PH.D. THESIS BEFORE FINAL CORRECTIONS

SHEAVES OF ORTHOMODULAR LATTICES AND MACNEILLE COMPLETIONS

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A Thesis
Submitted to the School of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree
Doctor of Philosophy

McMaster University

DOCTOR OF PHILOSOPHY (2007) MCMASTER UNIVERSITY
(Mathematics) Hamilton, Ontario

TITLE: Sheaves of Orthomodular Lattices and MacNeille Completions

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NUMBER OF PAGES: iv, 38

Acknowledgements

B. Banaschewski, G. Bruns, T. H. Choe, B. Mueller and M. Valeriote deserve my thanks for the help they have given me in preparing this thesis. Courses given by B. Banaschewski on category theory and frames and M. Valeriote on universal algebra have been tremendously helpful. Conversations with each of these people have been illuminating, and at times quite lengthy. Their interest, time and expertise was generously donated.

Günter Bruns has taught me a large part of the mathematics that I know. The amount and quality of the time, insight, guidance, humour and liquor that he has given to me will always be appreciated.

Contents

1 Introduction

Little is known about completions of orthomodular lattices (abbreviated OMLs). A complete modular ortholattice cannot contain an infinite pairwise perspective orthogonal set (see Amemiya-Halperin [3]), so the finite or cofinite dimensional subspaces of an infinite dimensional Hilbert space is a modular ortholattice which cannot be embedded into a complete modular ortholattice. It remains an open question whether every OML can be embedded into a complete OML.

It is known that the MacNeille completion of an OML is not necessarily orthomodular. The standard example of this, and previously the only one known, is based on a theorem of Amemiya and Araki [2]. This theorem states that for an inner product space V, if we consider the ortholattice $\mathcal{L}(V, \perp) = \{A \subseteq V : A = A^{\perp\perp}\}$ where $A^{\perp}$ is the set of elements orthogonal to A, then $\mathcal{L}(V, \perp)$ is an OML if and only if V is complete. Taking the OML L of finite or cofinite dimensional subspaces of an incomplete inner product space V, the ortholattice $\mathcal{L}(V, \perp)$ is a MacNeille completion of L which is not orthomodular. However, L can be embedded into the complete OML $\mathcal{L}(\bar{V}, \perp)$, where $\bar{V}$ is the completion of the inner product space V.

The only positive results about MacNeille completions of OMLs are given by Janowitz [14] and Bruns *et. al.* [7]. Janowitz showed that the MacNeille completion of an indexed OML is again an indexed OML. Bruns *et. al.* showed that a variety generated by a single finite OML is closed under MacNeille completions. There is still no useful characterization of the OMLs which have orthomodular MacNeille completions. That there are so few positive results about MacNeille completions of OMLs is not surprising, the misbehavior of the MacNeille completion has been well documented for the case of distributive lattices: in [10] Funayama produces a distibutive lattice whose MacNeille completion is not modular and in [13] I show that any lattice can be embedded into the MacNeille completion of some distributive lattice.

The contents of this thesis are in large devoted to extending the results of Bruns *et. al.*. Specifically, it is shown that a variety generated by a set of OMLs, a chain in any one having at most $n + 1$ elements, is closed under MacNeille completions. The crucial first step is taken in the second section, where it is shown that an OML in such a variety is directly irreducible if and only if it is simple. The mechanism of this proof is to provide certain polynomials over the given OML which *locally* return the least central upper bound of a given element.

These polynomials allow use to be made of the Pierce sheaf representation of an OML (a construction almost identical to the creature of the same name in ring theory). Of course, in the case of a directly irreducible OML this representation is entirely useless. But, in the setting of this thesis, the directly irreducibles OMLs considered are simple and of finite height. Therefore, the major drawback of this sheaf representation does not concern us.

The polynomials mentioned above are used to ensure that most of the stalks of the Pierce sheaf are well behaved. More precisely, the set of points where the stalks are directly irreducible and of height at most n contains a dense open set. This allows us to view the MacNeille completion of such an OML as the set of all sections on dense open sets modulo equivalence on dense open sets. As I have recently become aware (thanks in no small part to Prof. B. Mueller) this construction has an analogue in torsion theories.

In the final section, I have given a method to construct OMLs whose MacNeille completions are not orthomodular. Using examples constructed this way, it is shown that the primary result of this thesis cannot be extended to the variety generated by the finite OMLs (unfortunately negating our hope of proving that the variety of OMLs is not generated its finite members). These pathological OMLs can be embedded into complete OMLs via a process not unlike completing the underlying inner product space in the example described above.

2 Preliminaries

A fairly diverse range of subject material is used in this thesis. Most background material is briefly covered in this section, and even the reader with a good knowledge of these matters is advised to skim the following pages as the required notation is introduced here.

2.1 Ortholattices and orthomodular lattices

An orthocomplementation is a period two anti-isomophism of a bounded lattice which is also a complementation. An ortholattice (abbreviated: OL) is a pair $(L,')$ where L is a bounded lattice and $'$ is an orthocomplementation on L. It follows easily from this definition that an orthocomplementation satisfies the usual DeMorgan laws. As is customary, we refer to L as an ortholattice when no confusion is likely.

An orthomodular lattice (abbreviated: OML) is an ortholattice $(L,')$ which satisfies the following condition called the orthomodular law

$$\text{for all } a \leq b \in L,\ a \vee (a' \wedge b) = b, \tag{2.1}$$

or its equivalent form

$$\text{for all } a \leq b \in L,\ b \wedge a' = 0 \text{ if and only if } a = b. \tag{2.2}$$

Standard examples of OMLs include Boolean algebras and lattices of closed subspaces of Hilbert spaces, with orthogonality being the orthocomplementation. The standard reference for OLs and OMLs is [17].

A relation $\mathcal{C}$ is defined on an OML L by

$$a\mathcal{C}b \text{ if } (a \vee b) \wedge (a \vee b') = a.$$

We say that pairs in this relation are pairs of commuting elements. It can be shown that $\mathcal{C}$ contains the partial ordering of L and that $a\mathcal{C}b$ if and only if the sub-ortholattice generated by $\{a, b\}$ is Boolean. A block of L is defined to be a maximal set of pairwise commuting elements, or equivalently a maximal Boolean subalgebra of L. $\mathcal{C}(L)$, the centre of L, is defined to be the intersection of the blocks of L. This definition of the centre of an OML agrees with the usual definition of the centre of a lattice.

There are several interesting and important properties of the congruences of an OML, all of this information can be found in [17]. For an OML L, the congruences of L permute, that is to say that for congruences θ, ϕ of L $\theta \circ \phi = \phi \circ \theta$ where $\theta \circ \phi$ is the usual relational product. Equivalently, we say that θ and ϕ permute if $(a, b) \in \theta \vee \phi$ implies that there exists $c \in L$ with $(a, c) \in \theta$ and $(c, b) \in \phi$. Also, the congruences of L are exactly the binary relations which are congruences of the lattice reduct of L, so the congruence lattice of an OML is distributive as the congruence lattice of any lattice is distributive. Congruences of an OML share the same pleasant property exhibited by Boolean algebras, groups, etc., a congruence is completely determined by one of its equivalence classes. This relationship is given by

$$a\theta b \text{ if and only if } ((a \vee b) \wedge (a' \vee b'))\theta 0. \tag{2.3}$$

Certain congruences will play an important role in this thesis, the factor congruences. A congruence θ of an algebra A is called a factor congruence if there exists a congruence ϕ of A with A cannonically isomorphic to $A/\theta \times A/\phi$. It is easily seen that a congruence θ is a factor congruence if and only if there is a congruence ϕ with $\theta \wedge \phi = \Delta$, $\theta \vee \phi = A^2$ and θ, ϕ permuting. If the congruence lattice of an algebra A is distributive, then the factor congruences of A form a Boolean sublattice of the congruence lattice of A. For an OML L, the Boolean algebra of factor congruences of L is isomorphic to the centre of L. This isomorphism can be described by mapping the central element c to the factor congruence $\theta(c)$, where $\theta(c)$ is given by

$$\theta(c) = \{(a, b) \in L^2 : a \wedge c' = b \wedge c'\} = \{(a, b) \in L^2 : (a \vee b) \wedge (a' \vee b') \leq c\}. \tag{2.4}$$

The following simple observation will be quite useful

$$L \text{ is directly irreducible if and only if } \mathcal{C}(L) = \{0, 1\}. \tag{2.5}$$

An OML L is said to be of height at most n if every chain in L has at most $n + 1$ elements, and chain finite if every chain in L is finite. It follows from a result of

Dilworth's [9], and is proved explicitly in [17], that every congruence of a chain finite OML is a factor congruence. Therefore,

$$\text{a chain finite OML is directly irreducible if and only if it is simple.} \tag{2.6}$$

2.2 MacNeille completions

An ideal of a lattice L is the intersection of principal ideals if and only if it is equal to the set of lower bounds of the set of its upper bounds. Such ideals are called normal ideals. $\overline{L}$, the set of normal ideals of the lattice L, forms a complete lattice under set inclusion and is called the MacNeille completion [19] of L. It is easily seen that L can be join and meet densely embedded into $\overline{L}$. In fact, these properties determine the MacNeille completion of L up to isomorphism [5, 21]. That is, if C is a complete lattice into which L can be join and meet densely embedded then C is isomorphic to $\overline{L}$.

Given an ortholattice $(L,')$, an orthocomplementation $\perp$ may be defined on $\overline{L}$ as follows:

$$I^{\perp} = \{x \in L : x' \text{ is an upper bound of } I\}. \tag{2.7}$$

This orthocomplementation extends that of L and is uniquely determined by this property [18]. The OL $(\overline{L}, \perp)$ is called the MacNeille completion of the OL L. It is then easily seen that if C is a complete OL into which L can be join densely embedded then C is isomorphic to the MacNeille completion of L.

The following property of OMLs will be quite useful to us. For α an embedding of an OML L into an OML M, α is a join dense embedding if

$$\text{for each } 0 \neq m \in M \text{ there exists } 0 \neq l \in L \text{ with } \alpha(l) \leq m. \tag{2.8}$$

2.3 Universal algebra

A class of algebras of the same type is said to be a variety if it is closed under the formation of products, homomorphic images and subalgebras. Birkhoff has shown that a class of algebras of the same type is a variety if and only if it is the class of all models of some set of identities (first order formulas admitting only universal quantification) over the language of the algebras. Note that the orthomodular law is equivalent to the identity

$$x \vee (x' \wedge (x \vee y)) = x \vee y$$

so the class of all OMLs is a variety.

A subdirectly irreducible algebra is one with a least nontrivial congruence. The notions of directly irreducible and simple should be obvious from classical algebra. It easily follows that a simple algebra is subdirectly irreducible, and a subdirectly irreducible algebra is directly irreducible.

A subalgebra of a product of a family of algebras is said to be subdirect if each projection map is surjective. Birkhoff has also shown that any algebra in a given variety is isomorphic to a subdirect product of subdirectly irreducible algebras in that variety.

