Irreducible orthomodular lattices which are simple

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It is well known that for a chain finite orthomodular lattice, all congruences are factor congruences, so any directly irreducible chain finite orthomodular lattice is simple. In this paper it is shown that the notions of directly irreducible and simple coincide in any variety generated by a set of orthomodular lattices that has a uniform finite upper bound on the lengths of their chains. The prototypical example of such a variety is any variety generated by a set of n dimensional orthocomplemented projective geometries.

An orthomodular lattice (abbreviated: OML) L is an ortholattice which also satisfies the orthomodular law:

$$\text{for } a,\, b \in L \text{ if } a \leq b \text{ then } b = a \vee (b \wedge a').$$

We will give the few facts about OMLs which are needed for this paper. A standard reference for OMLs is [5]. Let L be an OML

$$\text{for } a,\, b \in L \;\; a = b \text{ if and only if } (a \vee b) \wedge (a' \vee b') = 0.$$

For a, $b \in L$, we say that a commutes with b if

$$(a \vee b) \wedge (a \vee b') = a$$

or equivalently, if

$$(a \wedge b) \vee (a \wedge b') = a.$$

A maximal set of pairwise commuting elements of L is called a block of L, and is a maximal Boolean subalgebra of L. The intersection of the blocks of L is called the centre of L. It will be very useful to know that L is directly irreducible if and only if its centre is $\{0, 1\}$.

It is an easy consequence of a result by Dilworth [3], and is explicitly states for OMLs in [5], that an OML in which every chain is finite is directly irreducible if and only if it is simple. We say that an OML L is of height at most n if every chain in L has at most $n+1$ elements. The main result of this paper is:

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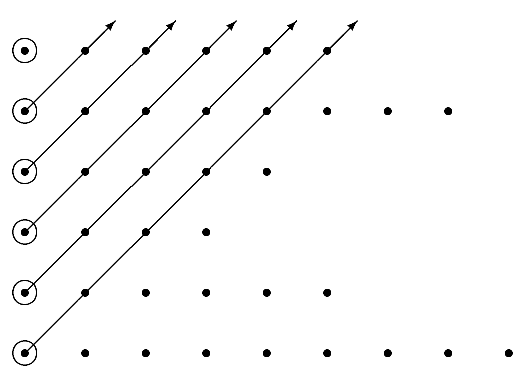
Theorem 1 *If $\mathcal{M}$ is a set of OMLs each of height at most n, then for any OML L in the variety generated by $\mathcal{M}$, L is directly irreducible if and only if it is simple.*

The proof of this theorem is rather technical, and is based on certain notions which first appeared in [1], but no prior knowledge of this paper is required. At first reading, it might be advantageous to skip Lemmas 3, 4 and 5 entirely, as they are required only for the proof of Lemma 6, and are rather messy besides.

For L an OML, we will say that M is a partial matrix in L if M is a rectangular matrix whose entries are elements of L. We do not require that each cell of M has an entry, but we do require a certain normal form. There must be an entry in each row and column of M, and the entries of a row must be an initial segment of that row. We say that a partial matrix M in an OML L is admissible if the following conditions are satisfied. For each row of M, the entries in that row are pairwise distinct and form a block of L. If we consider the Northeastern diagonals of M originating in the first column (these will be referred to simply as diagonals), there is an entry in each cell of the diagonal. Finally, we require that all of the entries on a given diagonal, which are not in the first column, are equal and do not commute with the entry in the first column of that diagonal.

For a partial matrix M, we will refer to the entry in the (i, j) cell of M, if there is one, by $M_{i,j}$. Define $N(M)$, the size of M, to be a sequence of natural numbers $< n_1, \ldots, n_r >$ where r is the number of rows in M and for each $1 \leq i \leq r$, n_i is the number of entries in the i^{th} row of M. If two partial matrices, M and P over the same OML have the same size, we say that $M \leq P$ if each entry of P dominates the corresponding entry of M. Finally, let $< \mathbb{N}^+, \leq_L >$ denote the set of sequences of positive natural numbers with the lexicographical ordering.

