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Orthmodular Lattices Whose MacNeille Completions Are Not Orthomodular

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Abstract. The only known example of an orthomodular lattice (abbreviated: OML) whose MacNeille completion is not an OML has been noted independently by several authors, see Adams [1], and is based on a theorem of Ameniya and Araki [2]. This theorem states that for an inner product space V, if we consider the ortholattice $\mathscr{L}(V, \perp) = \{A \subseteq V: A = A^{\perp\perp}\}$ where $A^{\perp}$ is the set of elements orthogonal to A, then $\mathscr{L}(V, \perp)$ is an OML if and only if V is complete. Taking the orthomodular lattice L of finite or cofinite dimensional subspaces of an incomplete inner product space V, the ortholattice $\mathscr{L}(V, \perp)$ is a MacNeille completion of L which is not orthomodular. This does not answer the longstanding question *Can every OML be embedded into a complete OML?* as L can be embedded into the complete OML $\mathscr{L}(\bar{V}, \perp)$, where $\bar{V}$ is the completion of the inner product space V.

Although the power of the Ameniya–Araki theorem makes the preceding example elegant to present, the ability to picture the situation is lost. In this paper, I present a simpler method to construct OMLs whose MacNeille completions are not orthomodular. No use is made of the Ameniya–Araki theorem. Instead, this method is based on a construction introduced by Kalmbach [7] in which the Boolean algebras generated by the chains of a lattice are *glued* together to form an OML. A simple method to complete these OMLs is also given.

The final section of this paper briefly covers some elementary properties of the Kalmbach construction. I have included this section because I feel that this construction may be quite useful for many purposes and virtually no literature has been written on it.

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1. Preliminaries

In this section, the elementary facts regarding ortholattices, orthomodular lattices, MacNeille completions of ortholattices, ultraproducts of algebras and Boolean algebras generated by chains are reviewed. A fairly detailed description of the Kalmbach construction is also given. As this construction is encountered throughout the entire paper, a good understanding of how it works is essential.

An orthocomplementation on a lattice L is a map $x \to x'$ of L which is an antimonotone complementation of period two. An ortholattice is a pair $(L, ')$ where L is a lattice and $'$ is an orthocomplementation on L. An OML is an ortholattice

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$(L, ')$ which satisfies the orthomodular law:

$$\text{for all } a \leqslant b \in L, a \vee (a' \wedge b) = b \tag{1.1}$$

or its equivalent form

$$\text{for all } a \leqslant b \in L, b \wedge a' = 0 \text{ if and only if } a = b. \tag{1.2}$$

It is customary to refer to an ortholattice $(L, ')$ by L when no confusion is possible. For further information on ortholattices and OMLs the reader should see [8].

An ideal of a lattice L is the intersection of a collection of principal ideals if and only if it is the set of lower bounds of its upper bounds. Such ideals are called normal ideals. $\bar{L}$, the set of all normal ideals of L, forms a complete lattice under set inclusion and is referred to as the completion by cuts or the MacNeille completion of L (see MacNeille [10]). It follows easily that $\bar{L}$ contains an isomorphic copy of L as a join and meet dense sublattice.

For an ortholattice $(L, ')$ an orthocomplementation $\perp$ may be defined on $\bar{L}$ by

$$I^{\perp} = \{x \in L : x' \text{ is an upper bound of } I\}.$$

This orthocomplementation extends that of L and is uniquely determined by this property (see MacLaren [9]). When we refer to the MacNeille completion of an ortholattice $(L, ')$, we will mean the ortholattice $(\bar{L}, \perp)$.

Given a family of algebras $(A_i)_{i \in I}$ of the same type, and a first order formula $\varphi(x_1, \ldots, x_n)$ in the language of these algebras and $a_1, \ldots, a_n \in \Pi A_i$ define

$$[\![\varphi(a_1, \ldots, a_n)]\!] = \{i \in I: A_i \vDash \varphi(a_1(i), \ldots, a_n(i))\}. \tag{1.3}$$

For $\mathscr{U}$ an ultrafilter over the set I the relation Θ on ΠA_i defined by

$$a \Theta b \text{ if and only if } [\![a = b]\!] \in \mathscr{U}$$

is a congruence on ΠA_i. We follow the customary practice of using $\mathscr{U}$ and the associated congruence Θ interchangibly. The ultraproduct of $(A_i)_I$ over $\mathscr{U}$ is defined to be $(\Pi A_i)/\Theta$ and is denoted by $\Pi A_i/\mathscr{U}$. A very useful theorem due to Łoś states

$$\prod_{i \in I} A_i/\mathscr{U} \vDash \varphi(a_1/\mathscr{U}, \ldots, a_n/\mathscr{U}) \text{ if and only if } [\![\varphi(a_1, \ldots, a_n)]\!] \in \mathscr{U}. \tag{1.4}$$

For further information on ultraproducts and universal algebra see [5].

