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ANY LATTICE CAN BE REGULARLY EMBEDDED INTO THE MACNEILLE COMPLETION OF A DISTRIBUTIVE LATTICE

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The completion by cuts of a totally ordered set was first introduced by Dedekind in his famous construction of the real numbers from the rationals. MacNeille [4] extended the method of completion by cuts to arbitrary partially ordered sets. For a partially ordered set P and $A \subseteq P$, defining

$$U(A) = \{x \in P : x \geq y \text{ for all } y \in A\} \text{ and}$$
$$L(A) = \{x \in P : x \leq y \text{ for all } y \in A\}$$

a cut, or normal ideal, of P is a subset A of P for which $A = LU(A)$. The set of all normal ideals of P partially ordered by set inclusion forms a complete lattice $\bar{P}$ where the supremum $\bigvee$ and infinum $\bigwedge$ of a subset S of $\bar{P}$ are given by

$$\bigvee S = LU\left(\bigcup S\right) \qquad \text{and}$$
$$\bigwedge S = \bigcap S$$

The partially ordered set P can be embedded into its MacNeille completion $\bar{P}$ and this embedding is both supremum and infimum dense. That is to say that every element of $\bar{P}$ is the supremum of elements in the image of P and the infimum of elements in the image of P. It has been shown (see[1,5]) that any complete lattice into which P can be supremum and infimum densely embedded is isomorphic to the MacNeille completion of P.

It is well known that the MacNeille completion is not particularly well behaved with respect to preserving lattice identities. Funayama [3]

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has shown that the MacNeille completion of a distributive lattice need not even be modular. His proof follows by showing that the non-modular five element lattice N_5 can be embedded into the MacNeille completion of a distributive lattice. It is the purpose of this paper to give the following result. Recall that a regular embedding is an embedding that preserves all existing joins and meets.

Theorem. *Any lattice can be regularly embedded into the MacNeille completion of a distributive lattice.*

We begin by constructing the various objects required in the proof. For a lattice L, define the following for each $y \in L$ and each integer m (the set of integers will be denoted by $\mathcal{Z}$).

P is the set of all non-empty finite subsets of L.

$X = L \times P \times \mathcal{Z}$.

X_y is the slab of X for which the first components are dominated by y, i.e. $X_y = pr_L^{-1}[\leftarrow, y]$.

$X_y^m = \{(x, A, n) \in X : x \leq y, n \leq m\} \cup \{(x, A, n) : x \text{ is a zero of } L\}$.

$U_y^m = \{(x, A, n) \in X : y \notin A\} \cup \{(x, A, n) \in X : x \leq \bigvee A\} \cup \{(x, A, n) \in X : n \leq m\}$

D is the sublattice of the power set lattice of X generated by $\{X_y^m, U_y^m : y \in L, m \in \mathcal{Z}\}$. In particular, D is distributive.

N_y is the collection of all elements of D which are contained in X_y. In particular, N_y is a non-empty ideal of D.

Before diving into the details of the proof, let me give a brief outline of the plan. As might be guessed by the notation, the sets N_y are intended to be normal ideals of D. For each $y \in L$, the collection of sets $\{U_y^m\}_{m \in \mathcal{Z}}$ is intended to serve as a set of upper bounds of N_y, refined enough to ensure that N_y is a normal ideal of D. For a finite non-empty subset A of L, to have an embedding of L into $\bar{D}$ we want the supremum of $\{N_a : a \in A\}$ in $\bar{D}$ to be $N_{\bigvee A}$. The essential point is that $\bigcup_{a \in A} U_a^{m_a}$ should be an upper bound of $N_{\bigvee A}$. Intuitively this says that the sets U_y^m are reasonably full. It is the dual role played by the sets U_y^m that necessitates their complicated definition.

To simplify notation in the following Lemma, let $\mathcal{G}$ denote the set of generators of the lattice D, i.e. $\mathcal{G} = \{X_y^m, U_y^m : y \in L, m \in \mathcal{Z}\}$. Also, let $\mathcal{G}_\cap$ and $\mathcal{G}_\cup$ be the closure of $\mathcal{G}$ under finite non-empty intersections and finite non-empty unions respectively. As D is a distributive lattice generated by $\mathcal{G}$, any element of D can be expressed as a finite non-empty union of elements of $\mathcal{G}_\cap$ or dually, as a finite non-empty intersection of members of $\mathcal{G}_\cup$.