Given a family of algebras $(A_i)_{i \in I}$ of the same type, and a first order formula $\varphi(x_1, \ldots, x_n)$ in the language of these algebras and $a_1, \ldots, a_n \in \prod A_i$ define

$$[\![\varphi(a_1, \ldots, a_n)]\!] = \{i \in I : A_i \models \varphi(a_1(i), \ldots, a_n(i))\}.$$

For $\mathcal{U}$ an ultrafilter over the set I the relation Θ on $\prod A_i$ defined by

$$a \Theta b \text{ if and only if } [\![a = b]\!] \in \mathcal{U}$$

is a congruence on $\prod A_i$. We follow the customary practice of using $\mathcal{U}$ and the associated congruence Θ interchangeably. The ultraproduct of $(A_i)_I$ over $\mathcal{U}$ is defined to be $(\prod A_i)/\Theta$ and is denoted by $\prod A_i/\mathcal{U}$. A very useful theorem due to Łoś states

$$\prod_{i \in I} A_i/\mathcal{U} \models \varphi(a_1/\mathcal{U}, \ldots, a_n/\mathcal{U}) \text{ if and only if } [\![\varphi(a_1, \ldots, a_n)]\!] \in \mathcal{U}. \tag{2.9}$$

A variety of algebras is called congruence distributive if the congruence lattice of each algebra in the variety is distributive. The variety of OMLs is congruence distributive. The full usefulness of the ultraproduct construction is realized in congruence distributive varieties. Jónsson [15] has shown that if a set A of algebras of the same type generates a congruence distributive variety, then the subdirectly irreducible algebras in that variety are homomorphic images of subalgebras of ultraproducts of families of algebras in A.

A particular application of the above theory will be of importance to us. If $\mathcal{V}$ is a variety generated by a set of OMLs, each having height at most n, then

$$\text{the subdirectly irreducibles in } \mathcal{V} \text{ have height at most } n. \tag{2.10}$$

The proof of this fact follows easily from the observation that being of height at most n is a first order property, and that every variety of OMLs is congruence distributive.

All of the above information can be found in [8].

2.4 Boolean algebras and Stone spaces

Given a Boolean algebra B, we define B^* to be the set of all maximal proper ideals of B, and for each $c \in B$ define c^* to be the set of all maximal proper ideals of B which contain c. For $c, d \in B$

$$c^* \cup d^* = (c \wedge d)^*, c^* \cap d^* = (c \vee d)^*, (c')^* = B^* - c^*, 0^* = B^* \text{ and } 1^* = \emptyset,$$

so $\{c^* : c \in B\}$ is a basis for a topology on B^* and B^* with this topology is called the Stone space of B. It is well known that the Stone space of B is a compact zero-dimensional (has a basis of sets which are both open and closed) Hausdorff space for which the sets which are both open and closed (often called clopen) are exactly the sets c^* where $c \in B$.

For a bounded chain C, let $\mathcal{F}$ be the field of subsets of $C - \{1\}$ generated by the sets $A_x = \{y \in C : y < x\}$ where x ranges over C. As every element of $\mathcal{F}$ has a unique representation of the form

$$\bigcup_{i=1}^{n} (A_{x_{2i}} - A_{x_{2i-1}}) \text{ where } x_1 < x_2 < \ldots < x_{2n} \in C,$$

the set $\mathcal{B}(C)$ of all finite, even length chains in C carries a natural Boolean structure. For $x \in \mathcal{B}(C)$ let $l(x)$ be half of the length of x and let $x_1, \ldots, x_{2l(x)}$ be the elements of x in order of increasing size. Then if $\preceq$ and $\perp$ are the induced partial ordering and orthocomplementation on $\mathcal{B}(C)$ we have for x, $y \in \mathcal{B}(C)$

$$\begin{aligned} x \preceq y \quad \text{if and only if} \quad & \text{for each } 1 \le i \le l(x) \text{ there exists } 1 \le j \le l(y) \\ & \text{such that } y_{2j-1} \le x_{2i-1} < x_{2i} \le y_{2j}, \end{aligned} \tag{2.11}$$

and that $x^\perp$ is defined by

$$x \cup x^\perp = x \cup \{0, 1\} \text{ and } x \cap x^\perp = x - \{0, 1\}. \tag{2.12}$$

It is immediately evident that $\mathcal{B}(C)$, with the natural Boolean structure, is generated by a sub-chain which is isomorphic to C. We call $\mathcal{B}(C)$ the Boolean algebra generated by the chain C. We will make use of the fact that a Boolean algebra generated by a chain is complete if and only if it is finite.

For further information on Boolean algebras see [4].

2.5 Kalmbach's construction

In [16] Kalmbach introduced a method of constructing an OML containing a given lattice as a sublattice, showing that the variety of OMLs does not satisfy any particular lattice identities. This construction will be exploited in the final section of this thesis to produce OMLs whose MacNeille completions are not orthomodular.

For a bounded lattice L, define the set $\mathcal{K}(L)$ to be the union of the sets $\mathcal{B}(C)$ where C ranges over all 0, 1 sub-chains of L. Define a map $\perp: \mathcal{K}(L) \longrightarrow \mathcal{K}(L)$ to be the union of the complementations on the $\mathcal{B}(C)$ and define a relation $\preceq$ on $\mathcal{K}(L)$ to be the union of the partial orderings on the $\mathcal{B}(C)$. Then, $(\mathcal{K}(L), \preceq, \perp)$ is an OML, which will be referred to simply as $\mathcal{K}(L)$.

Later, we will need certain recursive methods for finding joins and meets in $\mathcal{K}(L)$. As a description of these methods amounts to a proof that $\mathcal{K}(L)$ is an OML, a proof of this is given.

Theorem 2.1 *For L a bounded lattice, $\mathcal{K}(L)$ is an OML.*

Proof.

From (2.11) it follows that $\preceq$ is a partial ordering and from (2.12) it follows that $\perp$ is indeed a function. For elements x, y of $\mathcal{K}(L)$, if $x \cup y$ is a chain of L then the supremum and infinum of these elements in $\mathcal{B}(x \cup y \cup \{0, 1\})$ are their supremum and infinum in $\mathcal{K}(L)$. Therefore, it follows immediately from (2.11) that $\mathcal{K}(L)$ satisfies (2.1), so if $\mathcal{K}(L)$ is indeed a lattice, it is an OML.

Claim. For x, y elements of $\mathcal{K}(L)$, if $l(x) = 1$ then $x \vee y$ exists.

Proof. The proof is by induction on $l(y)$. Assume that $x \cup y$ is not a chain of L, and therefore that $l(y) > 0$. It is easily verified that

$$\{x_1, x_2\} \vee \{y_1, y_2\} = \{x_1 \wedge y_1, x_2 \vee y_2\} \text{ if } \{x_1, x_2, y_1, y_2\} \text{ is not a chain.} \tag{2.13}$$

Setting $z_i = \{y_{2i-1}, y_{2i}\}$ for each $1 \leq i \leq l(y)$ and taking k least such that $x \cup z_k$ is not a chain, by (2.13) $x \vee z_k$ exists and by inductive hypothesis $(x \vee z_k) \vee (y - z_k)$ exists, so

$$x \vee y = (x \vee z_k) \vee (y - z_k). \tag{2.14}$$

It is now easily verified that joins exist in $\mathcal{K}(L)$. For x an y nonzero elements of $\mathcal{K}(L)$, setting $w_i = \{x_{2i-1}, x_{2i}\}$ for each $1 \leq i \leq l(x)$, using the previous claim we have

$$x \vee y = (((y \vee w_1) \vee w_2) \ldots) \vee w_{l(x)}. \tag{2.15}$$

For x and y elements of $\mathcal{K}(L)$, to see that the infinum of x and y exists first note that

$$\text{if } l(x) = l(y) = 1 \text{ then } x \wedge y = \begin{cases} \{x_1 \vee y_1, x_2 \wedge y_2\} & \text{if } x_1 \vee y_1 < x_2 \wedge y_2 \\ \emptyset & \text{otherwise} \end{cases} \tag{2.16}$$

But, z is a lower bound of $\{x, y\}$ if and only if for each $1 \leq k \leq l(z)$ there exists $1 \leq i \leq l(x)$, and $1 \leq j \leq l(y)$ such that $x_{2i-1} \vee y_{2j-1} \leq z_{2k-1} < z_{2k} \leq x_{2i} \wedge y_{2j}$. So

$$x \wedge y = \bigvee \{\{x_{2i-1}, x_{2i}\} \wedge \{y_{2j-1}, y_{2j}\} : 1 \leq i \leq n, 1 \leq j \leq m\}. \tag{2.17}$$

If z is the result of an operation on elements x, y of $\mathcal{K}(L)$ then z is a chain in the sublattice of L generated by $x \cup y \cup \{0, 1\}$. This follows immediately from (2.12) for the operation $\perp$ and by a simple induction using (2.13) through (2.15) for join. Then (2.15) and (2.17) provide the result for meets. As a corollary of this observation,

$$\text{if } M \text{ is a 0,1 sublattice of } L \text{ then } \mathcal{K}(M) \text{ is a subalgebra of } \mathcal{K}(L). \tag{2.18}$$

2.6 Sheaves

The notion of a sheaf of sets has several wildly varying descriptions: as a local homeomorphism between two topological spaces, or as a contravarient functor from a frame to the category of sets satisfying certain conditions. The notion of a sheaf of groups, rings etc. can be similarly described by requiring that the stalks of the local homeomorphism each have a group structure compatible with the topology, or by a contravarient functor from a frame to the category of groups satisfying certain conditions.

For many applications of sheaves to universal algebra, the simpler notion of a Boolean product given by Burris and Werner suffices. A different definition of a sheaf of algebras will be used in this thesis as I believe it makes for a simpler and more natural presentation. The notion used here might be more appropriately called a topological product of a family of algebras. The global sections of a sheaf of algebras are a natural example of such a topological product.

Such matters aside, the only sheaf actually considered in this thesis is the Pierce sheaf of an algebra, and readers familiar with standard approaches to sheaves will have no trouble putting our definitions and results into standard terminology.

Definition 2.2 *A sheaf of τ-algebras is a triple $\tilde{S} = ((L_x)_{x\in X}, \gamma, \delta)$ where*

i) $(L_x)_{x\in X}$ *is a family of algebras of type τ whose underlying sets are pairwise disjoint,*

ii) γ *is a topology on X,*

iii) δ *is a topology on $\bigcup_{x\in X} L_x$,*

iv) $\{f \in \prod_{x\in X} L_x : f \text{ is continuous } \}$ *is a subalgebra of $\prod_{x\in X} L_x$.*

The algebra $\{f \in \prod_{x\in X} L_x : f \text{ is continuous } \}$ is called the global sections of $\tilde{S}$ and will be denoted by $\Gamma\tilde{S}$, the union of $(L_x)_{x\in X}$ with its topology δ is called the sheaf space of $\tilde{S}$ and will be denoted by S, and the algebra L_x is called the stalk of $\tilde{S}$ at x.

The following result was originally obtained by Pierce (in the setting of rings), but the presentation here draws largely on results of Burris and Werner for Boolean products [8]. This result will be essential later in the thesis and the notation introduced here will be used freely.

Theorem 2.3 *If the factor congruences of an algebra A are pairwise commuting and form a Boolean sublattice B of the congruence lattice of A then*

i) for each $I \in B^*$, $\bigcup I$ is a congruence on A,

ii) for each $a \in A$ and $\theta \in B$, if we set $\mathcal{O}(a,\theta) = \{a/\bigcup I : \theta \in I\}$ we have that $\{\mathcal{O}(a,\theta) : a \in A, \theta \in B\}$ is a basis for a topology δ on $\bigcup_{I\in B^*} A/\bigcup I$,

iii) with γ the Stone topology on B^*, $\tilde{S} = ((A/\bigcup I)_{I\in B^*}, \gamma, \delta)$ is a sheaf of algebras of the type of A,

iv) A is isomorphic to $\Gamma\tilde{S}$ by the map $a \rightsquigarrow \tilde{a}$ where $\tilde{a}(I) = a/\bigcup I$.