For a bounded chain C, let $\mathscr{F}$ be the field of subsets of $C - \{1\}$ generated by the sets $A_x = \{y \in C: y < x\}$ where x ranges over C. As every element of $\mathscr{F}$ has a unique representation of the form

$$\bigcup_{i=1}^{n} (A_{x_{2i}} - A_{x_{2i-1}}) \text{ where } x_1 < x_2 < \cdots < x_{2n} \in C$$

the set $\mathscr{B}(C)$ of all finite, even length chains in C carries a natural Boolean structure. For $x \in \mathscr{B}(C)$ let $l(x)$ be half of the length of x and let $x_1, \ldots, x_{2l(x)}$ be the elements of x in order of increasing size. Then if $\preccurlyeq$ and $\perp$ are the induced partial ordering and orthocomplementation on $\mathscr{B}(C)$ we have for $x, y \in \mathscr{B}(C)$

$$x \preccurlyeq y \text{ if and only if for each } 1 \leqslant i \leqslant l(x) \text{ there exists } 1 \leqslant j \leqslant l(y) \text{ such that } y_{2j-1} \leqslant x_{2i-1} < x_{2i} \leqslant y_{2j},$$

and that $x^{\perp}$ is defined by

$$x \cup x^{\perp} = x \cup \{0, 1\} \quad \text{and} \quad x \cap x^{\perp} = x - \{0, 1\}. \tag{1.6}$$

It is immediately evident that $\mathscr{B}(C)$, with the natural Boolean structure, is generated by a sub-chain which is isomorphic to C. We call $\mathscr{B}(C)$ the Boolean algebra generated by the chain C. For basic information on Boolean algebras generated by chains see [3, pages 105–109].

With the above discussion of Boolean algebras generated by chains, it is a simple matter to describe the Kalmbach construction. For a bounded lattice L, define the set $\mathscr{K}(L)$ to be the union of the sets $\mathscr{B}(C)$ where C ranges over all 0, 1 sub-chains of L. Define a map $\perp: \mathscr{K}(L) \to \mathscr{K}(L)$ to be the union of the complementations on the $\mathscr{B}(C)$ and define a relation $\preccurlyeq$ on $\mathscr{K}(L)$ to be the union of the partial orderings on the $\mathscr{B}(C)$. It is shown in [7] (a more accessible reference may be [8, page 230]) that $(\mathscr{K}(L), \perp, \preccurlyeq)$ is an OML. The proof follows by giving a recursive method for calculating joins and meets in $\mathscr{K}(L)$ and is outlined below.

From (1.5) it follows that $\preccurlyeq$ is a partial ordering and from (1.6) it follows that $\perp$ is indeed a function. For elements x, y of $\mathscr{K}(L)$, if $x \cup y$ is a chain of L then the supremum and infinum of these elements in $\mathscr{B}(x \cup y \cup \{0, 1\})$ are their supremum and infinum in $\mathscr{K}(L)$. Therefore, it follows immediately from (1.5) that $\mathscr{K}(L)$ satisfies (1.1), so if $\mathscr{K}(L)$ is indeed a lattice, it is an OML.

For x, y elements of $\mathscr{K}(L)$, if $l(x) = 1$ then $x \vee y$ exists. The proof is by induction on $l(y)$. Assume that $x \cup y$ is not a chain of L, and therefore that $l(y) > 0$. It is easily verified that

$$\{x_1, x_2\} \vee \{y_1, y_2\} = \{x_1 \wedge y_1, x_2 \vee y_2\} \text{ if } \{x_1, x_2, y_1, y_2\} \text{ is not a chain.} \tag{1.7}$$

Setting $z_i = \{y_{2i-1}, y_{2i}\}$ for each $1 \leqslant i \leqslant l(y)$ and taking k least such that $x \cup z_k$ is not a chain, by (1.7) $x \vee z_k$ exists and by inductive hypothesis $(x \vee z_k) \vee (y - z_k)$ exists, so

$$x \vee y = (x \vee z_k) \vee (y - z_k). \tag{1.8}$$

For x an y nonzero elements of $\mathscr{K}(L)$, setting $w_i = \{x_{2i-1}, x_{2i}\}$ for each $1 \leqslant i \leqslant l(x)$, we have

$$x \vee y = (((y \vee w_1) \vee w_2) \cdots) \vee w_{l(x)}. \tag{1.9}$$

For x and y elements of $\mathscr{K}(L)$, to see that the infinum of x and y exists first note that

$$\text{if } l(x) = l(y) = 1 \text{ then } x \wedge y = \begin{cases} \{x_1 \vee y_1, x_2 \wedge y_2\} & \text{if } x_1 \vee y_1 < x_2 \wedge y_2 \\ \emptyset & \text{otherwise} \end{cases} \tag{1.10}$$

But, z is a lower bound of $\{x, y\}$ if and only if for each $1 \leqslant k \leqslant l(z)$ there exists $1 \leqslant i \leqslant l(x)$, and $1 \leqslant j \leqslant l(y)$ such that $x_{2i-1} \vee y_{2j-1} \leqslant z_{2k-1} < z_{2k} \leqslant x_{2i} \wedge y_{2j}$. So

$$x \wedge y = \bigvee \{\{x_{2i-1}, x_{2i}\} \wedge \{y_{2j-1}, y_{2j}\}: 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}. \tag{1.11}$$