A set G in $\mathcal{G}_\cap$ has a representation as

$$G = \bigcap_{i=1}^{p} X_{a_i}^{m_i} \cap \bigcap_{j=1}^{q} U_{b_j}^{n_j}$$

where p, q are positive integers, not both 0, and $a_i, b_j \in L, m_i, n_j \in \mathcal{Z}$ for each $1 \leq i \leq p, 1 \leq j \leq q$. It is not difficult to see that if $a_i = a_k$ for some $1 \leq i, k \leq p$ then one of the terms $X_{a_i}^{m_i}, X_{a_k}^{m_k}$ is redundant. Following this reasoning, we can represent G by

$$G = \bigcap_{a \in A} X_a^{m_a} \cap \bigcap_{b \in B} U_b^{n_b}$$

where A, B are finite subsets of L, not both empty, and m_a, n_b are integers for each $a \in A, b \in B$. Of course, similar statements hold for $\mathcal{G}_\cup$.

We will have need to use such representations frequently, and as no confusion is possible as to the nature of the entities A, B, m_a, n_b, references to their nationalities will be omitted.

Lemma 1.

i) *For $a, b \in L$ and n, m integers, $X_a^m \cap X_b^n = X_{a \wedge b}^{\min\{n,m\}}$.*

ii) *If y is a zero of L then $X_y \subseteq X_a^m$ and $X_y \subseteq U_a^m$ for each $a \in L, m \in \mathcal{Z}$.*

iii) *If A, B are finite subsets of L, not both empty, then*

$$\bigcup_{a \in A} X_a^{m_a} \cup \bigcup_{b \in B} U_b^{n_b} \supseteq X_y \text{ if and only if } \bigvee B \text{ exists and } \bigvee B \geq y.$$

iv) *For $G \in \mathcal{G}_\cup$ and $S \subseteq L$, if $\vee S$ exists and $G \supseteq \bigcup_{s \in S} X_s$ then $G \supseteq X_{\vee S}$.*

v) *For $y \in L$ and $G \in \mathcal{G}_\cap$, if $G \subseteq U_y^m$ for each integer m, then $G \subseteq X_y$.*

Proof.

i) This is a straight forward calculation.

ii) For $(x, A, n) \in X_y$, if y is a zero of L then $x = y$, so x is a zero of L and $x \leq \vee A$ giving $(x, A, n) \in X_a^m$ and $(x, A, n) \in U_a^m$.

iii) First we check that special case that y is a zero of L. By part ii) we have $X_y \subseteq \bigcup_{a \in A} X_a^{m_a} \cup \bigcup_{b \in B} U_b^{n_b}$ since not both of A, B are empty. As y is a zero of L, even if B is empty B has a supremum in L and $\vee B \geq y$.

Assume that y is not a zero of L and that $\bigcup_{a \in A} X_a^{m_a} \cup \bigcup_{b \in B} U_b^{n_b} \supseteq X_y$. This implies that B must be non-empty. Setting $t = \max\{m_a, n_b : a \in A, b \in B\} + 1$, we have $(y, B, t) \in X_y$, so for some $b \in B$ we have $(y, B, t) \in U_b^{n_b}$. Then either $b \notin B$, or $t \leq n_b$ or $\vee B \geq y$. The first two conditions are obviously false, giving $\vee B \geq y$.

Assume that y is not a zero of L, that $\vee B$ exists and that $\vee B \geq y$. This implies that B is non-empty. Take $(x, C, p) \in X_y$ and consider two cases; that B is contained in C and that B is not contained in C. In the first case $\vee C \geq \vee B \geq y \geq x$, giving that $(x, C, p) \in U_b^{n_b}$ for each $b \in B$. In the second case, there is some element $b \in B$ with $b \notin C$, giving $(x, C, p) \in U_b^{n_b}$. So, $X_y \subseteq \cup_{b \in B} U_b^{n_b}$.

iv) As $G \in \mathcal{G}_\cup$, there is a representation

$$G = \bigcup_{a \in A} X_a^{m_a} \cup \bigcup_{b \in B} U_b^{n_b}$$

where not both of A, B are empty. If $G \supseteq \cup_{s \in S} X_s$ then by part iii) we have that $\vee B$ exists and $\vee B \geq s$ for each $s \in S$. If $\vee S$ also exists, then $\vee B \geq \vee S$ and so by part iii) $G \supseteq X_{\vee S}$.

v) As $G \in \mathcal{G}_\cap$, there is a representation

$$G = \bigcap_{a \in A} X_a^{m_a} \cap \bigcap_{b \in B} U_b^{n_b}$$

where not both of A, B are empty. By part i) we may assume that A has at most one element.

