The MacNeille completion of a uniquely complemented lattice

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Problem 36 of the third edition of Birkhoff's *Lattice theory* [2] asks whether the MacNeille completion of uniquely complemented lattice is necessarily uniquely complemented. We show that the MacNeille completion of a uniquely complemented lattice need not be complemented.

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Questions regarding the axiomatics of Boolean algebras led Huntington to conjecture, in 1904, that every uniquely complemented lattice was distributive. By 1940, Huntington's conjecture had been verified for the classes of modular lattices, atomic lattices, and complemented lattices which satisfy DeMorgan's laws. Then, a 1945 paper of Dilworth [3] proved the quite unexpected result that any lattice could be embedded into a uniquely complemented lattice. It is presently unknown whether a complete uniquely complemented lattice must be distributive. This question has been answered in the affirmative for the classes of continuous lattices (and therefore algebraic lattices), complete lattices with compact unit, as well as the classes mentioned above. The construction of Dilworth seems to have shed little light on this subject, as the uniquely complemented lattices constructed by his method need not be complete. For a thorough description of the results mentioned above and of the history of Huntington's conjecture, see [6] and [1].

Glivenko's theorem states that the MacNeille completion (also known as the completion by cuts) of a Boolean algebra is a Boolean algebra. One might hope for a generalization of this result to uniquely complemented lattices. Indeed, Birkhoff raised this question in the third edition of *Lattice theory* [2] as did Saliĭ in *Lattices with unique complements* [6]. We show that the MacNeille completion of a uniquely complemented lattice is not necessarily complemented.

The example given here is based on Dilworth's original construction of uniquely complemented lattices given in [3], and we will assume a knowledge of this paper.

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For convenience, a result of [3] will be referred to simply by using italicized script, for example *Theorem 4.4*. I have attempted to keep the notation here consistent with [3]. Some of the results of [3] have however been rephrased to conform with modern terminology.

The object of particular interest here is what Dilworth would refer to as the free uniquely complemented lattice generated by a totally unordered set P. We give a brief outline of the construction given in [3].

The set O of *operator polynomials* over P *(Definition 1.1)* would commonly be referred to today as the term algebra of type $\cap$, $\cup$, $*$ over P, where $\cap$ and $\cup$ are binary operation symbols, and $*$ is a unary operation symbol. The symbol $\equiv$ *(Definition 1.3)* is used to denote equality between members of O. A rather complicated definition of a binary relation $\supseteq$ over O is given by *Definition 1.5, Definition 2.1* and *Definition 2.2*, and a relation $\simeq$ is defined on O by setting $A \simeq B$ if $A \supseteq B$ and $B \supseteq A$. *Theorem 2.2* gives an alternate description of the relation $\supseteq$ which is much better suited to our purposes. As the set P we are considering is totally unordered, by the opening remarks in the proof of *Theorem 2.24* we have the following version of *Theorem 2.2*.

Theorem 1. *(Theorem 2.2)* $A \supseteq B$ in O if and only if one of the following holds;

(1) $A \equiv B$.

(2) $A \equiv A_1 \cup A_2$ with $A_1 \supseteq B$ or $A_2 \supseteq B$.

(3) $A \equiv A_1 \cap A_2$ with $A_1 \supseteq B$ and $A_2 \supseteq B$.

(4) $B \equiv B_1 \cup B_2$ with $A \supseteq B_1$ and $A \supseteq B_2$.

(5) $B \equiv B_1 \cap B_2$ with $A \supseteq B_1$ or $A \supseteq B_2$.

(6) $A \equiv A_1^*$ and $B \equiv B_1^*$ with $A_1 \simeq B_1$.

There is a small clash between *Theorem 2.1* and what has become accepted terminology. In modern terms, the relation $\supseteq$ is a quasi-ordering of O and $O/\simeq$ is a lattice under the partial ordering inherited from $\supseteq$. The least upper bound of $A/\simeq$ and $B/\simeq$ being given by $(A \cup B)/\simeq$ and the greatest lower bound by $(A \cap B)/\simeq$.

An element A of O is defined to be *reflexive (Definition 3.2)* if $A \simeq (X^*)^*$ for some X in O. The set of all operator polynomials which contain no reflexive sub-polynomials is denoted by N. An operator polynomial $A \in N$ is *union singular (Definition 4.1, Lemma 4.1)* if $A \supseteq X, X^*$ for some $X, X^* \in N$ and A is *crosscut singular* if $X, X^* \supseteq A$ for some $X, X^* \in N$. A is called *singular* if it is either union or crosscut singular. We denote by M the set of all operator polynomials which contain no singular sub-polynomials together with the two symbols u and z. We extend the relation $\supseteq$ to M by setting $u \supseteq A \supseteq z$ for all $A \in M$. Again making allowances for differing terminology, we may state the results given in the proof of *Theorem 4.1* as

Theorem 2. *(Theorem 4.1)* $M/\simeq$ is a lattice. Furthermore the join of $A/\simeq$ and $B/\simeq$ is $(A \cup B)/\simeq$ if $A \cup B$ is nonsingular and is $u/\simeq$ if $A \cup B$ is singular, while the meet of $A/\simeq$ and $B/\simeq$ is given by $(A \cap B)/\simeq$ if $A \cap B$ is nonsingular and is $z/\simeq$ if $A \cap B$ is singular.