If z is the result of an operation on elements x, y of $\mathscr{K}(L)$ then z is a chain in the sublattice of L generated by $x \cup y \cup \{0, 1\}$. This follows immediately from (1.6) for the operation $\perp$ and by a simple induction using (1.7) through (1.9) for join. Then (1.10) and (1.11) provide the result for meets. As a corollary of this observation,

$$\text{if } M \text{ is a } 0, 1 \text{ sublattice of } L \text{ then } \mathscr{K}(M) \text{ is a subalgebra of } \mathscr{K}(L). \tag{1.12}$$

2. Completions of $\mathscr{K}(L)$

PROPOSITION 2.1. *For L a bounded lattice, the MacNeille completion of $\mathscr{K}(L)$ is an OML if and only if the condition* (†) *holds in L.*

(†) If $(C_i)_I, (D_j)_J$ are two families of closed intervals in L such that $\emptyset \neq \bigcap C_i \subset \bigcap D_j$, then there exists $x, y \in \bigcap D_j$ such that $x < y$ and either x is an upper bound of $\bigcap C_i$ or y is a lower bound of $\bigcap C_i$.

Proof. Assume that L is a bounded lattice and the MacNeille completion of $\mathscr{K}(L)$ is an OML. Take two families of non-degenerate closed intervals in L, say $([a_i, b_i])_I$ and $([c_j, d_j])_J$, and set $X = \bigcap \{[a_i, b_i]: i \in I\}$, $Y = \bigcap \{[c_j, d_j]: j \in J\}$. Assume that $\emptyset \neq X \subset Y$. For each $i \in I$ let $A_i = [\leftarrow, \{a_i, b_i\}]_{\mathscr{K}(L)}$ and for each $j \in J$ let $B_j = [\leftarrow, \{c_j, d_j\}]_{\mathscr{K}(L)}$. Set $A = \bigcap \{A_i: i \in I\}$, and $B = \bigcap \{B_j: j \in J\}$, then $A = \{x \in \mathscr{K}(L): x \subseteq X\}$ and $B = \{y \in \mathscr{K}(L): y \subseteq Y\}$. As A_i is a principal ideal of $\mathscr{K}(L)$ for each $i \in I$, A is a normal ideal of $\mathscr{K}(L)$, as is B.

But $\emptyset \neq X \subset Y$, so there exist $f, g \in L$ such that $f \in X$ and $g \in (Y - X)$. Then $\{f \wedge g, f \vee g\} \in B - A$, but we assumed the MacNeille completion of $\mathscr{K}(L)$ was on OML, so by (1.2) $B \cap A^\perp \neq \{0\}$. Take $z \in B \cap A^\perp$, such that $l(z) \neq 0$. Then $z^\perp$ is an upper bound of A. But X is a convex sublattice of L, so one of $\{0, z_1\}$, $\{z_{2l(z)}, 1\}$, or $\{z_{2k}, z_{2k+1}\}$ for some $1 \leqslant k < l(z)$, is an upper bound of A. As $z \in B$, $z \subseteq Y$, so if $\{0, z_1\}$ is an upper bound of A, then z_1, z_2 respectively serve the roles of x, y in (†). If $\{z_{2l(z)}, 1\}$ is an upper bound of A, then $z_{2l(z)-1}, z_{2l(z)}$ serve the roles of x, y; and if $\{z_{2k}, z_{2k+1}\}$ is an upper bound of A, then z_{2k+1}, z_{2k+2} serve the roles of x, y. So, if the MacNeille completion of $\mathscr{K}(L)$ is an OML then L satisfies (†).

For the converse, suppose that (†) is satisfied in L and that A, B are normal ideals of $\mathscr{K}(L)$ such that $\emptyset \neq A \subset B$. We must show that $B \cap A^\perp \neq \emptyset$.

Let $U(A)$, $U(B)$ be the set of upper bounds of A and B in $\mathscr{K}(L)$ respectively. Set

$$\mathscr{F} = \{f \in \mathbb{N}^{U(A)}: 1 \leqslant f(x) \leqslant l(x) \text{ for all } x \in U(A)\}$$
$$\mathscr{G} = \{g \in \mathbb{N}^{U(B)}: 1 \leqslant g(x) \leqslant l(x) \text{ for all } x \in U(B)\}$$

and for $f \in \mathscr{F}$, $g \in \mathscr{G}$ set

$$X_f = \bigcap_{x \in U(A)} [x_{2f(x)-1}, x_{2f(x)}]$$

and

$$Y_g = \bigcap_{x \in U(B)} [x_{2g(x)-1}, x_{2g(x)}].$$

Then for $x \in \mathscr{K}(L)$, as A is a normal ideal, $x \in A$ if and only if $x \preccurlyeq y$ for all $y \in U(A)$ if and only if for each $1 \leqslant i \leqslant l(x)$ and each $y \in U(A)$ there exists $1 \leqslant j \leqslant l(y)$ such that $[x_{2i-1}, x_{2i}] \subseteq [y_{2j-1}, y_{2j}]$ if and only if for each $1 \leqslant i \leqslant l(x)$ there exists $f \in \mathscr{F}$ such that $[x_{2i-1}, x_{2i}] \subseteq X_f$.