The diagram below may help to visualize a partial matrix and its diagonals.



Lemma 1 *For $\mathcal{K}$ a set of OMLs, each of which is directly irreducible and of height at most n, define a set A by $A = \{N(R) : R$ is admissible in some $L \in \mathcal{K}\}$, then*

i) A is a finite set and has a maximum in $< I\!N^+, \leq_L >$, say $< m_1, \ldots, m_q >$.

ii) If M is admissible in some $L \in \mathcal{K}$ and $N(M) = < m_1, \ldots, m_q >$, then for $y \in L$, y commutes with all the entries of M if and only if $y \in \{0, 1\}$.

Proof. i) If R is admissible in some $L \in \mathcal{K}$, as the entries of a row of L are pairwise distinct and form a block of L, each row of R has at most 2^n entries. As there is an entry in each cell of a diagonal of R, the number of rows of R cannot exceed the length of the first row of R. So A is a finite set, and as $< I\!N^+, \leq_L >$ is a chain, A has a maximum in $< I\!N^+, \leq_L >$.

ii) Take $M \in L$ as given. Assume that $y \in L - \{0, 1\}$ and y commutes with all the entries of M. As each row of M forms a block of L, y and 0 appear on each row of L and never on a diagonal of L (except possibly the one element diagonal). As $y \notin \{0, 1\}$ there exists $z \in L$ which does not commute with y, and a block B of L with $z \in B$. Form a new partial matrix M' by adding a row to the bottom of M, the entries being the elements of B, each listed only once, with z listed first. By switching at most two entries per row of M', we may form a new partial matrix M'' which agrees with M' on all the diagonals, except possibly the diagonal originating at z, such that the entries of the diagonals originating at z which are not in the first column are all equal to y. But M'' is admissible, contradicting the maximality of $< m_1, \ldots, m_q >$.

In the following, we will assume that $\mathcal{K}$ and $< m_1, \ldots, m_q >$ are as described in Lemma 1. T denotes the term algebra, of the type of ortholattices, over a countably infinite set S. We will assume that $a_{1,1}, \ldots, a_{q,m_q}$ are elements of S, and denote the vector $< a_{1,1}, \ldots, a_{q,m_q} >$ by $\vec{a}$. For M a partial matrix in an OML L with $N(M) = < m_1, \ldots, m_q >$, there exists a map $\varphi : S \longrightarrow L$ such that $\varphi(a_{i,j}) = M_{i,j}$ for all $1 \leq i \leq q$, $1 \leq j \leq m_i$, and a homomorphism $\overline{\varphi} : T \longrightarrow L$ extending φ. So, for $t(\vec{a}) \in T$ we may define $t(M)$ to be $\overline{\varphi}(\vec{a})$.

Lemma 2 *There exists $p(\vec{a}, b) \in T$ such that for any partial matrix M in any $L \in \mathcal{K}$ with $N(M) = < m_1, \ldots, m_q >$, we have*

i) $p(M, 0) = 0$.

ii) If M is admissible then for $z \in L$, $p(M, z) = 0$ if and only if $z = 0$, and $p(M, z) = 1$ otherwise.

Proof. Define recursively for each $k \geq 0$, $p^k(\vec{a}, b) \in T$ as follows:

$$p^0(\vec{a}, b) = \bigvee_{1 \leq i \leq q} \bigvee_{1 \leq j \leq m_i} [(b \vee a_{i,j}) \wedge (b \vee a'_{i,j})],$$