If y is a unit of L then $X_y = X$ so clearly $G \subseteq X_y$. Assume then that y is not a unit of L and that $z \not\leq y$.

If B is non-empty, setting $t = \min\{n_b : b \in B\}$ we have $(z, \{y\}, t) \in \cap_{b \in B} U_b^{n_b}$. But $z \nleq \vee\{y\}$, so $(z, \{y\}, t) \notin U_y^{t-1}$.

If $G \subseteq U_y^m$ for each integer m, from the above remarks we may conclude that A is non-empty, and consists of a single element, say a. Setting $p = \min\{m_a, n_b : b \in B\}$, we have that $(a, \{y\}, p) \in G$ so $(a, \{y\}, p) \in U_y^m$ for each integer m. However, this can only occur if $a \leq \vee\{y\} = y$ giving that $G \subseteq X_a^{m_a} \subseteq X_y$.

Lemma 2. *For each $y \in L, N_y$ is a normal ideal of D, and if $y \neq z$ then $N_y \neq N_z$.*

Proof. We must show that $N_y = LU(N_y)$. From general principles it follows that $N_y \subseteq LU(N_y)$. Note that by applying part iii) of Lemma 1 for the special case of A being empty and $B = \{y\}$, we have that $U_y^m \supseteq X_y$ for each integer m. So U_y^m is an upper bound of N_y for each integer m. Suppose $G \in LU(N_y)$ and that $G = G_1 \cup \ldots \cup G_n$, with $n \geq 1$, is a representation of G as a union of members of $\mathcal{G}_\cap$. Then for each $1 \leq i \leq n$ we have $G_i \in LU(N_y)$ and in particular $G_i \subseteq U_y^m$ for each integer m. Then by part v) of Lemma 1, for each $1 \leq i \leq n$ we have $G_i \subseteq N_y$ for each integer m. Then by part v) of Lemma 1, for each $1 \leq i \leq n$ we have $G_i \subseteq X_y$ so $G_i \in N_y$. Then as N_y is an ideal of $D, G \in N_y$.

To see the further remark, note that if $y \nleq z$ then $X_y^1 \in N_y$ but $X_y^1 \notin N_z$.

Lemma 3. *If $S \subseteq L$ and $\wedge S$ exists then $N_{\wedge s} = \cap_{s \in S} N_s$.*

Proof.

$$\text{Note that } \bigcap_{s\in S} X_s = \{(x, A, m) \in X : x \leq s \text{ for all } s \in S\}$$
$$= \{(x, A, m) \in X : x \leq \bigwedge S\}$$
$$= X_{\wedge s}.$$

$$\text{So } \bigcap_{s\in S} N_s = \{G \in D : G \subseteq X_s \text{ for each } s \in S\}$$
$$= \{G \in D : G \subseteq \bigcap_{s\in S} X_s\}$$
$$= \{G \in D : G \subseteq X_{\wedge S}\}$$
$$= N_{\wedge S}$$

Lemma 4. *If $S \subseteq L$ and $\bigvee S$ exists then $N_{\vee S} = LU(\cup_{s\in S} N_s)$.*

Proof. It will be sufficient to show that $U(N_{\vee S}) = U(\cup_{s\in S} N_s)$ since this statement implies that $LU(N_{\vee S}) = LU(\cup_{s\in S} N_S)$ and Lemma 2 has supplied the fact that $N_{\vee S}$ is a normal ideal of D so $LU(N_{\vee S}) = N_{\vee S}$.

As $N_{\vee S}$ contains $\cup_{s\in S} N_s$, it follows that $U(N_{\vee S}) \subseteq U(\cup_{s\in S} N_s)$. Suppose G is an upper bound of $\cup_{s\in S} N_s$ and that $G = G_1 \cap \ldots \cap G_n$, where $n \geq 1$, is a representation of G as a finite intersection of members of $\mathcal{G}_{\cup}$. Then for each $1 \leq i \leq n$ we have that G_i is an upper bound of $\cup_{s\in S} N_s$, so $G_i \supseteq \cup_{s\in S} X_s$. Then by part iv) of Lemma 1, for each $1 \leq i \leq n$ we have that $G_i \supseteq X_{\vee S}$. So $G \supseteq X_{\vee S}$ and G is an upper bound of $N_{\vee S}$.

Lemmas 2,3, and 4 show that the map which sends an element y of L to the subset N_y of D is a regular embedding of L into the MacNeille completion of the distributive lattice D.

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