As $A \subset B$, there exists $x \in (B - A)$ with $l(x) = 1$ and therefore $g \in \mathscr{G}$ with $[x_1, x_2] \subseteq Y_g$.

Assume that $h_{|U(B)} = g$ implies that $X_h = \emptyset$ for all $h \in \mathscr{F}$. For each $z \in A$ with $l(z) = 1$ we have $[z_1, z_2] \in X_h$ for some $h \in \mathscr{F}$. As $h(y) \neq g(y)$ for some $y \in U(B)$ and $z \preccurlyeq \{y_{2h(y)-1}, y_{2h(y)}\}$, $x \preccurlyeq \{y_{2g(y)-1}, y_{2g(y)}\}$ we have $x \preccurlyeq z^{\perp}$. So, $x \preccurlyeq z^{\perp}$ for all $z \in A$, and therefore $x \in B \cap A^{\perp}$.

If there exist $f, h \in \mathscr{F}$ such that $\emptyset \neq X_f \subseteq Y_g, \emptyset \neq X_h \subseteq Y_g$ then for some $y \in U(A) - U(B)$ $f(y) \neq h(y)$. So,

$$Y_g \cap [y_{2f(y)-1}, y_{2f(y)}] \supseteq X_f \neq \emptyset$$

and

$$Y_g \cap [y_{2h(y)-1}, y_{2h(y)}] \supseteq X_h \neq \emptyset.$$

Assuming that $f(y) < h(y)$, as Y_g is convex, $[y_{2f(y)}, y_{2f(y)+1}] \subseteq Y_g$. Then $\{y_{2f(y)}, y_{2f(y)+1}\} \in B$, and as $\{y_{2f(y)}, y_{2f(y)+1}\} \preccurlyeq y^{\perp} \in A^{\perp}$ we have finished.

As $\emptyset$ is not an upper bound of A, there exists an $f \in \mathscr{F}$ such that $f_{|U(B)} = g$. By the above discussion we may assume that f is the unique element of $\mathscr{F}$ such that $\emptyset \neq X_f \subseteq Y_g$. But $[x_1, x_2] \subseteq Y_g$ so $X_f \neq Y_g$. Applying (†) we find $a, b \in Y_g$ such that $a < b$ and either a is an upper bound of X_f or b is a lower bound of X_f. In either case, $\{a, b\} \in B \cap A^{\perp}$.

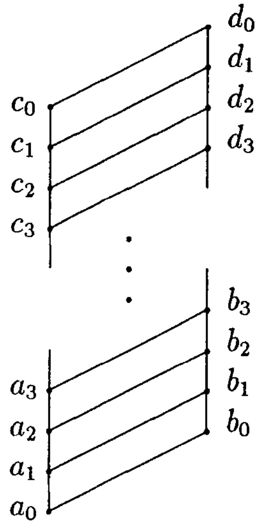
The following result is somewhat surprising in view of the fact that $\mathscr{K}(L)$ is complete if and only if it is finite.

COROLLARY 2.1. *If L is a bounded lattice, then $\mathscr{K}(L)$ can be embedded into a complete OML.*

Proof. It is easy to see that (†) is satisfied by any complete lattice. If we let $\bar{L}$ denote the MacNeille completion of L, then as L is a sublattice of $\bar{L}$, by (1.12) $\mathscr{K}(L)$ is a sub-OML of $\mathscr{K}(\bar{L})$. But $\bar{L}$ is complete, so $\mathscr{K}(\bar{L})$ is an OML and $\mathscr{K}(L)$ is a sub-OML of $\mathscr{K}(\bar{L})$.

COROLLARY 2.2. *There is a bounded lattice L such that the MacNeille completion of $\mathscr{K}(L)$ is not an OML.*

Proof. In the lattice L_0 depicted below, where it is to be understood that $a_n \leqslant c_m, d_m$ and $b_n \leqslant d_m$ for all $n, m \in \mathbb{N}$, the families $([a_0, c_n])_{n \in \mathbb{N}}$ and $([a_0, d_n])_{n \in \mathbb{N}}$ violate (†).