$$p^{k+1}(\vec{a}, b) = p^0(\vec{a}, p^k(\vec{a}, b)) \text{ for } k \geq 0.$$

For $z \in L$, $p^0(M, z) \geq z$, so $p^{k+1}(M, z) = p^0(M, p^k(M, z)) \geq p^k(M, z)$, therefore $\{p^k(M, z) : k \geq 0\}$ forms a chain in L. If $p^{k+1}(M, z) = p^k(M, z)$ then $p^{k+2}(M, z) = p^{k+1}(M, z)$, so $p^n(M, z) = p^{n+1}(M, z)$ since every chain in L has at most n elements. But, $p^0(M, z) = z$ if and only if z commutes with all of the entries of M, so $p^n(M, z)$ commutes with all the entries of M. In particular, if M is admissible then $p^n(M, z) \in \{0, 1\}$ (by Lemma 1). A simple induction shows that $p^n(M, z) = 0$ if and only if $z = 0$. Set $p(\vec{a}, b) = p^n(\vec{a}, b)$.

Lemma 3 *For each $1 \leq i \leq q$, $1 \leq j \leq m_i$ there exists $p_{i,j}(\vec{a}) \in T$ such that for any partial matrix M, in any $L \in \mathcal{K}$, with $N(M) = < m_1, \ldots, m_q >$, if we define a partial matrix Q in L, with $N(Q) = < m_1, \ldots, m_q >$, by setting $Q_{i,j} = p_{i,j}(M)$ for all $1 \leq i \leq q$, $1 \leq j \leq m_i$, then*

i) The entries of each row of Q are pairwise commuting.

ii) $Q \leq M$, and if the entries of each row of M are pairwise commuting then $Q = M$.

Proof. Define recursively for each $k \geq 0$, $p^k_{i,j}(\vec{a}) \in T$ for each $1 \leq i \leq q$, $1 \leq j \leq m_i$ as follows:

$$p^0_{i,j}(\vec{a}) = \bigwedge_{1 \leq l \leq m_i} [(a_{i,j} \wedge a_{i,l}) \vee (a_{i,j} \wedge a'_{i,l})],$$

$$p^{k+1}_{i,j}(\vec{a}) = p^0_{i,j}(p^k_{1,1}(\vec{a}), \ldots, p^k_{q,m_q}(\vec{a})) \text{ for } k \geq 0.$$

For each $k \geq 0$ define a partial matrix Q^k in L, with $N(Q^k) = N(M)$, by setting $Q^k_{i,j} = p^k_{i,j}(M)$ for each $1 \leq i \leq q$, $1 \leq j \leq m_i$. Note that $Q^{k+1}_{i,j} = p^0_{i,j}(Q^k)$, so if $Q^{k+1} = Q^k$, then $Q^{k+2} = Q^{k+1}$.

For any partial matrix R in L with $N(R) = N(M)$ we have for any $1 \leq i \leq q$, $1 \leq j \leq m_i$ that $p^0_{i,j}(R) \leq R_{i,j}$, and $p^0_{i,j}(R) = R_{i,j}$ if and only if $R_{i,j}$ commutes with all the entries on the i^{th} row of R. By an easy induction we have $M \geq Q^k \geq Q^{k+1}$ for all $k \geq 0$. As there are at most $q2^n$entries in M, and every chain of L has at most $n+1$ elements, $Q^N = Q^{N+1}$, where $N = (n+1)q2^n$. Then, setting $p_{i,j}(\vec{a}) = p^N_{i,j}(\vec{a})$ for each $1 \leq i \leq q$, $1 \leq j \leq m_i$ we are finished.

Lemma 4 *For each $1 \leq i \leq q$, $1 \leq j \leq m_i$ there exists $v_{i,j}(\vec{a}) \in T$ such that for any partial matrix M in any $L \in \mathcal{K}$, with $N(M) = < m_1, \ldots, m_q >$, if we define a partial matrix V in L, with $N(V) = N(M)$, by setting $V_{i,j} = v_{i,j}(M)$ for all $1 \leq i \leq q$, $1 \leq j \leq m_i$ then*

i) The entries of each row of V are pairwise commuting and the entries of each diagonal of V which are not in the first column, are equal.

ii) If M is admissible then $V = M$.

