md(B))$, * is a binary connective,

$md(\neg A)=md(A)$,

$md(\Box A)=md(A)+1$.

Def Formula and game *n-equivalence* relations between M,x and M',y.

- $M,x \equiv_n M',y$:= for any A (in the given language) of modal depth $\leq n$ $M,x \vDash A \Leftrightarrow M',y \vDash A$.
- $M,x \sim_n M',y$:= **E** has a winning strategy in $BG_n(M,M',x,y)$.

Remark $\equiv_n$ is clearly an equivalence; for $\sim_n$ this not so obvious. Anyway this follows from the next theorem.

Bisimulation games-4

Main Theorem on finite bisimulation games (Stirling, 1995)

On weak Kripke models with designated points

$$\equiv_n \, = \, \sim_n$$

Corollary $M,x \equiv M',y$ iff $M,x \sim_n M',y$ for all n.

We denote the latter relation by $\sim_\infty$ and call it

∞-bisimilarity ($\sim_n$ is *n-bisimilarity*).

Bisimulation games-5

Bisimilarity $\underline{\leftrightarrow}$ is also related to games, but now we need infinitely many rounds.

The *infinite* *bisimulation game* $BG_\omega(M,M',x_0,y_0)$ is played by the same rules as finite games, but the number of rounds is infinite. **E** wins, if the play does not stop. This game generates the equivalence $\sim$, and we have

Theorem $M,x \underline{\leftrightarrow} M',y$ iff $M,x \sim M',y$.

Canonical logics-1

Def A modal logic L is canonical if it is valid on its canonical frame $F_L := (W_L, R_{1,L}, \ldots, R_{N,L})$, i.e., $L \subseteq \mathbf{L}(F_L)$.

On the other hand, by Canonical model theorem,

$A \notin L \Rightarrow F_L \nvDash A$, i.e., $\mathbf{L}(F_L) \subseteq L$.

So *every canonical logic is complete*: $L = \mathbf{L}(F_L)$.

Many results on completeness for modal logics were obtained through canonicity. In fact, there are general theorems stating that modal logics with axioms of special form are canonical.

The most famous of them is Sahlqvist theorem.

Canonical logics-2

Def A positive modal formula is constructed from proposition letters, $\bot$, and $\top$, by using the connectives

$\Box_i$, $\Diamond_i$, $\wedge$, $\vee$. Negations of positive formulas are called negative.

- A boxed atom is a formula of the form $\blacksquare q$, where $q \in PL$, $\blacksquare$ is a (maybe, empty) sequence of boxes.
- A Sahlqvist antecedent is a formula obtained from boxed atoms and negative formulas by using $\Diamond_i$, $\wedge$, $\vee$.
- A Sahlqvist formula is of the form $\blacksquare(A \to B)$, where $\blacksquare$ is a sequence of boxes, A is a Sahlqvist antecedent, B is positive.

Canonical logics-3

Sahlqvist theorem (1975)

- Every logic axiomatized by Sahlqvist formulas is canonical.
- Every Sahlqvist formula is elementary.

Remark 1 There exists a further generalization of Sahlqvist theorem (Goranko & Vakarelov, 2002; Kikot, 2010).

Remark 2 Complete logics may be neither canonical, nor elementary. Well-known counterexamples are

GL (Gödel – Löb logic) = **L**(finite strict posets),

Grz (Grzegorczyk logic) = **L**(finite posets).

Remark 3 Canonical and elementary logics may be non-axiomatizable by Sahlqvist formulas. A counterexample:

S4.1 = **S4** + $\Box\Diamond p \to \Diamond\Box p$.

Formula depth-1

The *modal depth of a formula* A *in a* (maybe weak) *modal logic* L

$md_L(A) := \min\{md(B) \mid L \vdash A \leftrightarrow B\}$

The *modal depth of a logic L*

$md(L) := \max\{md_L(A) \mid A \text{ is in the language of } L\}$

It follows that $md(L) = \sup \{md(L\lceil k) \mid k \geq 0\}$.

A logic L is called *locally tabular* if every its weak fragment $L\lceil k$ is tabular.