Strictly speaking my statement of this theorem is incorrect as there is no guaranty that the underlying sets of the family $(A/\bigcup I)_{I\in B^*}$ are pairwise disjoint. However this difficulty could be easily remedied by considering the family $(A/\bigcup I \times \{I\})_{I\in B^*}$, where

$\{I\}$ is the trivial algebra, but the expense in notation would hardly seem worthwhile to correct this minor inaccuracy. It is to be assumed throughout that $(A/\bigcup I)_{I\in B^*}$ is a family whose underlying sets are pairwise disjoint.

Proof. The first assertion follows from the fact that I is an updirected subset of the congruence lattice of A. In fact $\bigcup I$ is just the join of the set I in the congruence lattice of A.

For the second assertion, for $a/\bigcup I \in \mathcal{O}(b_1,\theta_1)\cap\mathcal{O}(b_2,\theta_2)$, we will produce $\phi\in B$ with $a/\bigcup I \in \mathcal{O}(a,\phi) \subseteq \mathcal{O}(b_1,\theta_1)\cap\mathcal{O}(b_2,\theta_2)$. But if $a/\bigcup I \in \mathcal{O}(b_1,\theta_1)\cap\mathcal{O}(b_2,\theta_2)$ then $\theta_1,\theta_2\in I$ and $a/\bigcup I = b_1/\bigcup I = b_2/\bigcup I$, so for some $\chi_1,\chi_2\in I$, $a/\chi_1 = b_1/\chi_1$ and $a/\chi_2 = b_2/\chi_2$. Then for $\phi = \theta_1\vee\theta_2\vee\chi_1\vee\chi_2$, $\phi\in I$ so $a/\bigcup I\in\mathcal{O}(a,\phi)$. But, if $a/\bigcup J \in \mathcal{O}(a,\phi)$ then $\phi\in J$ so $\theta_1,\theta_2\in J$ and $a/\bigcup J = b_1/\bigcup J = b_2/\bigcup J$.

To show that $\Gamma\tilde{S} = \{f\in\prod_{I\in B^*} A/\bigcup I : f \text{ is continuous }\}$ is a subalgebra of $\prod_{I\in B^*} A/\bigcup I$ we must show that it is closed under the operations of $\prod_{I\in B^*} A/\bigcup I$. For nullary operations, we will show more than is necessary by showing that for each $a\in A$ the map $\tilde{a} : B^* \longrightarrow S$ defined by $\tilde{a}(I) = a/\bigcup I$ is continuous. To see this, suppose V is an open neighbourhood of $\tilde{a}(I)$, then as $\mathcal{O}(a,\Delta)$ is also an open neighbourhood of $\tilde{a}(I)$, for some $\phi\in B$, $\tilde{a}(I)\in\mathcal{O}(a,\phi)\subseteq\mathcal{O}(a,\Delta)\cap V$ giving $\tilde{a}[\phi^*]\subseteq V$.

For an n-ary operation t of A, with $n\geq 1$, and $f_1,\ldots,f_n\in\Gamma\tilde{S}$, to show that $t(f_1,\ldots,f_n)\in\Gamma\tilde{S}$ we must show that for a basic open neighbourhood $\mathcal{O}(a,\theta)$ of $t(f_1,\ldots,f_n)(I)$ there is $\phi\in I$ with $t(f_1,\ldots,f_n)[\phi^*]\subseteq\mathcal{O}(a,\theta)$. Say $f_i(I) = a_i/\bigcup I$ for each $1\leq i\leq n$, then as each f_i is continuous, for each $1\leq i\leq n$ there is $\theta_i\in I$ with $f_i[\theta_i^*]\subseteq\mathcal{O}(a_i,\theta_i)$. Setting $\chi = \theta\vee\bigvee_{i=1}^n\theta_i$ we have $\chi\in I$ and if $J\in\chi^*$ then $f_i(J)\in\mathcal{O}(a_i,\theta_i)$ for each $1\leq i\leq n$ giving

$$t(f_1,\ldots,f_n)(J) = t(a_1/\bigcup J,\ldots,a_n/\bigcup J) = t(a_1,\ldots,a_n)/\bigcup J.$$

But $t(f_1,\ldots,f_n)(I) = a/\bigcup I$, so for some $\psi\in I$ $a/\psi = t(a_1,\ldots a_n)/\psi$. Then for $\phi = \chi\vee\psi$ we have $\phi\in I$ and $t(f_1,\ldots,f_n)[\phi^*]\subseteq\mathcal{O}(a,\theta)$.

As $\tilde{a}$ is continuous for each $a\in A$, the map $a\rightsquigarrow\tilde{a}$ is into $\Gamma\tilde{S}$. To show this map is one to one we will show more, namely, for $a, b \in A$ and $\theta \in B$

$$\tilde{a} \text{ and } \tilde{b} \text{ agree on } \theta^* \text{ if and only if } (a,b) \in \theta. \tag{2.19}$$

Note that if $\tilde{a} = \tilde{b}$ then $(a,b) \in \Delta$, so $a = b$. To prove this result suppose $(a,b) \in \theta$, then $a/\bigcup I = b/\bigcup I$ for each $I \in \theta^*$ so $\tilde{a}$ and $\tilde{b}$ agree on θ^*. On the other hand, if $(a,b) \notin \theta$ then for ϕ the complement of θ in B and $F = \{\chi \in B : (a,b) \in \chi\}$, $\theta \notin F$ and F is a proper filter over B. We can extend $F \cup \{\phi\}$ to an ultrafilter $\mathcal{U}$ over B, then $I = B - \mathcal{U} \in B^*$, $I \in \theta^*$ and $(a,b) \notin \bigcup I$ giving $\tilde{a}(I) \neq \tilde{b}(I)$.

For $f \in \Gamma\tilde{S}$ there exists a natural number n and for each $1 \leq i \leq n$ a congruence $\theta_i \in B$ and $a_i \in A$ so that $f[\theta_i^*] = \mathcal{O}(a_i, \theta_i)$ and $\bigcup_{i=1}^n \theta_i^* = B^*$. Indeed, as f is continuous, for each $I \in B^*$ there exists $a \in A$ and $\theta \in B$ so that $I \in \theta^*$ and $f[\theta^*] = \mathcal{O}(a, \theta)$. The set of such θ^* forms an open cover of B^* which may be reduced to a finite subcover as B^* is compact, giving such a system mentioned above. If n is the least natural number for which there exists such a system we claim that $n = 1$ giving that $f[B^*] = \mathcal{O}(a, \Delta)$ for some $a \in A$ so $f = \tilde{a}$. To justify this claim, suppose $n \geq 2$, then as $f[\theta_1^* \cap \theta_2^*] = \mathcal{O}(a_1, \theta_1 \vee \theta_2) = \mathcal{O}(a_2, \theta_1 \vee \theta_2)$, $a_1(\theta_1 \vee \theta_2)a_2$. But the congruences in B permute, so for some $b \in A$ we have $a_1\theta_1 b\theta_2 a_2$. Then $f[\theta_1^*] = \mathcal{O}(a_1, \theta_1) = \mathcal{O}(b, \theta_1)$ and $f[\theta_2^*] = \mathcal{O}(a_2, \theta_2) = \mathcal{O}(b, \theta_2)$, so $f[(\theta_1 \wedge \theta_2)^*] = \mathcal{O}(b, \theta_1 \wedge \theta_2)$ contradicting the minimality of n.

The sheaf constructed above is usually called the Pierce sheaf of the algebra A as Pierce first proved this result for rings. As any OML is congruence distributive, the factor congruences form a Boolean sublattice of the congruence lattice, and as all congruences in an OML permute, the factor congruences are pairwise commuting. The following corollary was first stated by Graves and Selesnick.

Corollary 2.4 *An OML is isomorphic to the global sections of its Pierce sheaf.*

The following seems to be part of the folklore of the subject.

Proposition 2.5 *If the factor congruences of an algebra A are pairwise commuting and form a Boolean sublattice B of the congruence lattice of A then for $\tilde{S}$ the Pierce sheaf of A the following are equivalent*

i) for each $a, b \in A$, $[\![\tilde{a} = \tilde{b}]\!]$ is clopen,

ii) for each $a, b \in A$, there is a least congruence in B containing (a, b),

iii) the sheaf space of $\tilde{S}$ is Hausdorff.

Proof. To see that the first condition implies the second, for $a, b \in A$ as $[\![\tilde{a} = \tilde{b}]\!]$ is clopen, $[\![\tilde{a} = \tilde{b}]\!] = \theta^*$ for some $\theta \in B$. By 2.19 we have $(a, b) \in \theta$ and if $(a, b) \in \phi$ for some $\phi \in B$ then $\tilde{a}$ agrees with $\tilde{b}$ on ϕ^* so $\phi^* \subseteq [\![\tilde{a} = \tilde{b}]\!]$ giving that $\theta \leq \phi$. So θ is the least congruence in B containing (a, b).

To see that the second condition implies the third, suppose that $a/\bigcup I$ and $b/\bigcup J$ are distinct points in S. If $I \neq J$ then there is $\theta \in I$ with $\phi \in J$, where ϕ is the complement of θ in B. Then $\mathcal{O}(a, \theta)$ and $\mathcal{O}(b, \phi)$ are disjoint open sets separating $a/\bigcup I$ and $b/\bigcup J$. If $I = J$ then $a/\bigcup I \neq b/\bigcup I$, so for θ the least element of B containing (a, b), $\theta \notin I$. So $\phi \in I$ where ϕ is the complement of θ in B. Then $a/\bigcup I \in \mathcal{O}(a, \phi)$ and $b/\bigcup I \in \mathcal{O}(b, \phi)$, but if $c/\bigcup M \in \mathcal{O}(a, \phi) \cap \mathcal{O}(b, \phi)$ then $a/\bigcup M = b/\bigcup M$ giving that $\phi \in M$ an impossibility.

To see that the third condition implies the first, suppose $a, b \in A$. As $\tilde{a}, \tilde{b}$ are continuous maps into a Hausdorff space, $[\![\tilde{a} = \tilde{b}]\!]$ is closed. But $[\![\tilde{a} = \tilde{b}]\!] = \tilde{a}^{-1}[\mathcal{O}(b, \Delta)]$ which is open.

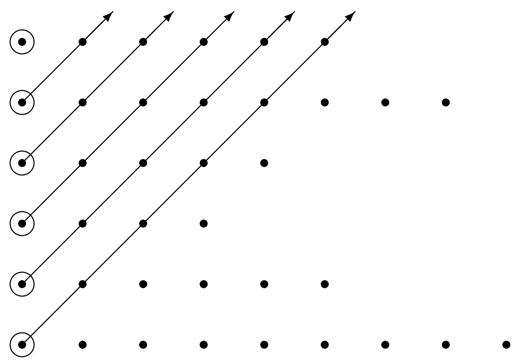
As a final remark about the various notions of sheaves briefly discussed at the start of this section, the definition given here of a sheaf (or topological product) is the most general. The usual definition of a sheaf is more general than that of a Boolean product [8], in fact the Pierce sheaf of an algebra is a Boolean product if and only if the algebra satisfies one of the three equivalent conditions of the previous proposition.

3 Some polynomials

For L an OML, we will say that M is a partial matrix in L if M is a rectangular matrix whose entries are elements of L. We do not require that each cell of M has an entry, but we do require a certain normal form. There must be an entry in each row and column of M, and the entries of a row must be an initial segment of that row. We say that a partial matrix M in an OML L is admissible if the following conditions are satisfied. For each row of M, the entries in that row are pairwise distinct and form a block of L. If we consider the Northeastern diagonals of M originating in the first column (these will be referred to simply as diagonals), there is an entry in each cell of the diagonal. Finally, we require that all of the entries on a given diagonal, which are not in the first column, are equal and do not commute with the entry in the first column of that diagonal.