Proof. Define recursively for each $k \geq 1$, $v^k_{i,j}(\vec{a}) \in T$ for each $1 \leq i \leq q$, $1 \leq j \leq m_i$ as follows:

$$v^1_{i,j}(\vec{a}) = \begin{cases} \bigwedge\{a_{l,m} : l+m = i+j, m \neq 1\} & \text{if } 2 \leq j \leq q-i+1 \\ a_{i,j} & \text{otherwise} \end{cases}$$

$$v^{2k}_{i,j}(\vec{a}) = p_{i,j}(v^{2k-1}_{1,1}(\vec{a}), \ldots, v^{2k-1}_{q,m_q}(\vec{a})) \text{ for } k \geq 1 \ (*)$$

$$v^{2k+1}_{i,j}(\vec{a}) = v^1_{i,j}(v^{2k}_{1,1}(\vec{a}), \ldots, v^{2k}_{q,m_q}(\vec{a})) \text{ for } k \geq 1$$

$(*)$ the $p_{i,j}(\vec{a})$ are described in Lemma 3.

For a cell (i,j) on a diagonal and not in the first column, we want $v^1_{i,j}(M)$ to be the meet of all entries of M on that diagonal which are not in the first column, hence the cryptic definition.

For each $k \geq 1$ define a partial matrix V^k in L, with $N(V^k) = N(M)$, by setting $V^k_{i,j} = v^k_{i,j}(M)$ for each $1 \leq i \leq q$, $1 \leq j \leq m_i$. Note that $V^{2k}_{i,j} = p_{i,j}(V^{2k-1})$ for $k \geq 1$, and $V^{2k+1}_{i,j} = v^1_{i,j}(V^{2k})$ for all $k \geq 0$. So, if $V^{2k+2} = V^{2k}$, then $V^{2k+4} = V^{2k+2}$. If R is any partial matrix in L with $N(R) = N(M)$, we have $p_{i,j}(R) \leq R_{i,j}$ and $v^1_{i,j}(R) \leq R_{i,j}$ for all $1 \leq i \leq q$, $1 \leq j \leq m_i$, so $V^{2k+2} \leq V^{2k+1} \leq V^{2k}$. As before, $V^{2N+2} = V^{2N}$, where $N = (n+1)q2^n$. So, the entries on each diagonal of V^{2N} which are not in the first column are equal, and the entries of each row of V^{2N} are pairwise commuting. Set $v_{i,j}(\vec{a}) = v^{2N}_{i,j}(\vec{a})$ for each $1 \leq i \leq q$, $1 \leq j \leq m_i$.

Lemma 5 *There exists $s(\vec{a}) \in T$ such that for any partial matrix M in any $L \in \mathcal{K}$, with $N(M) = < m_1, \ldots, m_q >$, if the entries of each row of M are pairwise commuting and the entries of each diagonal of M which are not in the first column are equal, then*

$$s(M) = \begin{cases} 1 & \textit{if } M \textit{ is admissible} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Using the polynomial $p(\vec{a}, b)$ from Lemma 2, define:

$$f(\vec{a}) = \bigwedge_{1 \leq i \leq q} \; \bigwedge_{1 \leq j < k \leq m_i} p(\vec{a}, ((a_{i,j} \vee a_{i,k}) \wedge (a'_{i,j} \vee a'_{i,k})))$$

$$g(\vec{a}) = \bigwedge_{2 \leq i \leq q} p(\vec{a}, (a_{1,i} \wedge ((a_{1,i} \wedge a_{i,1}) \vee (a_{1,i} \wedge a'_{i,1}))'))$$