Formula depth-2

Finite modal depth $\Rightarrow$ local tabularity $\Rightarrow$ FMP

- The first implication is easy, since there are finitely many k-formulas of bounded depth up to equivalence in the minimal logic (lemma 1, slide 16)
- The second implication: every locally tabular logic is determined by its weak canonical frames.

PROBLEM. *Does every locally tabular modal have a finite formula depth?*

Conjecture: no.

Formula depth and games-1

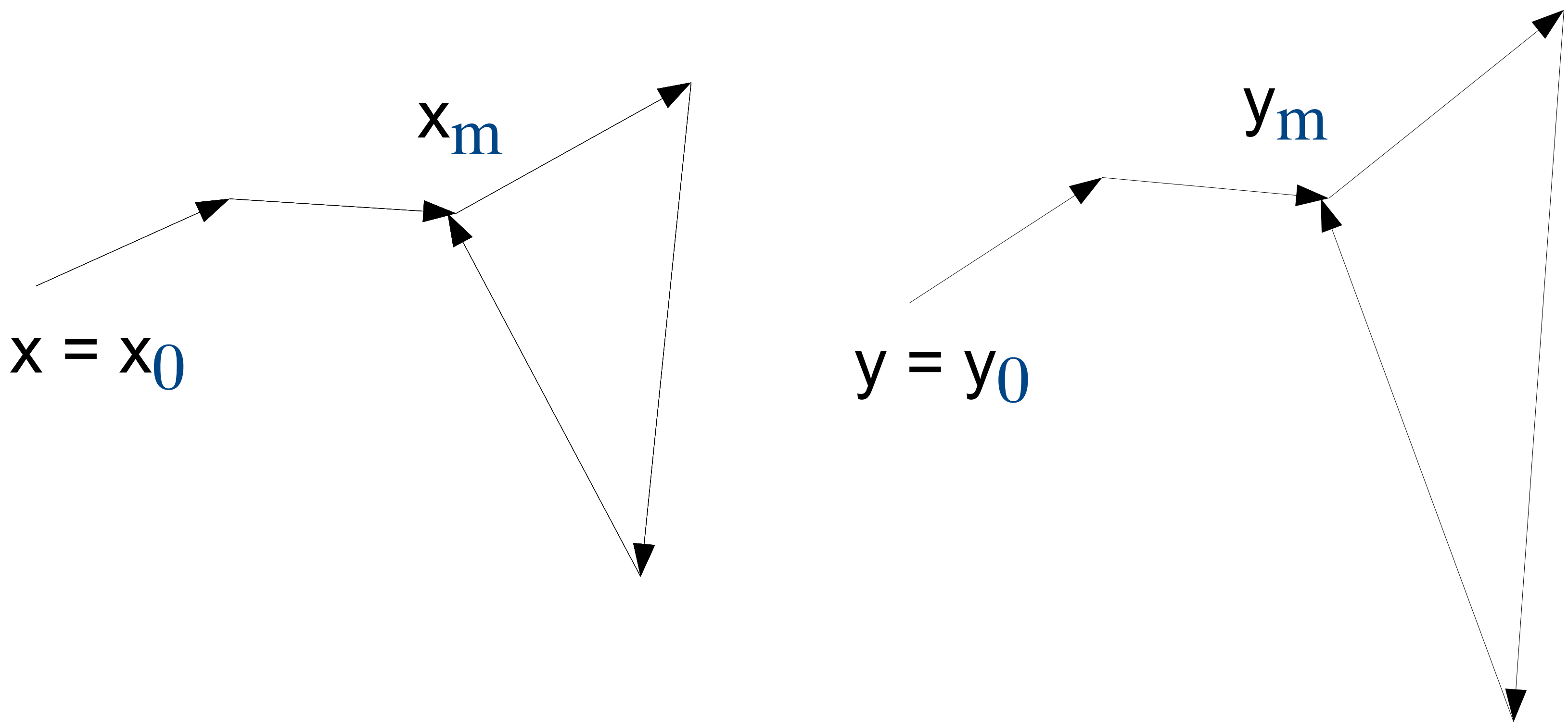
<u>Stabilization lemma</u>

$\equiv_n = \equiv_{n+1}$ in a weak canonical model $M_{L\lceil k}$ (bisimulation games *stabilize at round n*) iff $md(L\lceil k) \leq n$.