For a partial matrix M, we will refer to the entry in the (i, j) cell of M, if there is one, by $M_{i,j}$. Define $N(M)$, the size of M, to be a sequence of natural numbers $< n_1, \ldots, n_r >$ where r is the number of rows in M and for each $1 \leq i \leq r$, n_i is the number of entries in the i^{th} row of M. If two partial matrices, M and P over the same OML have the same size, we say that $M \leq P$ if each entry of P dominates the corresponding entry of M. Finally, let $< \mathbb{N}^+, \leq_L>$ denote the set of sequences of positive natural numbers with the lexicographical ordering.

The diagram below may help to visualize a partial matrix and its diagonals.



Lemma 3.1 *For $\mathcal{K}$ a set of OMLs, each of which is directly irreducible and of height at most n, define a set A by $A = \{N(R) : R$ is admissible in some $L \in \mathcal{K}\}$, then*

i) A is a finite set and has a maximum in $< I\!N^+, \leq_L >$, say $< m_1, \ldots, m_q >$.

ii) If M is admissible in some $L \in \mathcal{K}$ and $N(M) = < m_1, \ldots, m_q >$, then for $y \in L$, y commutes with all the entries of M if and only if $y \in \{0, 1\}$.

Proof. i) If R is admissible in some $L \in \mathcal{K}$, as the entries of a row of L are pairwise distinct and form a block of L, each row of R has at most 2^n entries. As there is an entry in each cell of a diagonal of R, the number of rows of R cannot exceed the length of the first row of R. So A is a finite set, and as $< I\!N^+, \leq_L >$ is a chain, A has a maximum in $< I\!N^+, \leq_L >$.

ii) Take $M \in L$ as given. Assume that $y \in L - \{0, 1\}$ and y commutes with all the entries of M. As each row of M forms a block of L, y and 0 appear on each row of L and never on a diagonal of L (except possibly the one element diagonal). As $y \notin \{0, 1\}$ there exists $z \in L$ which does not commute with y, and a block B of L with $z \in B$. Form a new partial matrix M' by adding a row to the bottom of M, the entries being the elements of B, each listed only once, with z listed first. By switching at most two entries per row of M', we may form a new partial matrix M'' which agrees with M' on all the diagonals, except possibly the diagonal originating at z, such that the entries of the diagonals originating at z which are not in the first column are all equal to y. But M'' is admissible, contradicting the maximality of $< m_1, \ldots, m_q >$.

In the following, we will assume that $\mathcal{K}$ and $< m_1, \ldots, m_q >$ are as described in lemma 3.1. T denotes the term algebra, of the type of ortholattices, over a countably infinite set S. We will assume that $a_{1,1}, \ldots, a_{q,m_q}$ are elements of S, and denote the vector $< a_{1,1}, \ldots, a_{q,m_q} >$ by $\vec{a}$. For M a partial matrix in an OML L with $N(M) = < m_1, \ldots, m_q >$, there exists a map $\varphi : S \longrightarrow L$ such that $\varphi(a_{i,j}) = M_{i,j}$ for all $1 \leq i \leq q$, $1 \leq j \leq m_i$, and a homomorphism $\overline{\varphi} : T \longrightarrow L$ extending φ. So, for $t(\vec{a}) \in T$ we may define $t(M)$ to be $\overline{\varphi}(t)$.

Lemma 3.2 *There exists $p(\vec{a}, b) \in T$ such that for any partial matrix M in any $L \in \mathcal{K}$ with $N(M) = < m_1, \ldots, m_q >$, we have*

i) $p(M, 0) = 0$.

ii) If M is admissible then for $z \in L$, $p(M, z) = 0$ if and only if $z = 0$, and $p(M, z) = 1$ otherwise.

Proof. Define recursively for each $k \geq 0$, $p^k(\vec{a}, b) \in T$ as follows:

$$p^0(\vec{a}, b) = \bigvee_{1 \leq i \leq q} \bigvee_{1 \leq j \leq m_i} [(b \vee a_{i,j}) \wedge (b \vee a'_{i,j})],$$

$$p^{k+1}(\vec{a}, b) = p^0(\vec{a}, p^k(\vec{a}, b)) \text{ for } k \geq 0.$$

For $z \in L$, $p^0(M, z) \geq z$, so $p^{k+1}(M, z) = p^0(M, p^k(M, z)) \geq p^k(M, z)$, therefore $\{p^k(M, z) : k \geq 0\}$ forms a chain in L. If $p^{k+1}(M, z) = p^k(M, z)$ then $p^{k+2}(M, z) = p^{k+1}(M, z)$, so $p^n(M, z) = p^{n+1}(M, z)$ since every chain in L has at most n elements. But, $p^0(M, z) = z$ if and only if z commutes with all of the entries of M, so $p^n(M, z)$ commutes with all the entries of M. In particular, if M is admissible then $p^n(M, z) \in \{0, 1\}$ (by lemma 3.1). A simple induction shows that $p^n(M, z) = 0$ if and only if $z = 0$. Set $p(\vec{a}, b) = p^n(\vec{a}, b)$.

Lemma 3.3 *For each $1 \leq i \leq q$, $1 \leq j \leq m_i$ there exists $p_{i,j}(\vec{a}) \in T$ such that for any partial matrix M, in any $L \in \mathcal{K}$, with $N(M) = < m_1, \ldots, m_q >$, if we define a partial matrix Q in L, with $N(Q) = < m_1, \ldots, m_q >$, by setting $Q_{i,j} = p_{i,j}(M)$ for all $1 \leq i \leq q$, $1 \leq j \leq m_i$, then*

i) The entries of each row of Q are pairwise commuting.

ii) $Q \leq M$, and if the entries of each row of M are pairwise commuting then $Q = M$.

Proof. Define recursively for each $k \geq 0$, $p^k_{i,j}(\vec{a}) \in T$ for each $1 \leq i \leq q$, $1 \leq j \leq m_i$ as follows:

$$p^0_{i,j}(\vec{a}) = \bigwedge_{1\leq l\leq m_i} [(a_{i,j} \wedge a_{i,l}) \vee (a_{i,j} \wedge a'_{i,l})],$$

$$p^{k+1}_{i,j}(\vec{a}) = p^0_{i,j}(p^k_{1,1}(\vec{a}), \ldots, p^k_{q,m_q}(\vec{a})) \text{ for } k \geq 0.$$

For each $k \geq 0$ define a partial matrix Q^k in L, with $N(Q^k) = N(M)$, by setting $Q^k_{i,j} = p^k_{i,j}(M)$ for each $1 \leq i \leq q$, $1 \leq j \leq m_i$. Note that $Q^{k+1}_{i,j} = p^0_{i,j}(Q^k)$, so if $Q^{k+1} = Q^k$, then $Q^{k+2} = Q^{k+1}$.

For any partial matrix R in L with $N(R) = N(M)$ we have for any $1 \leq i \leq q$, $1 \leq j \leq m_i$ that $p^0_{i,j}(R) \leq R_{i,j}$, and $p^0_{i,j}(R) = R_{i,j}$ if and only if $R_{i,j}$ commutes with all the entries on the i^{th} row of R. By an easy induction we have $M \geq Q^k \geq Q^{k+1}$ for all $k \geq 0$. As there are at most $q2^n$entries in M, and every chain of L has at most $n+1$ elements, $Q^N = Q^{N+1}$, where $N = (n+1)q2^n$. Then, setting $p_{i,j}(\vec{a}) = p^N_{i,j}(\vec{a})$ for each $1 \leq i \leq q$, $1 \leq j \leq m_i$ we are finished.

Lemma 3.4 *For each $1 \leq i \leq q$, $1 \leq j \leq m_i$ there exists $v_{i,j}(\vec{a}) \in T$ such that for any partial matrix M in any $L \in \mathcal{K}$, with $N(M) = < m_1, \ldots, m_q >$, if we define a partial matrix V in L, with $N(V) = N(M)$, by setting $V_{i,j} = v_{i,j}(M)$ for all $1 \leq i \leq q$, $1 \leq j \leq m_i$ then*

i) The entries of each row of V are pairwise commuting and the entries of each diagonal of V which are not in the first column, are equal.

ii) If M is admissible then $V = M$.

Proof. Define recursively for each $k \geq 1$, $v^k_{i,j}(\vec{a}) \in T$ for each $1 \leq i \leq q$, $1 \leq j \leq m_i$ as follows:

$$v^1_{i,j}(\vec{a}) = \begin{cases} \bigwedge\{a_{l,m} : l + m = i + j, m \neq 1\} & \text{if } 2 \leq j \leq q - i + 1 \\ a_{i,j} & \text{otherwise} \end{cases}$$

$$v^{2k}_{i,j}(\vec{a}) = p_{i,j}(v^{2k-1}_{1,1}(\vec{a}), \ldots, v^{2k-1}_{q,m_q}(\vec{a})) \text{ for } k \geq 1 \ (*)$$

$$v^{2k+1}_{i,j}(\vec{a}) = v^1_{i,j}(v^{2k}_{1,1}(\vec{a}), \ldots, v^{2k}_{q,m_q}(\vec{a})) \text{ for } k \geq 1$$

$(*)$ the $p_{i,j}(\vec{a})$ are described in lemma 3.3.

For a cell (i, j) on a diagonal and not in the first column, we want $v^1_{i,j}(M)$ to be the meet of all entries of M on that diagonal which are not in the first column, hence the cryptic definition.

For each $k \geq 1$ define a partial matrix V^k in L, with $N(V^k) = N(M)$, by setting $V^k_{i,j} = v^k_{i,j}(M)$ for each $1 \leq i \leq q$, $1 \leq j \leq m_i$. Note that $V^{2k}_{i,j} = p_{i,j}(V^{2k-1})$ for $k \geq 1$, and $V^{2k+1}_{i,j} = v^1_{i,j}(V^{2k})$ for all $k \geq 0$. So, if $V^{2k+2} = V^{2k}$, then $V^{2k+4} = V^{2k+2}$. If R is any partial matrix in L with $N(R) = N(M)$, we have $p_{i,j}(R) \leq R_{i,j}$ and $v^1_{i,j}(R) \leq R_{i,j}$ for all $1 \leq i \leq q$, $1 \leq j \leq m_i$, so $V^{2k+2} \leq V^{2k+1} \leq V^{2k}$. As before, $V^{2N+2} = V^{2N}$, where $N = (n + 1)q2^n$. So, the entries on each diagonal of V^{2N} which are not in the first column are equal, and the entries of each row of V^{2N} are pairwise commuting. Set $v_{i,j}(\vec{a}) = v^{2N}_{i,j}(\vec{a})$ for each $1 \leq i \leq q$, $1 \leq j \leq m_i$.

Lemma 3.5 *There exists $s(\vec{a}) \in T$ such that for any partial matrix M in any $L \in \mathcal{K}$, with $N(M) = < m_1, \dots, m_q >$, if the entries of each row of M are pairwise commuting and the entries of each diagonal of M which are not in the first column are equal, then*

$$s(M) = \begin{cases} 1 & \textit{if } M \textit{ is admissible} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Using the polynomial $p(\vec{a}, b)$ from lemma 3.2, define:

$$f(\vec{a}) = \bigwedge_{1 \le i \le q} \bigwedge_{1 \le j < k \le m_i} p(\vec{a}, ((a_{i,j} \vee a_{i,k}) \wedge (a'_{i,j} \vee a'_{i,k})))$$

$$g(\vec{a}) = \bigwedge_{2 \le i \le q} p(\vec{a}, (a_{1,i} \wedge ((a_{1,i} \wedge a_{i,1}) \vee (a_{1,i} \wedge a'_{i,1}))'))$$

$$s(\vec{a}) = f(\vec{a}) \wedge g(\vec{a}).$$

Take a partial matrix M in some $L \in \mathcal{K}$, with the entries of each row of M pairwise commuting and the entries of each diagonal of M which are not in the first column equal and $N(M) = < m_1, \dots, m_q >$. If M is not admissible then at least one of the following must be true; two entries in the same row are equal, an entry in the first column of some diagonal commutes with the entry in the first row of that diagonal, the entries of some row do not form a block. But, if M does not satisfy the first two conditions and satisfies the third, we can produce an admissible partial matrix in L of greater size than M, an impossibility. If two entries in some row of M are equal then by lemma 3.2 i), $f(M) = 0$. If the entry in the first column of some diagonal commutes with the entry in the first row of that diagonal then by lemma 3.2 i) $g(M) = 0$.