$$s(\vec{a}) = f(\vec{a}) \wedge g(\vec{a}).$$

Take a partial matrix M in some $L \in \mathcal{K}$, with the entries of each row of M pairwise commuting and the entries of each diagonal of M which are not in the first column equal and $N(M) = < m_1, \ldots, m_q >$. If M is not admissible then at least one of the following must be true; two entries in the same row are equal, an entry in the first column of some diagonal commutes with the entry in the first row of that diagonal, the entries of some row do not form a block. But, if M does not satisfy the first two conditions and does satisfy the third, we can produce an admissible partial matrix in L of greater size than M, an impossibility. If two entries in some row of M are equal then by Lemma 2 i), $f(M) = 0$. If the entry in the first column of some diagonal commutes with the entry in the first row of that diagonal then by Lemma 2 i), $g(M) = 0$.

Conversely, if M is admissible, then the entries in each row are pairwise distinct. So $(M_{i,j} \vee M_{i,k}) \wedge (M'_{i,j} \vee M'_{i,k}) \neq 0$ for all $1 \leq i \leq q$, $1 \leq j < k \leq m_i$. Also, the entry in the first column of a diagonal does not commute with the entry in the first row of that diagonal, so $M_{1,i} \wedge [(M_{1,i} \wedge M_{i,1}) \vee (M_{1,i} \wedge M'_{i,1})]' \neq 0$ for each $2 \leq i \leq q$. Then, by Lemma 2 ii), $f(M) = g(M) = 1$.

Lemma 6 *There exists $t(\vec{a}) \in T$ such that for any partial matrix M in any $L \in \mathcal{K}$, with $N(M) = < m_1, \ldots, m_q >$*

$$t(M) = \begin{cases} 1 & \textit{if } M \textit{ is admissible} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Using the polynomials $p(\vec{a}, b)$ from Lemma 2, $v_{i,j}(\vec{a})$ from Lemma 4, and $s(\vec{a})$ from Lemma 5, define

$$\begin{aligned} t(\vec{a}) &= \bigwedge_{1 \le i \le q} \bigwedge_{1 \le j \le m_i} [p(v_{1,1}(\vec{a}), \ldots, v_{q,m_q}(\vec{a}), ((a_{i,j} \vee v_{i,j}(\vec{a})) \wedge (a'_{i,j} \vee v_{i,j}(\vec{a})')))]' \\ &\quad \wedge s(v_{1,1}(\vec{a}), \ldots, v_{q,m_q}(\vec{a})). \end{aligned}$$

Define V from the $v_{i,j}(M)$ as in Lemma 4. If M is admissible, $M = V$ by Lemma 4 ii), giving $v_{i,j}(M) = M_{i,j}$ for each $1 \le i \le q$, $1 \le j \le m_i$. Then

$$t(M) = \bigwedge_{1 \le i \le q} \bigwedge_{1 \le j \le m_i} p(M, 0)' \wedge s(M)$$

which by Lemma 2 i) and Lemma 5, gives $t(M) = 1$.

If M is not admissible, then either V is not admissible, or V is admissible and $M \neq V$. In the first case, $s(V) = 0$, and in the second case, we have by Lemma 2 i) that $p(V, ((M_{i,j} \vee V_{i,j}) \wedge ((M'_{i,j} \vee V'_{i,j})))' = 0$ for some $1 \le i \le q$, $1 \le j \le m_i$. So $t(M) = 0$.

Proof (of the main theorem). Take $\mathcal{M}$ a set of OMLs each of height at most n, and assume that L is directly irreducible and in the variety generated by $\mathcal{M}$. By Łŏs' theorem [2, p. 210], an ultraproduct of OMLs in $\mathcal{M}$ has height at most n, so by Jónsson's theorem [4], the subdirectly irreducibles in the variety generated by $\mathcal{M}$ all have height at most n.

As L is in the variety generated by $\mathcal{M}$, by Birkhoff's theorem [2, p. 58], L is isomorphic to an OML L' which is a subdirect product of a family $(L_x)_{x \in X}$ of subdirectly irreducibles in the variety generated by $\mathcal{M}$. Let $\mathcal{K} = \{L_x : x \in X\}$, and $A = \{N(R) : R \text{ is an admissible partial matrix in } L_x, \text{ for some } x \in X\}$. By Lemma 1, A has a maximum in $< I\!N^+, \le_L >$, say $< m_1, \ldots, m_q >$.