Or: $md(L\lceil k) = \min \{n \mid \equiv_n = \equiv_{n+1}$ in $M_{L\lceil k}\}$.

Formula depth and games-2

Lemma on repeating positions Suppose in a Kripke model M $x \equiv_n y$ and the Duplicator has a winning strategy s in $BG_n(M,x,y)$ such that every play controlled by s has at least two repeating positions. Then $x \equiv_{n+1} y$.



Formula depth and games-3

tabularity ⇒ finite modal depth

<u>Theorem</u> If F is finite, then $md(L(F)) \leq |F|^2+1$.

Proof: The Pigeonhole principle gives repeating positions.

<u>Remark</u> In many cases we have a better (linear) upper bound.

Logics of finite depth

Def A frame $F=(W,R)$ is *of (intransitive) depth n* ($d(F)=n$) if in F there are paths of length n ($x_0Rx_1...Rx_n$), but there are no paths of length $(n+1)$.

$d(F)=\infty$ iff there are paths of any finite length.

Lemma $d(F) < n$ iff $F \vDash \Box^n\bot$.

Def A modal logic L is of *finite (intransitive) depth* if $\Box^n\bot \in L$ for some n.

Theorem 1 $md(\mathbf{K} + \Box^n\bot) = n-1$

Proof. For the upper bound: every play of a bisimulation game contains at most (n-1) rounds. For the lower bound: $md_L(\Box^{n-1}\bot)= n-1$.

Corollary 1.1 [>>Gabbay, Sh 1998] Every logic of finite depth is locally tabular.