Conversely, if M is admissible, then the entries in each row are pairwise distinct. So $(M_{i,j} \vee M_{i,k}) \wedge (M'_{i,j} \vee M'_{i,k}) \neq 0$ for all $1 \le i \le q$, $1 \le j < k \le m_i$. Also, the entry in the first column of a diagonal does not commute with the entry in the first row of that diagonal, so $M_{1,i} \wedge [(M_{1,i} \wedge M_{i,1}) \vee (M_{1,i} \wedge M'_{i,1})]' \neq 0$ for each $2 \le i \le q$. Then, by lemma 3.2 ii), $f(M) = g(M) = 1$.

Lemma 3.6 *There exists $t(\vec{a}) \in T$ such that for any partial matrix M in any $L \in \mathcal{K}$, with $N(M) = < m_1, \ldots, m_q >$*

$$t(M) = \begin{cases} 1 & \textit{if } M \textit{ is admissible} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Using the polynomials $p(\vec{a}, b)$ from lemma 3.2, $v_{i,j}(\vec{a})$ from lemma 3.4, and $s(\vec{a})$ from lemma 3.5, define

$$\begin{aligned} t(\vec{a}) \quad = \quad & \bigwedge_{1\le i\le q} \bigwedge_{1\le j\le m_i} [p(v_{1,1}(\vec{a}), \ldots, v_{q,m_q}(\vec{a}), ((a_{i,j} \vee v_{i,j}(\vec{a})) \wedge (a'_{i,j} \vee v_{i,j}(\vec{a})')))]' \\ & \wedge s(v_{1,1}(\vec{a}), \ldots, v_{q,m_q}(\vec{a})). \end{aligned}$$

Define V from the $v_{i,j}(M)$ as in lemma 3.5. If M is admissible, $M = V$ by lemma 3.4 ii), giving $v_{i,j}(M) = M_{i,j}$ for each $1 \le i \le q$, $1 \le j \le m_i$. Then

$$t(M) = \bigwedge_{1\le i\le q} \bigwedge_{1\le j\le m_i} p(M, 0)' \wedge s(M),$$

which by lemma 3.2 i) and lemma 3.5, gives $t(M) = 1$.

If M is not admissible, then either V is not admissible, or V is admissible and $M \neq V$. In the first case, lemma 3.5 gives $s(V) = 0$, and in the second case, we have by lemma 3.2 i) that $p(V, ((M_{i,j} \vee V_{i,j}) \wedge ((M'_{i,j} \vee V'_{i,j})))' = 0$ for some $1 \le i \le q$, $1 \le j \le m_i$. So $t(M) = 0$.

Theorem 3.7 *If $\mathcal{M}$ is a set of OMLs each of height at most n, then for any OML L in the variety generated by $\mathcal{M}$, L is directly irreducible if and only if it is simple.*

Proof. Take $\mathcal{M}$ a set of OMLs each of height at most n, and assume that L is directly irreducible and in the variety generated by $\mathcal{M}$. By Birkhoff's theorem, L is isomorphic to an OML L' which is a subdirect product of a family $(L_x)_{x\in X}$ of subdirectly irreducibles in the variety generated by $\mathcal{M}$. Let $\mathcal{K} = \{L_x : x \in X\}$, then by (2.10) we have that $\mathcal{K}$ is a set of subdirectly irreducible OMLs each having height at most n, and we may apply the results of this section.

For $A = \{N(R) : R$ is an admissible partial matrix in L_x, for some $x \in X\}$, lemma 3.1 gives that A has a maximum in $< I\!N^+, \leq_L>$, say $< m_1, \ldots, m_q >$. For R a partial matrix in L', we define for each $x \in X$ a partial matrix $R(x)$ in L_x of the same size as R, by setting $R(x)_{i,j} = R_{i,j}(x)$ (this is simply the x^{th} projection of R). As L' is a subdirect product of the family $(L_x)_{x \in X}$, there exists a partial matrix M in L', with $N(M) = < m_1, \ldots, m_q >$, such that $M(y)$ is admissible in L_y for some $y \in X$ since the maximum of A will be attained in some $L_y \in \mathcal{K}$.

By lemma 3.6, $t(M(x)) \in \{0,1\}$ for all $x \in X$, and $t(M(y)) = 1$. But, $t(M)(x) = t(M(x))$ for all $x \in X$, so $t(M)$ is in the centre of L'. We assumed that L' was irreducible, so its centre is just $\{0,1\}$, but $t(M(y)) = 1$, so $t(M) = 1$. Again by lemma 3.6, we have that $M(x)$ is admissible in L_x for all $x \in X$, so by lemma 3.2 $p(M, z) \in \{0,1\}$ for all $z \in L'$.

Assume that L' has a chain with $n+2$ elements, say $f_1, \ldots, f_{n+2}$. Chose $y \in X$, then for some $1 \leq i < j \leq n+2$ we have $f_i(y) = f_j(y)$. Setting $g = (f_i \vee f_j) \wedge (f_i' \vee f_j')$, we have $p(M, g)(y) = 0$, so $p(M, g)(x) = p(M(x), g(x)) = 0$ for all $x \in X$. But $M(x)$ is admissible in L_x for each $x \in X$, so by lemma 3.2 ii) $g = 0$, giving $f_i = f_j$ a contradiction.

Then as L' is of height at most n and directly irreducible, it follows from (2.6) that L' is simple.

Before we put the technical aspects of these polynomials to rest, we must prove a Lemma which will be of importance in the next section.

Lemma 3.8 *For L an OML in the variety $\mathcal{V}$, there exists an orthogonal subset A of the centre of L and for each $z \in A$ a polynomial $p_z(\vec{a_z}, x)$ such that*

i) $\bigvee A = 1$

ii) For each $z \in A$, $a \in L$, $p_z(\vec{a_z}, a \wedge z)$ is the least central element of L which dominates $a \wedge z$

iii) For any OML M, and any vector $\vec{m}$ of appropriate length in M, if x is central in M then $p_z(\vec{m}, x) = x$

Proof. As $L \in \mathcal{V}$, L is a subdirect product of a family $(L_i)_{i\in I}$ of OMLs, each of which is subdirectly irreducible and has height at most n. Then for any $0 \neq x \in L$ there exists a partial matrix R in L such that

$$N(R) = max\{N(M):\ M \text{ is admissible in } L_i \text{ for some } i \in I \text{ with } x(i) \neq 0\}$$

and for some $j \in I$ with $x(j) \neq 0$, $R[j]$ is admissible in L_j.

Define $\mathcal{C}$ to be the collection of all orthogonal subsets B of the centre of L such that for each $b \in B$ there exists a partial matrix R_b in L with $R_b[j]$ admissible for each $j \in b^{-1}[1]$ and

$$N(R_b) = max\{N(M):\ M \text{ is admissible in } L_i \text{ for some } i \in I \text{ with } b(i) \neq 0\}.$$

As $\mathcal{C}$ is inductive, it has a maximal member, say A. If u is an upper bound of A in L, we claim that $u = 1$. Assume that $u \neq 1$, then there exists a partial matrix $R_{u'}$ in L such that

$$N(R_{u'}) = \max\{N(M) : M \text{ is admissible in } L_i \text{ for some } i \in I \text{ with } u'(i) \neq 0\}$$

and for some $j \in I$ with $u'(j) \neq 0$, $R_{u'}[j]$ is admissible in L_j.

Using the terms t from lemma 3.6 (for $\mathcal{K} = \{L_i : i \in u'^{-1}[1]\}$) and p from lemma 3.2 set $v = p(R_{u'}, u') \wedge t(R_{u'})$. Then

$$v(i) = \begin{cases} 1 & \text{if } R_{u'}[i] \text{ is admissible in } L_i \text{ and } u'(i) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

As, $u'(j) \neq 0$ and $R_{u'}[j]$ is admissible in L_j, $v(j) = 1$. But v' is an upper bound of A, since $z \in A$ and $z(i) = 1$ imply $u'(i) = 0$ and $v'(i) = 1$. However, we now have that $A \cup \{v\} \in \mathcal{C}$. This contradiction implies that $u = 1$, and so $\bigvee A = 1$.

For each $z \in A$, set $p_z(\vec{a_z}, x)$ to be $p(R_z, x)$, where p is the polynomial from lemma 3.2. As $R_z[i]$ is admissible in L_i for each $i \in z^{-1}[1]$, it follows immediately that A satisfies all necessary conditions.

To conclude this section we will show that the assumption *each OML in $\mathcal{M}$ has height at most n* cannot be weakened to *each OML in $\mathcal{M}$ is of finite height*. Take a non-principal ultrafilter over the natural numbers, and use this to form an ultraproduct of the n-dimensional real projective geometries, where n ranges over the natural numbers. This ultraproduct is an atomic modular ortholattice, and the subalgebra of this consisting of the elements of finite height and their complements is subdirectly irreducible but not simple. In fact, its congruence lattice is a three element chain.

For an example which is directly irreducible but not subdirectly irreducible, consider F, the OML in the variety generated by the finite OMLs which is freely generated by the countably infinite set $\{x_1, x_2, \ldots\}$. If $p(x_1, \ldots, x_n) \notin \{0_F, 1_F\}$ then there are finite OMLs L, M and $l_1, \ldots, l_n \in L$, $m_1, \ldots, m_n \in M$ so that $p(l_1, \ldots, l_n) \neq 0$ and $p(m_1, \ldots, m_n) \neq 1$. In the horizontal sum of 2^2 and $L \times M$, $p((l_1, m_1), \ldots, (l_n, m_n))$ is not central, for convenience assume it does not commute with q. The map which sends x_i to (l_i, m_i) for $1 \leq i \leq n$ and x_i to q for $i > n$ extends to a homomorphism, showing that $p(x_1, \ldots, x_n)$ does not commute with x_{n+1} and therefore F is directly irreducible.

For each natural number n, define a map f_n from $\{x_1, x_2, \ldots\}$ to F by setting $f_n(x_i) = x_i$ if $i \leq n$ and $f_n(x_i) = 0$ otherwise. The map f_n extends to a homomorphism from F into F, which is the identity on the subalgebra of F generated by $\{x_1, \ldots x_n\}$. Therefore F is not subdirectly irreducible.

4 The MacNeille completion of a sheaf of OMLs

The aim of this section is to demonstrate that a variety generated by a set of OMLs, each having height at most n, is closed under MacNeille completions. The main tool will be the Pierce sheaf representation of an OML. In the remainder of this section L is an OML, B is the Boolean algebra of factor congruences of L and $\tilde{S}$ is the Pierce sheaf of L.