L_0

PROPOSITION 2.2. *For a family $(L_i)_{i \in I}$ of bounded lattices and an ultrafilter $\mathscr{U}$ over the set I, $\mathscr{K}(\Pi L_i/\mathscr{U})$ can be embedded into $\Pi \mathscr{K}(L_i)/\mathscr{U}$.*

Proof. Let S be the collection of all subsets of ΠL_i which have even cardinality. With π_i being i^{th} projection of ΠL_i, define a map $p: S \to \Pi \mathscr{K}(L_i)$ as follows

$$p(X)(i) = \begin{cases} \pi_i[X] & \text{if } \pi_i[X] \in \mathscr{K}(M_i) \text{ and } |\pi_i[X]| = |X| \\ \emptyset & \text{otherwise} \end{cases}$$

If $z \in \mathscr{K}(\Pi L_i/\mathscr{U})$ and $a_1, \ldots, a_{2l(z)}, b_1, \ldots, b_{2l(z)} \in \Pi L_i$ are such that $a_i/\mathscr{U} = b_i/\mathscr{U} = z_i$ for each $1 \leqslant i \leqslant 2l(z)$ then, as $\mathscr{U}$ is closed under finite intersections, $p(\{a_i: 1 \leqslant i \leqslant 2l(z)\})/\mathscr{U} = p(\{b_i: 1 \leqslant i \leqslant 2l(z)\})/\mathscr{U}$. Therefore, we may define a map $\beta: \mathscr{K}(\Pi L_i/\mathscr{U}) \to \Pi \mathscr{K}(L_i)/\mathscr{U}$ by

$$\beta(z) = p(\{x_1, \ldots, x_{2l(z)}\})/\mathscr{U} \text{ if } z_i = x_i/\mathscr{U} \text{ for all } 1 \leqslant i \leqslant 2l(z).$$

For $x, y \in \mathscr{K}(\Pi L_i/\mathscr{U})$ choose $a_1, \ldots, a_{2l(x)} \in \Pi L_i$ such that $a_i/\mathscr{U} = x_i$ for each $1 \leqslant i \leqslant 2l(x)$ and choose $b_1, \ldots, b_{2l(y)} \in \Pi L_i$ such that $b_j/\mathscr{U} = y_j$ for each $1 \leqslant j \leqslant 2l(y)$. Let $\varphi(p_1, \ldots, p_{2l(x)}, q_1, \ldots, q_{2l(y)})$ be the first order formula which says that for each $1 \leqslant i \leqslant l(x)$ there exists $1 \leqslant j \leqslant l(y)$ such that $q_{2j-1} \leqslant p_{2i-1} < p_{2i} \leqslant q_{2j}$. Then using Łoś' theorem (1.4) and the fact that

$[\![a_1 < \cdots < a_{2l(x)}]\!]$, $[\![b_1 < \cdots < b_{2l(y)}]\!] \in \mathcal{U}$ we have

$$\begin{aligned} x \preccurlyeq y &\text{ if and only if } \prod L_i/\mathcal{U} \vDash \varphi(x_1, \ldots, x_{2l(x)}, y_1, \ldots, y_{2l(y)}) \\ &\text{ if and only if } [\![\varphi(a_1, \ldots, a_{2l(x)}, b_1, \ldots, b_{2l(y)})]\!] \in \mathcal{U} \\ &\text{ if and only if } [\![\varphi(a_1, \ldots, a_{2l(x)}, b_1, \ldots, b_{2l(y)})]\!] \\ &\qquad \cap [\![a_1 < \cdots < a_{2l(x)}]\!] \cap [\![b_1 < \cdots < b_{2l(y)}]\!] \in \mathcal{U} \\ &\text{ if and only if } [\![p(\{a_i : 1 \leqslant i \leqslant 2l(x)\}) \preccurlyeq p(\{b_j : 1 \leqslant j \leqslant 2l(y)\})]\!] \in \mathcal{U} \\ &\text{ if and only if } \beta(x) \leqslant \beta(y). \end{aligned}$$

Therefore β is an order embedding. By similar methods we can easily check that β is compatible with $\perp$.

To put the next result in the proper context, it is shown in [4] that any variety of OML's which is generated by a single finite OML, is closed under the formation of MacNeille completions. It is not unreasonable to hope that the variety which is generated by all the finite OML's would also have this property. This would settle a basic question about OML's. *Is the variety of OML's generated by its finite members*? Unfortunately, the following result nullifies this approach.

COROLLARY 2.3. *There is an OML in the variety generated by the finite OMLs whose MacNeille completion is not an OML.*

Proof. For each natural number n, let M_n be the interval $[a_0, b_n]$ in L_0, and for each $m \geqslant 0$ define $A_m, B_m, C_m, D_m \in \Pi\, M_n$ by $A_m(n) = a_{\min\{m,n\}}$, $B_m(n) = b_{\min\{m,n\}}$, $C_m(n) = a_{\max\{n-m,0\}}$ and $D_m(n) = b_{\max\{n-m,0\}}$. Let $\mathcal{U}$ be a non-principal ultrafilter over the natural numbers and define $\alpha : L_0 \to \Pi\, M_n/\mathcal{U}$ by setting $\alpha(a_m) = A_m/\mathcal{U}$, $\alpha(b_m) = B_m/\mathcal{U}$, $\alpha(c_m) = C_m/\mathcal{U}$ and $\alpha(d_m) = D_m/\mathcal{U}$. It is an easy matter to check that α is a 0, 1 lattice embedding. Then $\mathcal{K}(L_0)$, whose MacNeille completion is not an OML, can be embedded into $\mathcal{K}(\Pi\, M_n/\mathcal{U})$ which can be embedded into $\Pi\, \mathcal{K}(M_n)/\mathcal{U}$. As $\mathcal{K}(M_n)$ is finite for each choice of n, the proof is finished.