For R a partial matrix in L', we define for each $x \in X$ a partial matrix $R(x)$ in L_x of the same size as R, by setting $R(x)_{i,j} = R_{i,j}(x)$ (this is simply the x^{th} projection of R). As L' is a subdirect product of the family $(L_x)_{x \in X}$, there exists a partial matrix M in L', with $N(M) = < m_1, \ldots, m_q >$, such that $M(y)$ is admissible in L_y for some $y \in X$ since the maximum of A will be attained in some $L_y \in \mathcal{K}$.

By Lemma 6, $t(M(x)) \in \{0, 1\}$ for all $x \in X$, and $t(M(y)) = 1$. But, $t(M)(x) = t(M(x))$ for all $x \in X$, so $t(M)$ is in the centre of L'. We assumed that L' was irreducible, so its centre is just $\{0,1\}$, but $t(M(y)) = 1$, so $t(M) = 1$. Again by Lemma 6, we have that $M(x)$ is admissible in L_x for all $x \in X$, so by Lemma 2 $p(M,z) \in \{0,1\}$ for all $z \in L'$.

Assume that L' has a chain with $n+2$ elements, say $f_1, \ldots, f_{n+2}$. Chose $y \in X$, then for some $1 \leq i < j \leq n+2$ we have $f_i(y) = f_j(y)$. Setting $g = (f_i \vee f_j) \wedge (f_i' \vee f_j')$, we have $p(M,g)(y) = 0$, so $p(M,g)(x) = p(M(x), g(x)) = 0$ for all $x \in X$. But $M(x)$ is admissible in L_x for each $x \in X$, so by Lemma 2 ii) $g = 0$, giving that $f_i = f_j$ a contradiction.

Then as L' is of height at most n and directly irreducible, it is simple.

In conclusion, I will show that the assumption *each OML in $\mathcal{M}$ has height at most n* cannot be weakened to *each OML in $\mathcal{M}$ is of finite height*. Take a non-principal ultrafilter over the natural numbers, and use this to form an ultraproduct of the n-dimensional real projective geometries, where n ranges over the natural numbers. This ultraproduct is an atomic modular ortholattice, and the subalgebra of this consisting of the elements of finite height and their complements is subdirectly irreducible but not simple. In fact, its congruence lattice is a three element chain.

For an example which is directly irreducible but not subdirectly irreducible, consider F, the OML in the variety generated by the finite OMLs which is freely generated by the countably infinite set $\{x_1, x_2, \ldots\}$. If $p(x_1, \ldots, x_n) \notin \{0_F, 1_F\}$ then there are finite OMLs L, M and $l_1, \ldots, l_n \in L$, $m_1, \ldots, m_n \in M$ so that $p(l_1, \ldots, l_n) \neq 0$ and $p(m_1, \ldots, m_n) \neq 1$. In the horizontal sum of 2^2 and $L \times M$, $p((l_1, m_1), \ldots, (l_n, m_n))$ is not central, for convenience assume it does not commute with q. The map which sends x_i to (l_i, m_i) for $1 \leq i \leq n$ and x_i to q for $i > n$ extends to a homomorphism, showing that $p(x_1, \ldots, x_n)$ does not commute with x_{n+1} and therefore F is directly irreducible.

For each natural number n, define a map f_n from $\{x_1, x_2, \ldots\}$ to F by setting $f_n(x_i) = x_i$ if $i \leq n$ and $f_n(x_i) = 0$ otherwise. The map f_n extends to a homomorphism from F into F, which is the identity on the subalgebra of F generated by $\{x_1, \ldots x_n\}$. Therefore F is not subdirectly irreducible.

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