Definition 4.1 *Set $\Gamma_D\tilde{S}$ to be the set of all functions $f : E \longrightarrow S$ where E is a dense open subset of B^*, f is continuous with respect to the subspace topology on E, and $f(I) \in L/\bigcup I$ for each $I \in E$. To emphasize the point, the functions in $\Gamma_D\tilde{S}$ do not all have the same domain, their domains range over all dense open subsets of B^*.*

Proposition 4.2 *With operations defined componentwise on the common domain of the arguments, $\Gamma_D\tilde{S}$ is an algebra of the type of ortholattices and $\Gamma\tilde{S}$ is a subalgebra of $\Gamma_D\tilde{S}$.*

Proof. For $f_1, \ldots, f_n \in \Gamma_D\tilde{S}$, using the notation Df_i for the domain of f_i, we have $\bigcap_{i=1}^n Df_i = E$ is open and dense in B^* since the intersection of a finite number of dense open sets is dense open. If t is an n-ary operation symbol in the type of ortholattices, by definition $t(f_1, \ldots, f_n)(I) \in L/\bigcup I$ for each $I \in E$, so we need only show that $t(f_1, \ldots, f_n) : E \longrightarrow S$ is continuous with respect to the subspace topology of E. Say $I \in E$ and $f_i(I) = a_i/\bigcup I$ for each $1 \le i \le n$. If V is an open neighbourhood of $t(a_1, \ldots, a_n)/\bigcup I$ then as $\{\mathcal{O}(a, \theta) : a \in L, \theta \in B\}$ is a basis for the topology of S, there exists $\phi \in B$ with $t(a_1, \ldots, a_n)/\bigcup I \in \mathcal{O}(t(a_1, \ldots, a_n), \phi) \subseteq V$. But $f_i : Df_i \longrightarrow S$ is continuous for each $1 \le i \le n$ so there exists A_i open in Df_i with $I \in A_i$ and $f_i[A_i] \subseteq \mathcal{O}(a_i, \phi)$ for each $1 \le i \le n$. Then for $A = \bigcap_{i=1}^n$, A is open in E, $I \in A$ and $t(f_i, \ldots, f_n)[A] \subseteq \mathcal{O}(t(a_1, \ldots, a_n), \phi) \subseteq V$.

Note that the algebra $\Gamma_D\tilde{S}$ is not necessarily an ortholattice. If E is a proper dense open subset of B^* then for $a \in L$, the restriction of $\tilde{a}$ to E, denoted by $\tilde{a}_{|E}$, is an element of $\Gamma_D\tilde{S}$ but $\tilde{a}_{|E} \vee (\tilde{a}_{|E})' = \tilde{1}_{|E}$, however the nullary operation 1 of $\Gamma_D\tilde{S}$ has domain $B^* \neq E$. So $\tilde{a}_{|E} \vee (\tilde{a}_{|E})' \neq 1$.

Definition 4.3 *The Pierce sheaf $\tilde{S}$ of an OML L will be called weakly Hausdorff if any two global sections of $\tilde{S}$ that agree on a dense open subset are equal.*

Proposition 4.4 *For $\tilde{S}$ the Pierce sheaf of L, the following are equivalent*

i) $\tilde{S}$ is weakly Hausdorff,

ii) if $\mathcal{F}$ is a filter over B and the meet of $\mathcal{F}$ in B exists and is equal to Δ then $\bigcap \mathcal{F} = \Delta$,

iii) all existing meets in B agree with those in the congruence lattice of L.

To see that the first condition implies the second, suppose $\mathcal{F}$ is a filter over B and $\bigwedge \mathcal{F} = \Delta$ (this meet is taken in B). Then for $E = \bigcup\{\theta^* : \theta \in \mathcal{F}\}$, E is dense open. If $(a, b) \in \bigcap \mathcal{F}$ then $\tilde{a}_{|E} = \tilde{b}_{|E}$ so $a = b$.

To see that the second condition implies the first, if E is a dense open subset of B^* then for $\mathcal{F} = \{\theta \in B : \theta^* \subseteq E\}$, $\mathcal{F}$ is a filter over B and $\bigwedge \mathcal{F} = \Delta$ (meet taken in B), so $\bigcap \mathcal{F} = \Delta$. For $a, b \in L$ if $\tilde{a}_{|E} = \tilde{b}_{|E}$ then by 2.19, $(a, b) \in \theta$ for all $\theta \in \mathcal{F}$ so $a = b$.

Obviously the third condition implies the second. To see that the second condition implies the third, take $T \subseteq B$. If the meet in B of T exists and is θ then for ϕ the complement of θ in B and $\mathcal{F}$ the filter in B generated by $T \cup \{\phi\}$, $\bigwedge \mathcal{F} = \Delta$ (meet taken in B). So $\Delta = \bigcap \mathcal{F} = \bigcap T \cap \phi$, but as $\theta \circ \phi = L^2$ and $\theta \subseteq \bigcap T$, we have $(\bigcap T) \circ \phi = L^2$ so $\bigcap T \in B$ so $\bigwedge T = \bigcap T$.

Proposition 4.5 *Defining a relation Θ on $\Gamma_D\tilde{S}$ by setting $f\Theta g$ if f and g agree on a dense open subset of B^*, then if $\tilde{S}$ is weakly Hausdorff*

i) *Θ is a congruence on $\Gamma_D\tilde{S}$,*

ii) *the map $a \rightsquigarrow \tilde{a}/\Theta$ is an embedding of L into $\Gamma_D\tilde{S}/\Theta$,*

iii) *$\Gamma_D\tilde{S}/\Theta$ satisfies exactly the same identities as L,*

iv) *the map $a \rightsquigarrow \tilde{a}/\Theta$ is a join dense embedding.*

Proof. As the intersection of a finite number of dense open sets is dense open, it follows that Θ is an equivalence relation, and as operations in $\Gamma_D\tilde{S}$ are defined componentwise on the common domain of the arguments, Θ is a congruence.

As the map $a \rightsquigarrow \tilde{a}/\Theta$ is the composition of the homomorphisms $a \rightsquigarrow \tilde{a}$ and the natural quotient homomorphism, $a \rightsquigarrow \tilde{a}/\Theta$ is a homomorphism. That this map is an embedding is given by the definition of weakly Hausdorff.

As L can be embedded into $\Gamma_D\tilde{S}/\Theta$, all identities valid in $\Gamma_D\tilde{S}/\Theta$ are valid in L. For an identity $t(x_1,\ldots,x_n) \approx q(y_1,\ldots,y_m)$ valid in L, $t(x_1,\ldots,x_n) \approx q(y_1,\ldots,y_m)$ is valid in each stalk of $\tilde{S}$ since each stalk is a homomorphic image of L. Then for $f_1,\ldots,f_n,g_1,\ldots,g_m \in \Gamma_D\tilde{S}$ and E the common domain of these functions, E is dense open and $t(f_1,\ldots,f_n)$ agrees with $q(g_1,\ldots,g_m)$ on E. So $\Gamma_D\tilde{S}$ also satisfies $t(x_1,\ldots,x_n) \approx q(y_1,\ldots,y_m)$.

To show that the map $a \rightsquigarrow \tilde{a}/\Theta$ is join dense, we may make use of 2.8 as we know that $\Gamma_D\tilde{S}/\Theta$ is an OML. Suppose $f \in \Gamma_D\tilde{S}$ and that $f/\Theta \neq 0/\Theta$, then for some $I \in Df$ and $a \in L$, $f(I) = a/\bigcup \neq 0/\bigcup I$. Then as Df is open, f is continuous and B^* is zero dimensional, there exists a set K clopen in B^* with $I \in K \subseteq Df$ and $f[K] \subseteq \mathcal{O}(a,\Delta)$. Setting $g = f_{|K} \cup 0_{|B^*-K}$, we have $g \in \Gamma\tilde{S}$ and $g/\Theta \leq f/\Theta$. As $L \cong \Gamma\tilde{S}$, $g = \tilde{b}$ for some $b \in L$ and as $g(I) \neq 0/\bigcup I$, $b \neq 0$, but $\tilde{b}/\Theta \leq f/\Theta$ showing that $a \rightsquigarrow \tilde{a}/\Theta$ is a join dense map.

This construction is reminiscent of a reduced product construction. It is natural to ask what first order sentences are preserved by this construction, unfortunately I do not know the answer to this question.

Proposition 4.6 *If $\tilde{S}$ is weakly Hausdorff and there is a dense open subset of B^* where the stalks each have height at most n, then $\Gamma_D\tilde{S}/\Theta$ is complete and therefore the MacNeille completion of L.*

Proof. From the above discussion, if $\Gamma_D\tilde{S}/\Theta$ is complete, then it is the MacNeille completion of L. If it is shown that for any non-empty subset A of $\Gamma_D\tilde{S}$ that $\{f/\Theta : f \in A\}$ has a least upper bound in $\Gamma_D\tilde{S}/\Theta$ then it follows that $\Gamma_D\tilde{S}/\Theta$ is complete. We assume that G is a dense open subset of B^* where the stalks each have height at most n.

Given $\emptyset \neq A \subseteq \Gamma_D\tilde{S}$, put $C = G \cap \bigcup\{Df : f \in A\}$. Then C is dense open and as each stalk of G has height at most n, we may define a map $g : C \longrightarrow S$ by setting

$$g(I) = \bigvee\{f(I) : f \in A \text{ and } I \in Df \cap G\}.$$

If g is an element of $\Gamma_D\tilde{S}$ then it follows that g/Θ is the least upper bound of $\{f/\Theta : f \in A\}$. Unfortunately, g may not be continuous, however we will produce a dense open subset E of C with the restriction of g to E continuous. Then $g_{|E}/\Theta$ is the least upper bound of $\{f/\Theta : f \in A\}$.

Specifically, we claim that for any non-empty open set $N \subseteq B^*$ there exists a nonempty open set $M \subseteq N \cap Dg$ and $a \in L$ with $g[M] \subseteq \mathcal{O}(a, \Delta)$. Notice that for such a set M, g is continuous at each point in M by a familiar argument. Setting E to be the interior of the set of all points at which g is continuous, the claim then states that E is dense, which gives $g_{|E} \in \Gamma_D\tilde{S}$.

To verify the claim, suppose that N is a non-empty open set in B^*. Consider a tower of non-empty open sets $N \supseteq M_1 \supseteq M_2 \supseteq \ldots \supseteq M_p$ satisfying conditions i) and ii)

i) for each $1 \leq i \leq p$ there exists $f_i \in A$ and $a_i \in L$ with $M_i \subseteq Df_i \cap Dg$ and $f_i[M_i] \subseteq \mathcal{O}(a_i, \Delta)$,

ii) for each $I \in M_p$, $f_1(I), f_1(I) \vee f_2(I), \ldots, \bigvee_{i=1}^{p} f_i(I)$ is a strictly increasing chain in $L/\bigcup I$.

As each stalk of G has height at most n, any such tower can be of length at most $n+1$, it is possible then to choose such a tower of maximal length, say $N \supseteq M_1 \supseteq \ldots \supseteq M_p$. Note that as A is non-empty and G is dense open, $p \geq 1$. We will show that $g[M_p] \subseteq \mathcal{O}(\bigvee_{i=1}^{p} a_i, \Delta)$ finishing the proof of the proposition.