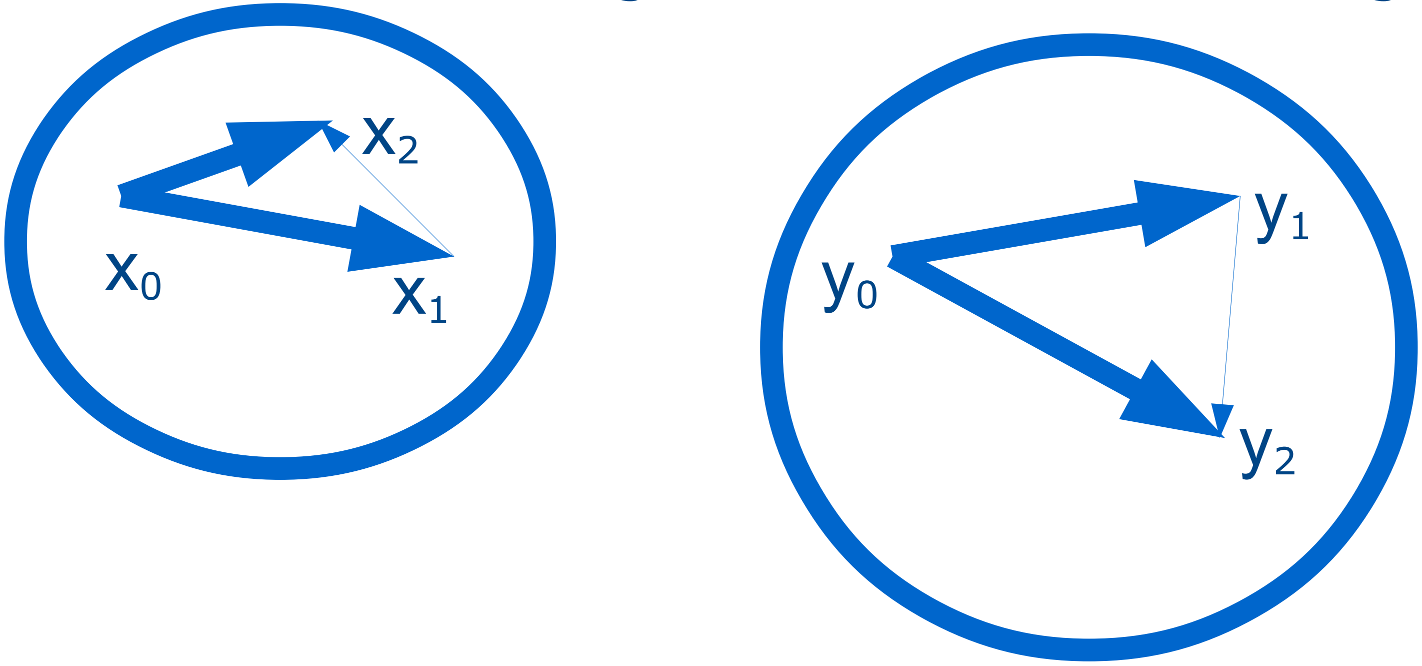
Examples of finite modal depth-1

- **S5** := **S4** + $\Diamond\Box p \to p$ = **L**(all equivalence frames)

  = **L**(all universal frames [clusters])

- md(**S5**) = 1 (a well-known fact)