The fundamental question as to whether every OML can be embedded into a complete OML is still open, as is the problem of characterizing the OML's that have orthomodular MacNeille completions. I hope that the above examples will be of some help in answering these questions.

3. Properties of the Construction

This section is primarily intended for the reader who may wish to make use of the Kalmbach construction. Several elementary properties of the OMLs arising from this construction are given. This is by no means intended to be a thorough exposition, but it does illustrate some techniques which are useful when working with the construction.

The reader should see [8] for information on orthostructures, [5] for universal algebra and [3] for Boolean algebras.

PROPOSITION 3.1. *Applying the Kalmbach construction to a bounded poset produces an orthomodular poset.*

Proof. This is entirely trivial.

PROPOSITION 3.2. *Let L, M be bounded lattices,*

(i) *L is a 0, 1 sub-lattice of $\mathcal{K}(L)$.*

(ii) *If L is a 0, 1 sub-lattice of M, then $\mathcal{K}(L)$ is a sub-OML of $\mathcal{K}(M)$.*

(iii) *$\mathcal{K}(L) \times \mathcal{K}(M) \cong \mathcal{K}(L \oplus M/\Theta(1_L, 0_M))$, where $\oplus$ is ordinal sum, and $\Theta(x, y)$ is the congruence generated by $\{x, y\}$.*

(iv) *$\mathcal{K}(L)$ has a full set of dispersion free states.*

(v) *For $x, y \in \mathcal{K}(L)$, x commutes with y if and only if $x \cup y$ is a chain in L.*

(vi) *The blocks of $\mathcal{K}(L)$ are exactly the $\mathcal{B}(C)$ where C is a maximal chain of L.*

(vii) *The centre of $\mathcal{K}(L)$ is $\mathcal{B}(D)$, where D is the set elements of L which are comparable to all others.*

Proof. (i) For $0 \neq x \in L$, set $\varphi(x) = \{0, x\}$ and let $\varphi(0) = \emptyset$. Then φ is the required lattice embedding.

(ii) This was already noted in (1.12).

(iii) For $x \in \mathcal{K}(L), y \in \mathcal{K}(M)$ define

$$\gamma(x, y) = \begin{cases} \{[x_1], \ldots, [x_{2l(x)-1}], [y_2], \ldots, [y_{2l(y)}]\} & \text{if } [x_{2l(x)}] = [y_1] \\ \{[z] : z \in x \cup y\} & \text{otherwise} \end{cases}$$

where $[x]$ is the $\Theta(1_L, 0_M)$ equivalence class of x. Then γ is the required isomorphism.

(iv) For each $a < 1 \in L$ define $\alpha_a : L \to \{0, 1\}$ and $\beta_a : \mathcal{K}(L) \to \{0, 1\}$ by setting

$$\alpha_a(c) = \begin{cases} 0 & \text{if } c \leqslant a \\ 1 & \text{otherwise} \end{cases} \quad \text{and } \beta_a(x) = \sum_{i=1}^{l(x)} (\alpha_a(x_{2i}) - \alpha_a(x_{2i-1})).$$

Then $\{\beta_a : a < 1 \in L\}$ is a full set of dispersion free states.

(v) If $x \cup y$ is a chain of L, then $x, y \in \mathcal{B}(x \cup y)$ and therefore commute. If $a \in x$ is incomparable to $b \in y$, then the commutator of x and y, $\gamma(x, y) \succcurlyeq \{a \wedge b, a \vee b\}$.

(vi) and (vii) follow easily from (v).

The following proposition will give a characterization of the congruences on $\mathcal{K}(L)$ in terms of the underlying lattice L. The proof is messy, and it may be of help to first think of the proposition in the case that L is chain finite. In this case, the p-ideals of $\mathcal{K}(L)$ correspond to the intervals $[0, c]$ where c is central in $\mathcal{K}(L)$.

DEFINITION 3.1. For a bounded lattice L, we say that $M \subseteq L$ is a middle set of L if M is a convex sublattice of L and for all $x \in L - M$ either x is an upper bound of M or x is a lower bound of M. We define a middle system of L to be a set of pairwise disjoint middle sets of L. For $\mathcal{M}$ a middle system of L, set

$$\beta_{\mathcal{M}} = \bigcup \{M^2 : M \in \mathcal{M}\} \cup \Delta_L$$

and

$$P_{\mathcal{M}} = \{x \in \mathcal{K}(L)\colon \text{for each } 1 \leqslant i \leqslant l(x), \{x_{2i-1}, x_{2i}\} \subseteq M \text{ for some } M \in \mathcal{M}\}.$$

PROPOSITION 3.3. *If L is a bounded lattice, then for each middle system $\mathcal{M}$ of L, $\beta_{\mathcal{M}}$ is a congruence on L, $P_{\mathcal{M}}$ is a p-ideal of $\mathcal{K}(L)$ and $\mathcal{K}(L)/P_{\mathcal{M}} \cong \mathcal{K}(L/\beta_{\mathcal{M}})$. Further, each p-ideal of $\mathcal{K}(L)$ is equal to $p_{\mathcal{M}}$ for some middle system $\mathcal{M}$ of L.*

CLAIM 1: $P_{\mathcal{M}}$ is a p-ideal of $\mathcal{K}(L)$.