Arguing by contradiction, if $g[M_p] \not\subseteq \mathcal{O}(\bigvee_{i=1}^{p} a_i, \Delta)$ then for some $f_{p+1} \in A$ and some $J \in M_p$, $\bigvee_{i=1}^{p} f_i(J) \neq \bigvee_{i=1}^{p+1} f_i(J)$. As $f_{p+1} \in \Gamma_D\tilde{S}$ and B^* is zero dimensional, there exists a clopen set K and $a_{p+1} \in L$ with $J \in K \subseteq M_p$ and $f_{p+1}[K] \subseteq \mathcal{O}(a_{p+1}, \Delta)$. For convenience, let $c = \bigvee_{i=1}^{p+1} a_i$ and $d = \bigvee_{i=1}^{p} a_i$. Setting $k = \tilde{c}_{|K} \cup \tilde{d}_{|B^*-K}$, we have $k, d \in \Gamma\tilde{S}$ then as $k \neq \tilde{d}$ and $\tilde{S}$ is weakly Hausdorff, $k/\Theta \neq \tilde{d}/\Theta$. Since $[\![k = \tilde{d}]\!] = k^{-1}[\mathcal{O}(d, \Delta)]$ is open, it cannot be dense, otherwise $k/\Theta = \tilde{d}/\Theta$. So there exists an open set M_{p+1} with $\emptyset \neq M_{p+1} \subseteq [\![k \neq \tilde{d}]\!] \subseteq K \subseteq M_p$, then $N \supseteq M_1 \supseteq \ldots \supseteq M_p \supseteq M_{p+1}$ is a tower of non-empty open sets satisfying conditions i) and ii), contradicting the maximality of $N \supseteq M_1 \supseteq \ldots \supseteq M_p$.

Theorem 4.7 *If $\mathcal{V}$ is a variety generated by a set of OMLs, each of which has height at most n, then for $L \in \mathcal{V}$ with Pierce sheaf $\tilde{S}$, $\Gamma_D\tilde{S}/\Theta$ is the MacNeille completion of L and $\Gamma_D\tilde{S}/\Theta \in \mathcal{V}$.*

Proof. In view of the above proposition, it is enough to show that $\tilde{S}$ is weakly Hausdorff and that $\{I \in B^* : L/\bigcup I$ has height at most $n\}$ contains a dense open set. However, by 2.10, the subdirectly irreducibles in $\mathcal{V}$ each have height at most n, and by Theorem 3.7, the directly irreducibles in $\mathcal{V}$ are simple (and therefore subdirectly irreducible), so it is enough to show that $\tilde{S}$ is weakly Hausdorff and that $\{I \in B^* : L/\bigcup I$ is directly irreducible $\}$ contains a dense open set.

By Lemma 3.8, there is a subset A of the centre of L and for each $z \in A$ a polynomial $p_z(\vec{a}_z, x)$ of L so that

i) $\bigvee A = 1$,

ii) for each $z \in A$, $a \in L$, $p_z(\vec{a}_z, a \wedge z)$ is the least central element of L which dominates $a \wedge z$,

iii) for any OML M and any vector $\vec{m}$ in M of appropriate length, if x is central in M then $p_z(\vec{m}, x) = x$ for each $z \in \mathcal{Z}$.

Recall 2.4 that $\mathcal{C}(L)$, the centre of L, is isomorphic to the Boolean algebra B of factor congruences of L by the map $c \rightsquigarrow \theta(c)$ where

$$\theta(c) = \{(a, b) \in L^2 : a \wedge c' = b \wedge c'\} = \{(a, b) \in L^2 : (a \vee b) \wedge (a' \vee b') \leq c\}.$$

To show that $\tilde{S}$ is weakly Hausdorff, by Proposition 4.4 it is enough to show that if $\mathcal{F}$ is a filter over B and the meet of $\mathcal{F}$ in B is Δ, then $\bigcap \mathcal{F} = \Delta$. Suppose $\mathcal{F}$ is a filter over B and the meet of $\mathcal{F}$ in B is Δ, setting $T = \{c \in \mathcal{C}(L) : \theta(c) \in \mathcal{F}\}$ we have the meet of T in $\mathcal{C}(L)$ is 0. If $(a, b) \in \bigcap \mathcal{F}$, then for $y = (a \vee b) \wedge (a' \vee b')$, $y \leq c$ for each $c \in T$. But for each $z \in A$, $p_z(\vec{a}_z, y \wedge z)$ is the least central element of L dominating $y \wedge z$, so $p_z(\vec{a}_z, y \wedge z) \leq c$ for each $c \in T$ giving $p_z(\vec{a}_z, y \wedge z) = 0$ and therefore $y \wedge z = 0$ for each $z \in A$. Then

$$y = y \wedge 1 = y \wedge \bigvee A = \bigvee_{z \in A} (y \wedge z) = 0$$

(we may distribute as $\{y\} \cup A$ is contained in a block of L), so by the orthomodular law 2.1, $a = b$. Therefore $\bigcap \mathcal{F} = \Delta$, so $\tilde{S}$ is weakly Hausdorff.

Setting $D = \{\theta(z') : z \in A\}$, as the join of A in L is 1, the join of A in $\mathcal{C}(L)$ is 1, so the meet of D in B is Δ. So $E = \{\theta(z')^* : z \in A\}$ is a dense open subset of B^*. We claim that $L/\bigcup I$ is directly irreducible for each $I \in E$, which will conclude the proof of the Theorem. For $I \in E$, by 2.5 it is enough to show that $\mathcal{C}(L/\bigcup I) = \{0/\bigcup I, 1/\bigcup I\}$. For $y \in L$, $z \in A$, $p_z(\vec{a}_z, y \wedge z) \in \mathcal{C}(L)$ so either $\theta(p_z(\vec{a}_z, y \wedge z)) \in I$ or $\theta(p_z(\vec{a}_z, y \wedge z)') \in I$. In the first case $p_z(\vec{a}_z, y \wedge z)/\bigcup I = 0/\bigcup I$ and in the second case $p_z(\vec{a}_z, y \wedge z)/\bigcup I = 1/\bigcup I$. Suppose $y/\bigcup I \in \mathcal{C}(L/\bigcup I)$. As $I \in E$, for some $w \in A$, $\theta(w') \in I$ so $w/\bigcup I = 1/\bigcup I$. Using this fact and property iii) above in conjunction with the assumption $y/\bigcup I \in \mathcal{C}(L/\bigcup I)$ we have

$$p_w(\vec{a}_w, y \wedge w)/\bigcup I = p_w(\vec{a}_w/\bigcup I, (y \wedge w)/\bigcup I) = p_w(\vec{a}_w/\bigcup I, y/\bigcup I) = y/\bigcup I.$$

But we have seen above that $p_w(\vec{a}_w, y \wedge w)/\bigcup I \in \{0/\bigcup I, 1/\bigcup I\}$ giving that $L/\bigcup I$ is directly irreducible.

5 Kalmbach's construction and completions

We will make use of the Kalmbach construction described in section 2.5 to give a method of constructing OMLs whose MacNeille completions are not orthomodular. In particular, we show that the results of the previous section are reasonably sharp, and our assumption that *the variety is generated by a set of OMLs each of height at most n* can not be weakened to *the variety is generated by a set of finite OMLs.* A simple method to complete the examples is also given.

Proposition 5.1 *For L a bounded lattice, the MacNeille completion of $\mathcal{K}(L)$ is an OML if and only if the condition* (†) *holds in L.*

(†) If $(C_i)_I, (D_j)_J$ are two families of closed intervals in L such that $\emptyset \neq \bigcap C_i \subset \bigcap D_j$, then there exists $x, y \in \bigcap D_j$ such that $x < y$ and either x is an upper bound of $\bigcap C_i$ or y is a lower bound of $\bigcap C_i$.

Proof. Assume that L is a bounded lattice and the MacNeille completion of $\mathcal{K}(L)$ is an OML. Take two families of non-degenerate closed intervals in L, say $([a_i, b_i])_I$ and $([c_j, d_j])_J$, and set $X = \bigcap\{[a_i, b_i] : i \in I\}$, $Y = \bigcap\{[c_j, d_j] : j \in J\}$. Assume that $\emptyset \neq X \subset Y$. For each $i \in I$ let $A_i = [\leftarrow, \{a_i, b_i\}]_{\mathcal{K}(L)}$ and for each $j \in J$ let $B_j = [\leftarrow, \{c_j, d_j\}]_{\mathcal{K}(L)}$. Set $A = \bigcap\{A_i : i \in I\}$, and $B = \bigcap\{B_j : j \in J\}$, then $A = \{x \in \mathcal{K}(L) : x \subseteq X\}$ and $B = \{y \in \mathcal{K}(L) : y \subseteq Y\}$. As A_i is a principal ideal of $\mathcal{K}(L)$ for each $i \in I$, A is a normal ideal of $\mathcal{K}(L)$, as is B.

But $\emptyset \neq X \subset Y$, so there exist $f, g \in L$ such that $f \in X$ and $g \in (Y - X)$. Then $\{f \wedge g, f \vee g\} \in B - A$, but we assumed the MacNeille completion of $\mathcal{K}(L)$ was on OML, so by (2.2) $B \cap A^{\perp} \neq \{0\}$. Take $z \in B \cap A^{\perp}$, such that $l(z) \neq 0$. Then by (2.7), $z^{\perp}$ is an upper bound of A. But X is a convex sublattice of L, so one of $\{0, z_1\}$, $\{z_{2l(z)}, 1\}$, or $\{z_{2k}, z_{2k+1}\}$ for some $1 \leq k < l(z)$, is an upper bound of A. As $z \in B$, $z \subseteq Y$, so if $\{0, z_1\}$ is an upper bound of A, then z_1, z_2 respectively serve the roles of x, y in (†). If $\{z_{2l(z)}, 1\}$ is an upper bound of A, then $z_{2l(z)-1}, z_{2l(z)}$ serve the roles of x, y; and if $\{z_{2k}, z_{2k+1}\}$ is an upper bound of A, then z_{2k+1}, z_{2k+2} serve the roles of x, y. So, if the MacNeille completion of $\mathcal{K}(L)$ is an OML then L satisfies (†).

For the converse, suppose that (†) is satisfied in L and that A, B are normal ideals of $\mathcal{K}(L)$ such that $\emptyset \neq A \subset B$. We must show that $B \cap A^\perp \neq \emptyset$.

Let $U(A), U(B)$ be the set of upper bounds of A and B in $\mathcal{K}(L)$ respectively. Set

$$\mathcal{F} = \{f \in I\!N^{U(A)} : 1 \leq f(x) \leq l(x) \text{ for all } x \in U(A)\}$$

$$\mathcal{G} = \{g \in I\!N^{U(B)} : 1 \leq g(x) \leq l(x) \text{ for all } x \in U(B)\}$$

and for $f \in \mathcal{F}$, $g \in \mathcal{G}$ set $X_f = \bigcap_{x \in U(A)} [x_{2f(x)-1}, x_{2f(x)}]$ and $Y_g = \bigcap_{x \in U(B)} [x_{2g(x)-1}, x_{2g(x)}]$. Then for $x \in \mathcal{K}(L)$, as A is a normal ideal, $x \in A$ if and only if $x \preceq y$ for all $y \in U(A)$ if and only if for each $1 \leq i \leq l(x)$ and each $y \in U(A)$ there exists $1 \leq j \leq l(y)$ such that $[x_{2i-1}, x_{2i}] \subseteq [y_{2j-1}, y_{2j}]$ if and only if for each $1 \leq i \leq l(x)$ there exists $f \in \mathcal{F}$ such that $[x_{2i-1}, x_{2i}] \subseteq X_f$.

As $A \subset B$, there exists $x \in (B - A)$ with $l(x) = 1$ and therefore $g \in \mathcal{G}$ with $[x_1, x_2] \subseteq Y_g$.

Assume that $h_{|U(B)} = g$ implies that $X_h = \emptyset$ for all $h \in \mathcal{F}$. For each $z \in A$ with $l(z) = 1$ we have $[z_1, z_2] \in X_h$ for some $h \in \mathcal{F}$. As $h(y) \neq g(y)$ for some $y \in U(B)$ and $z \preceq \{y_{2h(y)-1}, y_{2h(y)}\}$, $x \preceq \{y_{2g(y)-1}, y_{2g(y)}\}$ we have $x \preceq z^\perp$. So, $x \preceq z^\perp$ for all $z \in A$, and therefore $x \in B \cap A^\perp$.