  Proof. If **E** can win the 1-game, she can win the 2-game

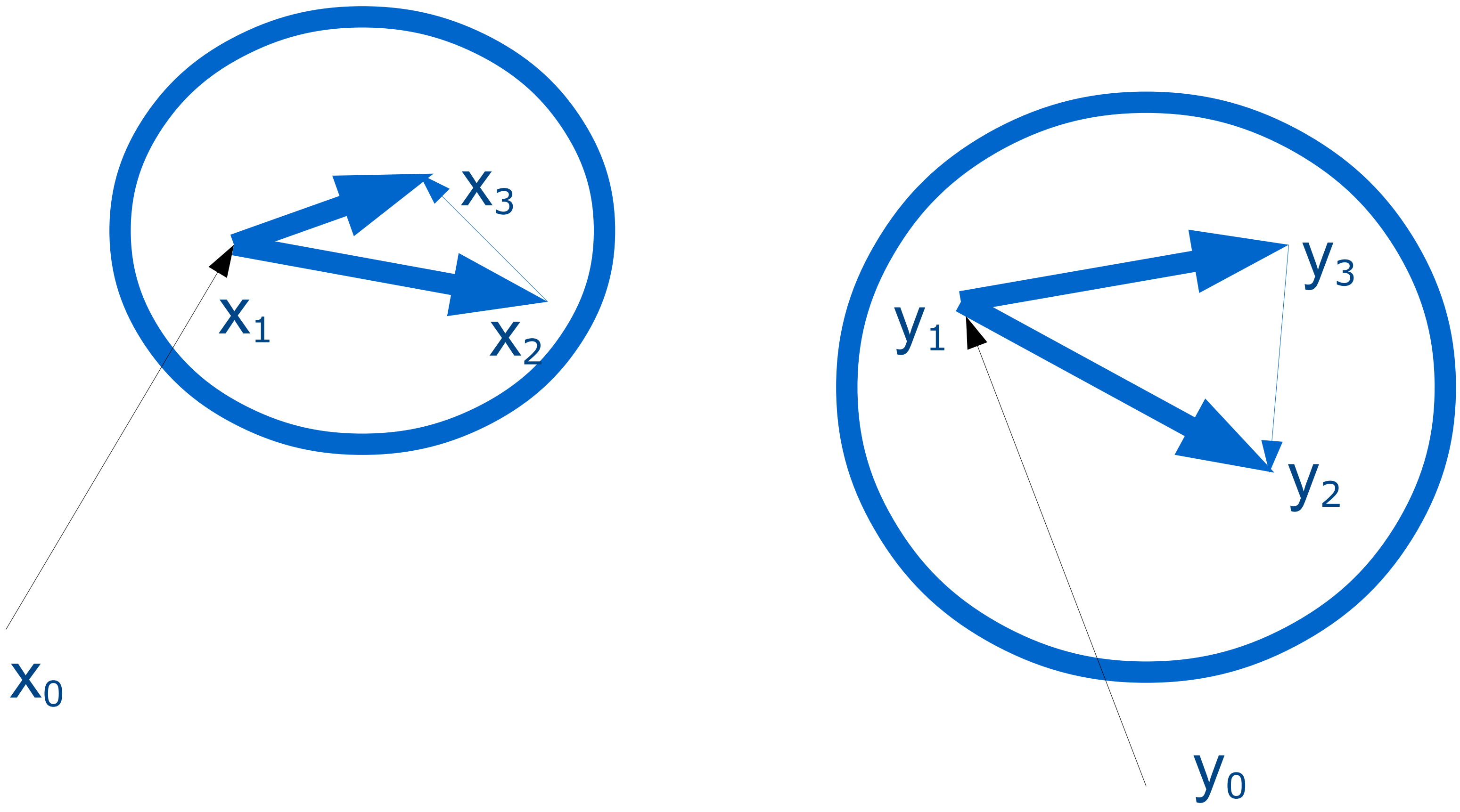


Examples of finite modal depth-2

K5 := **K** + $\Diamond\Box p \rightarrow \Box p$ = **L**(all Euclidean frames)

md(K**5**) = 2

Proof. If **E** can win the 2-game, she can win the 3-game



Examples of finite modal depth-3

$$\mathrm{md}(\mathbf{DL}) = 2$$

DL is the *difference logic*

$$\mathbf{DL} = \mathbf{K} + \Diamond\Box p \to p + \Diamond\Diamond p \to p \vee \Diamond p$$

- **DL** is complete w.r.t inequality frames $(W, \neq_W)$.
- Arbitary **DL-**frames are obtained from **S5-**frames (equivalence frames) by making some points irreflexive.

Examples of finite modal depth-4

Def A transitive frame F=(W,R) is *of transitive depth n* (d_{tr}(F)=n) if in F there are *ascending chains of length n*
($x_0 R x_1 \ldots R x_n$, where $\neg\, x_i R x_{i+1}$ for each i),
but there are no ascending chains of length (n+1).
Let $bd_n = \neg\Diamond(Q_0 \wedge \Diamond(Q_1 \wedge \ldots \wedge \Diamond Q_n))$,
where
$Q_i = p_i \wedge \bigwedge\{\neg\Diamond p_j \mid 0 \leq j < i\}$.

Lemma $d_{tr}(F) < n$ iff $F \vDash bd_n$

Examples of finite modal depth-5

K4 := **K** + $\Diamond\Diamond p \rightarrow \Diamond p$ = **L**(all transitive frames)

Теорема md(**K4**+bd_n) ≤ 4n – 3

Theorem (Segerberg 1971;Maksimova 1975) For L ⊇ **K4**

L is locally tabular iff L is of finite transitive depth.

Def L is of *finite transitive depth* if $L \vdash bd_n$ for some n.

Corollary For extensions of **K4** local tabularity is equivalent to finite modal depth.

This is a unique known criterion of local tabularity for families of modal logics. We do not know any criterion fro extenmsions of **K**.

Examples of finite modal depth-6

Theorem [Sh 2016]

$md(\mathbf{Grz}+bd_n) \le 2n-2$,

$md(\mathbf{Grz3}+bd_n) = n-1$.

Grz is the logic of finite partial orders,

Grz3 is the logic of finite chains,

Grz3 + bd_n = **L**(n-element chain),

i.e., a chain of depth n-1

Problem. Find $md(\mathbf{Grz}+bd_n)$.

An example of transitive depth 2

$md(\mathbf{Grz}+bd_2) = 2$

(0,1 show the truth values of p)

Here $x \equiv_1 y$, but $x \not\equiv_2 y$: Duplicator wins after 1 round. Spoiler wins after 2 rounds.

A distinguishing formula is $\Box\Diamond p$. So it has depth 2 in $\mathbf{Grz}+bd_2$

But note that $md(\mathbf{Grz3}+bd_2) = 1$, and

$\mathbf{Grz3}+bd_2 \vdash \Box\Diamond p \leftrightarrow (\Box p \vee (\neg p \wedge \Diamond p))$.

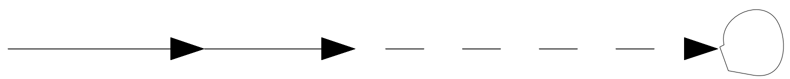
Examples of finite modal depth-7

Theorem $md(\mathbf{K} + \Box^n(\Box p \leftrightarrow p)) = n$

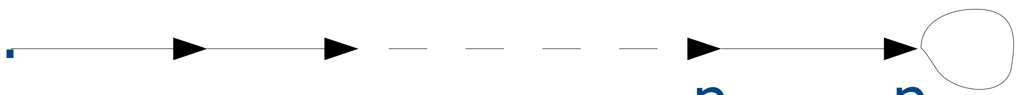
Proof.