Proof. Take $x, y \in \mathcal{K}(L)$. If $x \preccurlyeq y$ and $y \in P_{\mathcal{M}}$, then for each $1 \leqslant i \leqslant l(x)$ there exists $1 \leqslant j \leqslant l(y)$ such that $y_{2j-1} \leqslant x_{2i-1} < x_{2i} \leqslant y_{2j}$; as each $M \in \mathcal{M}$ is convex, $x \in P_{\mathcal{M}}$. If x, and y are both in $P_{\mathcal{M}}$ then, by a simple inductive argument using (1.7) through (1.9), $x \vee y \in P_{\mathcal{M}}$. So, $P_{\mathcal{M}}$ is an ideal of $\mathcal{K}(L)$.

We must show that for $x \in P_{\mathcal{M}}, y \wedge (y^{\perp} \vee x) \in P_{\mathcal{M}}$. For $M \in \mathcal{M}$ let $S_M = (x \cup y) \cap M$, $a_M = \bigwedge S_M$, $b_M = \bigvee S_M$, and $z = \bigvee\{\{a_M, b_M\}\colon a_M < b_M\}$. Note that $z \in P_{\mathcal{M}}$ and since $x \in P_{\mathcal{M}}$, $x \preccurlyeq z$. Also, $z \cup y$ is a chain, so z commutes with y. Therefore

$$y \wedge (y^{\perp} \vee x) \preccurlyeq y \wedge (y^{\perp} \vee z) = y \wedge z \in P_{\mathcal{M}}.$$

CLAIM 2: If I is a p-ideal of $\mathcal{K}(L)$, there exists a middle system $\mathcal{M}$ of L such that $I = P_{\mathcal{M}}$.

Proof. Given a p-ideal I of $\mathcal{K}(L)$, define a relation θ on L by

$$a\theta b \text{ if and only if } \{a \wedge b, a \vee b\} \in I \text{ or } a = b.$$

It is clear that θ is reflexive and symmetric. If $a, b, c \in L$ are all distinct and $a\theta b$, $b\theta c$ then as

$$\{a \wedge c, a \vee c\} \preccurlyeq \{a \wedge b \wedge c, a \vee b \vee c\} = \{a \wedge b, a \vee b\} \vee \{b \wedge c, b \vee c\} \in I$$

we have $a\theta c$. So θ is an equivalence relation.

Next, suppose that $a < b$ and $a\theta b$. If $a < c < b$, then as $\{a, c\} \preccurlyeq \{a, b\} \in I$, we have that $a\theta c$. So each block of θ is convex. Take $b, c \in a/\theta$ and assume that b is incomparable to c. As θ is an equivalence relation, $b\theta c$, so $\{b \wedge c, b \vee c\} \in I$. But as a/θ is convex, $c\theta(b \wedge c)$, $c\theta(b \vee c)$. Therefore, each block of θ is a convex sublattice of L.

If $|a/\theta| \geqslant 2$ and $x \in L$ is incomparable to a, then one of the following four cases must apply. There exists $b \in a/\theta$ with $a < b$ and $x \wedge a \neq 0$; there exists $b \in a/\theta$ with $a < b$ and $x \wedge a = 0$; there exists $b \in a/\theta$ with $a < b$ and $x \vee a \neq 1$; or there exists $b \in a/\theta$ with $b < a$ and $x \vee a = 1$.

If $a < b$ and $x \wedge a \neq 0$ then

$$\begin{aligned}\{a \wedge x, x\} \wedge (\{a \wedge x, x\}^{\perp} \vee \{a, b\}) &= \{a \wedge x, x\} \wedge (\{0, a \wedge x, x, 1\} \\ &\qquad \wedge \{a, b\}) \\ &= \{a \wedge x, x\} \wedge \{0, 1\} = \{a \wedge x, x\} \in I \\ \{a \wedge x, a\} \wedge (\{a \wedge x, a\}^{\perp} \vee \{a \wedge x, x\}) &= \{a \wedge x, a\} \wedge (\{0, a \wedge x, a, 1\} \\ &\qquad \vee \{a \wedge x, x\}) \\ &= \{a \wedge x, a\} \wedge \{0, 1\} = \{a \wedge x, a\} \in I\end{aligned}$$

So $\{a \wedge x, a\} \vee \{a \wedge x, x\} = \{a \wedge x, a \vee x\} \in I$ giving that $x \in a/\theta$.