If there exist f, $h \in \mathcal{F}$ such that $\emptyset \neq X_f \subseteq Y_g$, $\emptyset \neq X_h \subseteq Y_g$ then for some $y \in U(A) - U(B)$ $f(y) \neq h(y)$. So, $Y_g \cap [y_{2f(y)-1}, y_{2f(y)}] \supseteq X_f \neq \emptyset$ and $Y_g \cap [y_{2h(y)-1}, y_{2h(y)}] \supseteq X_h \neq \emptyset$. Assuming that $f(y) < h(y)$, as Y_g is convex, $[y_{2f(y)}, y_{2f(y)+1}] \subseteq Y_g$. Then $\{y_{2f(y)}, y_{2f(y)+1}\} \in B$, and as $\{y_{2f(y)}, y_{2f(y)+1}\} \preceq y^\perp \in A^\perp$ we have finished.

As $\emptyset$ is not an upper bound of A, there exists an $f \in \mathcal{F}$ such that $f_{|U(B)} = g$. By the above discussion we may assume that f is the unique element of $\mathcal{F}$ such that $\emptyset \neq X_f \subseteq Y_g$. But $[x_1, x_2] \subseteq Y_g$ so $X_f \neq Y_g$. Applying (†) we find a, $b \in Y_g$ such that $a < b$ and either a is an upper bound of X_f or b is a lower bound of X_f. In either case, as f is unique such that $\emptyset \neq X_f \subseteq Y_g$, $\{a, b\} \in B \cap A^\perp$.

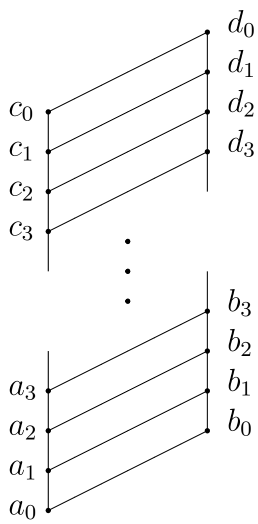
The following result is somewhat surprising in view of the fact that $\mathcal{K}(L)$ is complete if and only if it is finite.

Corollary 5.2 *If L is a bounded lattice, then $\mathcal{K}(L)$ can be embedded into a complete OML.*

Proof. It is easy to see that (†) is satisfied by any complete lattice. If we let $\overline{L}$ denote the MacNeille completion of L, then as L is a sublattice of $\overline{L}$, by (2.18) $\mathcal{K}(L)$ is a sub-OML of $\mathcal{K}(\overline{L})$. But $\overline{L}$ is complete, so $\overline{\mathcal{K}(\overline{L})}$ is an OML and $\mathcal{K}(L)$ is a sub-OML of $\overline{\mathcal{K}(\overline{L})}$.

Corollary 5.3 *There is a bounded lattice L such that the MacNeille completion of $\mathcal{K}(L)$ is not an OML.*

Proof. In the lattice L_0 depicted below, where it is to be understood that $a_n \leq c_m, d_m$ and $b_n \leq d_m$ for all $n, m \in I\!N$, the families $([a_0, c_n])_{n \in I\!N}$ and $([a_0, d_n])_{n \in I\!N}$ violate (†).



L_0

Proposition 5.4 *For a family $(L_i)_{i \in I}$ of bounded lattices and an ultrafilter $\mathcal{U}$ over the set I, $\mathcal{K}(\prod L_i/\mathcal{U})$ can be embedded into $\prod \mathcal{K}(L_i)/\mathcal{U}$.*

Proof. Let $\mathcal{S}$ be the collection of all subsets of $\prod L_i$ which have even cardinality. With π_i being i^{th} projection of $\prod L_i$, define a map $p : \mathcal{S} \longrightarrow \prod \mathcal{K}(L_i)$ as follows

$$p(X)(i) = \begin{cases} \pi_i[X] & \text{if } \pi_i[X] \in \mathcal{K}(M_i) \text{ and } |\pi_i[X]| = |X| \\ \emptyset & \text{otherwise} \end{cases}$$

If $z \in \mathcal{K}(\prod L_i/\mathcal{U})$ and $a_1, \ldots, a_{2l(z)}, b_1, \ldots, b_{2l(z)} \in \prod L_i$ are such that $a_i/\mathcal{U} = b_i/\mathcal{U} = z_i$ for each $1 \leq i \leq 2l(z)$ then, as $\mathcal{U}$ is closed under finite intersections, $p(\{a_i : 1 \leq i \leq 2l(z)\})/\mathcal{U} = p(\{b_i : 1 \leq i \leq 2l(z)\})/\mathcal{U}$. Therefore, we may define a map $\beta : \mathcal{K}(\prod L_i/\mathcal{U}) \longrightarrow \prod \mathcal{K}(L_i)/\mathcal{U}$ by

$$\beta(z) = p(\{x_1, \ldots, x_{2l(z)}\})/\mathcal{U} \text{ if } z_i = x_i/\mathcal{U} \text{ for all } 1 \leq i \leq 2l(z).$$

For x, $y \in \mathcal{K}(\prod L_i/\mathcal{U})$ chose $a_1, \ldots, a_{2l(x)} \in \prod L_i$ such that $a_i/\mathcal{U} = x_i$ for each $1 \leq i \leq 2l(x)$ and chose $b_1, \ldots, b_{2l(y)} \in \prod L_i$ such that $b_j/\mathcal{U} = y_j$ for each $1 \leq j \leq 2l(y)$. Let $\varphi(p_1, \ldots, p_{2l(x)}, q_1, \ldots, q_{2l(y)})$ be the first order formula which says that for each $1 \leq i \leq l(x)$ there exists $1 \leq j \leq l(y)$ such that $q_{2j-1} \leq p_{2i-1} < p_{2i} \leq q_{2j}$. Then using Łoś' theorem (2.9) and the fact that $[\![a_1 < \ldots < a_{2l(x)}]\!]$, $[\![b_1 < \ldots < b_{2l(y)}]\!] \in \mathcal{U}$ we have

$$\begin{aligned} x \preceq y \quad & \text{if and only if} \quad \prod L_i/\mathcal{U} \models \varphi(x_1, \ldots, x_{2l(x)}, y_1, \ldots, y_{2l(y)}) \\ & \text{if and only if} \quad [\![\varphi(a_1, \ldots, a_{2l(x)}, b_1, \ldots b_{2l(y)})]\!] \in \mathcal{U} \\ & \text{if and only if} \quad [\![\varphi(a_1, \ldots, a_{2l(x)}, b_1, \ldots b_{2l(y)})]\!] \cap \\ & \qquad\qquad\qquad\quad [\![a_1 < \ldots < a_{2l(x)}]\!] \cap [\![b_1 < \ldots < b_{2l(y)}]\!] \in \mathcal{U} \\ & \text{if and only if} \quad [\![p(\{a_i : 1 \leq i \leq 2l(x)\}) \preceq p(\{b_j : 1 \leq j \leq 2l(y)\})]\!] \in \mathcal{U} \\ & \text{if and only if} \quad \beta(x) \leq \beta(y). \end{aligned}$$

Therefore β is an order embedding. By similar methods we can easily check that β is compatible with $\perp$.

To put the next result in the proper context, it is shown in [7] that any variety of OMLs which is generated by a single finite OML, is closed under the formation of MacNeille completions. It is not unreasonable to hope that the variety which is generated by all the finite OMLs would also have this property. This would settle a basic question about OMLs. *Is the variety of OMLs generated by its finite members ?* Unfortunately, the following result nullifies this approach.

Corollary 5.5 *There is an OML in the variety generated by the finite OMLs whose MacNeille completion is not an OML.*

Proof. For each natural number n, let M_n be the interval $[a_0, b_n]$ in L_0, and for each $m \geq 0$ define $A_m, B_m, C_m, D_m \in \prod M_n$ by $A_m(n) = a_{min\{m,n\}}$, $B_m(n) = b_{min\{m,n\}}$, $C_m(n) = a_{max\{n-m,0\}}$ and $D_m(n) = b_{max\{n-m,0\}}$. Let $\mathcal{U}$ be a non-principal ultrafilter over the natural numbers and define $\alpha : L_0 \longrightarrow \prod M_n/\mathcal{U}$ by setting $\alpha(a_m) = A_m/\mathcal{U}$, $\alpha(b_m) = B_m/\mathcal{U}$, $\alpha(c_m) = C_m/\mathcal{U}$ and $\alpha(d_m) = D_m/\mathcal{U}$. It is an easy matter to check that α is a 0, 1 lattice embedding. Then $\mathcal{K}(L_0)$, whose MacNeille completion is not an OML, can be embedded into $\mathcal{K}(\prod M_n/\mathcal{U})$ which can be embedded into $\prod \mathcal{K}(M_n)/\mathcal{U}$. As $\mathcal{K}(M_n)$ is finite for each choice of n, the proof is finished.

References

[1] D. H. Adams, *The completion by cuts of an orthocomplemented modular lattice.* Bull. Austral. Math. Soc. 1 (1969), 279-280.

[2] I. Amemiya and H. Araki, *A remark on Piron's paper.* Pub. Res. Inst. Math. Ser., Kyoto Univ., Ser. A2 (1966), 423-427.

[3] I. Amemiya and I. Halperin, *Complemented modular lattices.* Can. J. Math 11 (1959), 481-520.

[4] D. R. Balbes and P. Dwinger, *Distributive lattices,* University of Missouri Press, Columbia, Missouri, 1974.

[5] B. Banaschewski, *Hullensysleme und Erweiterungen von Quasi-Ordnungen,* Z. Math. Logik Grundl. Math. 2 (1956), 35-46.

[6] G. E. Bredon, *Sheaf theory.* McGraw-Hill, New York, New York (1967).

[7] G. Bruns, R. J. Greechie, J. Harding and M. Roddy *Completions of orthomodular lattices.* Order 7 (1990), 67-76.

[8] S. Burris and H.P. Sankappanavar, *A Course in Universal Algebra.* Springer-Verlag, New York, 1981.

[9] R. P. Dilworth *The structure of relatively complemented lattices.* Ann. of Math. 51 (1950), 348-359.

[10] N. Funayama, *On the completion by cuts of a distributive lattice.* Proc. Imp. Acad. Tokyo 20 (1944), 1-2.

[11] W. H. Graves and S. A. Selesnick, *An extension of the Stone representation for orthomodular lattices.* Colloq. Math. 27 (1973), 21-30.

[12] R. Godowski and R. J. Greechie, *Some equations related to the states on orthomodular lattices.* Dem. Math. 17 (1984), 241-250.

[13] J. Harding, *Any lattice can be embedded into the MacNeille completion of a distributive lattice.* manuscript.

[14] M. F. Janowitz, *Indexed orthomodular lattices.* Math. Z. 119 (1971), 28-32.

[15] B. Jónsson, *Algebras whose congruence lattices are distributive.* Math. Scand. 21 (1967), 110-121.

[16] G. Kalmbach, *Orthomodular lattices do not satisfy any special lattice equations.* Arch. Math. (Basel), 28 (1977), 7-8.

[17] ——, *Orthomodular lattices,* Academic Press, London, (1983).

[18] M. D. MacLaren, *Atomic orthocomplemented lattices.* Pac. J. Math. 14 (1964), 597-612.

[19] H. M. MacNeille *Partially ordered sets.* Trans. AMS 42 (1937), 416-460.

[20] R. S. Pierce *Modules over commutative regular rings.* Mem. Amer. Math. Soc. 70 (1967).

[21] J. Schmidt, *Zur Kennzeichnung der Dedekind MacNeilleschen Hulle einer geordneten Menge,* Archiv d. Math. 7 (1956), 241-149.