- The axiom $\Box^n(\Box p \leftrightarrow p)$ says that after at most n steps we arrive at a maximal reflexive point ($R(x)=\{x\}$). So every play of a bisimulation game contains at most n nontrivial moves of **A**. After that he should stay in a maximal point, so the game stabilizes at n.
  Hence $md(L) \leq n$.
- For the lower bound: consider $A=\Box^{n-1}(\Box p \leftrightarrow p)$. $md_L(A)=n$, since it can distinguish two (n-1)-equivalent points:

Examples of finite modal depth-8



$y \models \Box^{n-1}(\Box p \leftrightarrow p)$



$x \not\models \Box^{n-1}(\Box p \leftrightarrow p)$

$x \equiv_{n-1} y$

Examples of finite modal depth-9

For a temporal frame $F=(W,R_1,R_2)$, where $R_2=(R_1)^{-1}$, we can define a special notion of depth.

- A *temporal path* of length n in F is a sequence of points related by R_1 or R_2: $x_{10}R_{i_1}x_1 \ldots x_{n-1}R_{i_n}x_n$ *without retreats*, i.e., it cannot go backwards along the same way: if $R_{i_{j+1}}=(R_{i_j})^{-1}$, then $x_{j+1} \neq x_{j-1}$.

  The *temporal depth* of F is the maximal length of temporal paths in F.

  The following modal formula expresses that the temporal depth of F is at most n-1:

  $Rd_n := \neg(Q_0 \wedge \Diamond(Q_1 \wedge \ldots \wedge \Diamond Q_n))$,

  where
- $Q_0 = p_0$,$Q_{j+1} = p_{j+1} \wedge \neg p_{j-1}$ if $i_{j+1} = - i_{j-1}$,
- $Q_{j+1} = p_{j+1}$ otherwise.

Examples of finite modal depth-10

Theorem md(**K.t** + Rd_n) ≤ n.

Proof Every play of a bisimulation game without retreats has the length at most n-1. So a play of length n should have retreats.

If **A** makes a retreat, **E** can also make a retreat, which leads to an earlier position. Thus Lemma on repeating positions can be applied.

QED.

Examples of finite modal depth-11

Theorem Every logic **K** + α_n (Chagrov's formula) is locally tabular.

Remarks:

- The proof does not give the FMD
- This theorem was conjectured in 1994 by Chagrov.

Intuitionistic Kripke models-1

An intuitionistic Kripke frame is a poset

$F = (W, \leq)$.

A intuitionistic valuation on F $\theta : PL \to 2^W$ satisfying

$x \in \theta(p_i)$ & $x \leq y \Rightarrow y \in \theta(p_i)$

Intuitionistic Kripke models-2

$(M,x \Vdash A)$

- $M,x \Vdash p_i \Leftrightarrow x \in \theta(p_i)$
- $M,x \Vdash A \vee B \Leftrightarrow (M,x \Vdash A$ and $M,x \Vdash B)$
- $M,x \Vdash A \wedge B \Leftrightarrow (M,x \Vdash A$ or $M,x \Vdash B)$
- $M,x \Vdash A \rightarrow B \Leftrightarrow \forall y \geq x\ (M,y \Vdash A \Rightarrow M,y \Vdash B)$

Then

- $M,x \Vdash \neg A \Leftrightarrow \forall y \geq x\ M,y \nVdash A$

A is valid on F ($F \Vdash A$), if it is true at all points of all Kripke models on F.

Examples of finite intuitionistic depth-1

$di(\mathbf{H}+ibd_n) \leq 2n-1$

In posets ibd_n forbids *chains* *of length n+1* : $x_1<x_2 \ldots <x_{n+1}$.

$ibd_1 = p_1 \vee \neg p_1$,

$ibd_{n+1} = p_{n+1} \vee (p_{n+1} \rightarrow ibd_n)$.