If $a < b$ and $x \wedge a = 0$ then

$$\begin{aligned}\{0, x\} \wedge (\{0, x\}^{\perp} \vee \{a, b\}) &= \{0, x\} \wedge (\{x, 1\} \vee \{a, b\}) \\ &= \{0, x\} \wedge \{0, 1\} = \{0, x\} \in I\end{aligned}$$

and by the above argument, $\{0, a\} \in I$ so $x \in a/\theta$. The remaining cases are identical to the ones described replacing $a \wedge x$ with $a \vee x$.

We have shown that if $|a/\theta| \geqslant 2$, then a/θ forms a middle set of L. To prove our claim, set $\mathcal{M} = \{a/\theta : a \in L, |a/\theta| \geqslant 2\}$. By the above discussion, $\mathcal{M}$ is a middle system of L, but $\{x_1, x_2\} \in P_{\mathcal{M}}$ if and only if $\{x_1, x_2\} \in I$. So $P_{\mathcal{M}} = I$.

CLAIM 3: If $\mathcal{M}$ is a middle system of L, then $\mathcal{K}(L/\beta_{\mathcal{M}})$ is isomorphic to $\mathcal{K}(L)/P_{\mathcal{M}}$.

Proof. By [8, page 76], for $x, y \in \mathcal{K}(L)$, $x/P_{\mathcal{M}} \leqslant y/P_{\mathcal{M}}$ if and only if there exists $t \in /P_{\mathcal{M}}$ such that $x \preccurlyeq y \vee z$ and therefore $x/P_{\mathcal{M}} \leqslant y/P_{\mathcal{M}}$ if and only if there exists $t \in P_{\mathcal{M}}$ such that for each $1 \leqslant i \leqslant l(x)$, $\{x_{2i-1}, x_{2i}\} \preccurlyeq y \vee t$. It is not too difficult to show that $x/P_{\mathcal{M}} \leqslant y/P_{\mathcal{M}}$ if and only if for each $1 \leqslant i \leqslant l(x)$ there exists $1 \leqslant j \leqslant l(y)$ such that $x_{2i-1}/\beta_{\mathcal{M}} = x_{2i}/\beta_{\mathcal{M}}$ or $y_{2j-1}/\beta_{\mathcal{M}} \leqslant x_{2i-1}/\beta_{\mathcal{M}} < x_{2i}/\beta_{\mathcal{M}} \leqslant y_{2j}/\beta_{\mathcal{M}}$.

If $x \in \mathcal{K}(L)$ and a, b are of minimal length in $x/\beta_{\mathcal{M}}$, say n, then the above condition shows that $a_i/\beta_{\mathcal{M}} = b_i/\beta_{\mathcal{M}}$ for each $1 \leqslant i \leqslant 2n$. It is therefore possible to define a map $\alpha : \mathcal{K}(L)/P_{\mathcal{M}} \to \mathcal{K}(L/\beta_{\mathcal{M}})$ by setting

$$\alpha(x/P_{\mathcal{M}}) = \{a_i/\beta_{\mathcal{M}} : 1 \leqslant i \leqslant 2l(a)\} \text{ where } a \text{ is of minimal length in } x/P_{\mathcal{M}}.$$

It is obvious that α is onto $\mathcal{K}(L/\beta_{\mathcal{M}})$ and the condition for $x/P_{\mathcal{M}}$ to be dominated by $y/P_{\mathcal{M}}$ then exactly says that α is an order embedding. For $x \in \mathcal{K}(L)$ if $0/\beta_{\mathcal{M}} \in \alpha(x/P_{\mathcal{M}})$ then choose $y \in x/P_{\mathcal{M}}$ of minimal length such that $0 \in y$, and do similarly if $1/\beta_{\mathcal{M}} \in \alpha(x/P_{\mathcal{M}})$. It is then a straightforward application of (1.6) to verify that α is compatible with the orthocomplementations.

Piecing together the previous chains proves the proposition.

Let **K** denote the class of all OMLs which are isomorphic to $\mathcal{K}(L)$ for some bounded lattice L. To conclude this section, we will make a few remarks about the variety generated by the class **K**. A Boolean algebra is in **K** if and only if it is generated by a chain, and a Boolean algebra generated by a chain is complete if and only if it is finite. The class **K** cannot possibly be closed under arbitrary products as the two element Boolean algebra is in **K**, but $2^{\mathbb{N}}$ is not in **K**. It is also easy to see that **K** is not closed under subalgebras. There is a Boolean algebra B generated by an uncountable chain. If we take B' to be the subalgebra of B consisting of all elements of finite height and their complements, then every chain of B' is countable. But as B' is uncountable it could not possibly be generated by one of it's maximal chains.

Although **K** is not a variety, we have already seen that it is closed under finite products and homomorphic images. The variety generated by **K** is not the variety of OMLs, since there are identities satisfied by all OMLs with a full set of dispersion free states that are not satisfied by all OMLs [6].

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