Def Intermediate logics of finite transitive depth: extensions of $\mathbf{H}+ibd_n$

Theorem (Kuznetsov – Komori) These logics are locally tabular.

Examples of finite intuitionistic depth-2

di(**LC**) = 2

LC = H+ $(p \to q) \vee (q \to p)$ is the intermediate logic of arbitrary chains.

Proof: if $x \equiv_1 y$, then $x \equiv_2 y$: ignore the first move.

Proving the FMP-1

Def A *tree* is a rooted frame, in which every point is accessible from the root by a unique path.

- A frame F is *serial*, if it has no endpoints, i.e., $F \vDash \Diamond\top$.

Theorem If F is a serial tree, then **L**(F) has the FMP.

- Proof Let 0 be the root of F. Let M be a weak Kripke model over our frame F. Then $M,0 \equiv_n M',0$ for M' obtained by truncation to height n + reflexivity at endpoints. There is a p-morphism $f: F \to F'$ sending every x of height n and lower to itself, and every x of a higher level to a unique y of level n on the path from 0 to x. We have $F' \vDash \Box^n(\Box p \leftrightarrow p)$, so L(F') is locally tabular (Theorem on slide 50).

Proving the FMP-2

L(F) $\subseteq$ **L**(F'), since f is p-morphic.

- It follows that every formula refutable in F is refutable in some locally tabular logic containing **L**(F). Thus **L**(F) is an intersection of locally tabular logics, so it has the FMP. QED.

References-1

Guram Bezhanishvili and Revaz Grigolia. Locally finite varieties of Heyting algebras. Algebra universalis, 54 (2005), 465–473.

P. Blackburn, M. De Rijke, Y. Venema. Modal Logic. Cambridge University Press, 2001.

A. Chagrov, M. Zakharyaschev. Modal Logic. Oxford University Press, 1996.

K. Segerberg. An Essay in Classical Modal Logic. Uppsala, 1971.

D. Gabbay, A. Kurucz, F. Wolter, M. Zakharyaschev. Many-dimensional Modal Logics: Theory and Applications. Elsevier, 2003.

D. Gabbay, V. Shehtman. Products of modal logics, part 1. Logic Journal of the IGPL, v. 6, pp. 73-146, 1998.

References-2

V. Shehtman. Canonical filtrations and local tabularity. In: Advances in Modal Logic, v.10, 498-512. College Publications, 2014.

V. Shehtman. Bisimulation games and locally tabular logics. Russian Mathematical Surveys, v. 71, No. 5 (2016), 979-981.

V. Shehtman. Segerberg squares of modal logics and theories of relation algebras. In: Larisa Maksimova on Implication, Interpolation, and Definability, Springer 2018, p. 245-296.

G. Bezhanishvili. Locally finite varieties. Algebra Universalis, v. 46(2001), 531-548.

G. Bezhanishvili, R. Grigolia. Locally tabular extensions of MIPC. Advances in Modal Logic 1998, 101—120.

Logics

- $\mathbf{K} = \mathbf{L}(\text{all frames})$
- $\mathbf{K4} := \mathbf{K} + \Diamond\Diamond p \to \Diamond p = \mathbf{L}(\text{all transitive frames})$
- $\mathbf{K5} := \mathbf{K} + \Diamond\Box p \to \Box p = \mathbf{L}(\text{all Euclidean frames})$
- $\mathbf{S4} := \mathbf{K4} + p \to \Diamond p = \mathbf{L}(\text{all transitive reflexive frames})$

  $= \mathbf{L}(\text{all partial orders})$

- $\mathbf{Grz} := \mathbf{S4} + \neg(p \wedge \Box(p \to \Diamond(\neg p \wedge \Diamond p)))$

  $= \mathbf{L}(\text{all finite partial orders})$

- $\mathbf{Grz3} := \mathbf{Grz} + \Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond p)$

  $= \mathbf{L}(\text{all finite chains})$

- $\mathbf{S5} := \mathbf{S4} + \Diamond\Box p \to p = \mathbf{L}(\text{all equivalence frames})$

  $= \mathbf{L}(\text{all universal frames [clusters]})$

All these logics have the FMP, so they are decidable.