In fact, each element of $M/\simeq$ has exactly one complement *(Theorem 4.2)*, and $M/\simeq$ is the *free lattice with unique complements generated by the unordered set P (Theorem 4.5)*. An alternate characterization of $M/\simeq$ is given by considering the variety V of lattices with an additional unary operation $\perp$ which satisfies $x + x^{\perp} \geq y$, $x \cdot x^{\perp} \leq y$ and $x^{\perp\perp} = x$. Then the proof of *Theorem 4.5* shows that $M/\simeq$ is freely generated in V by the set P[1].

We recall the construction of the MacNeille completion [5] of a partially ordered set Q. For a subset S of Q, define $L(S) = \{x \in Q : x \leq s$ for each $s \in S\}$ and $U(S) = \{x \in Q : s \leq x$ for each $s \in S\}$. The subset S is called a normal ideal of Q if $S = LU(S)$. It is well known that S is a normal ideal of Q if and only if S is the intersection of principal ideals of Q. Therefore the collection of normal ideals of Q, partially ordered by set inclusion, forms a complete lattice which is called the MacNeille completion of Q (sometimes this is referred to as the completion by cuts). For normal ideals I and J of Q, the join of I and J in the MacNeille completion is $LU(I \smile J)$, while the meet is given by $I \frown J$ (the symbols $\smile$, $\frown$ denote set union and intersection).

We focus our attention on the MacNeille completion of the lattice $M/\simeq$ constructed above. In the following, I and J will be normal ideals of $M/\simeq$, neither containing the unit $u/\simeq$ of $M/\simeq$ and both distinct from the zero ideal $\{z/\simeq\}$.

Lemma 1. Let A, B be operator polynomials in M.

i) If $A/\simeq$, $B/\simeq \in I$ then $A \cup B \in M$ and $(A \cup B)/\simeq \in I$.

ii) If $A/\simeq$, $B/\simeq \in J$ then $A \cup B \in M$ and $(A \cup B)/\simeq \in J$.

iii) If $A/\simeq$, $B/\simeq \in U(I)$ then $A \cap B \in M$ and $(A \cap B)/\simeq \in U(I)$.

iv) If $A/\simeq$, $B/\simeq \in U(J)$ then $A \cap B \in M$ and $(A \cap B)/\simeq \in U(J)$.

Proof. Each of these is a simple consequence of Theorem 2 since I and J are ideals distinct from $\{z/\simeq\}$ which do not contain $u/\simeq$.

Lemma 2. If $U(I \smile J) = \{u/\simeq\}$, then for $B, C \in M$ with $B/\simeq \in U(I)$ and $C/\simeq \in U(J)$, $B \cup C$ is singular.

[1] Saliĭ expresses concern [6, p. viii] that there are no explicit examples of uniquely complemented lattices outside the class of Boolean algebras. Free algebras in a variety as simply described as V seem to be quite explicit. Of course this is a matter of opinion.

Proof. The join of $B/\simeq$ and $C/\simeq$ is $u/\simeq$, therefore by Theorem 2 $B \cup C$ is singular.

Lemma 3. For $A, B \in M$, if $A \cup B$ is singular, then there is $X^* \in M$ with $A \cup B \supseteq X, X^*$.

Proof. Let $A, B \in M$ with $A \cup B$ singular. As A, B are nonsingular, $A \cup B$ must be union singular, so $A \cup B \supseteq Y, Y^*$ for some $Y, Y^* \in N$. By *Theorem 2.7* either $A \supseteq Y^*$ or $B \supseteq Y^*$; we assume that $A \supseteq Y^*$. By *Theorem 2.11* there is a sub-polynomial X^* of A with $X \simeq Y$. As X^* is a sub-polynomial of A and $A \in M$, by definition $X, X^* \in M$. But $X \simeq Y$, which implies that $X^* \simeq Y^*$. Therefore $A \cup B \supseteq X, X^*$.

Definition 1. For $A \in M$ let $\tilde{A}$ be $\{X^*/\simeq \,:\, X^* \in M \text{ and } A \supseteq X^*\}$ and for $T \subseteq M/\simeq$ let $\tilde{T}$ be $\{X^*/\simeq \,\in T : X^* \in M\}$.

Lemma 4. For each $A \in M$, $\tilde{A}$ is finite.

Proof. By *Theorem 2.11* if $A \supseteq X^*$ there is a sub-polynomial A_1^* of A with $A_1 \simeq X$, so $A_1^* \simeq X^*$.

Lemma 5. There is $B_0 \in M$ with $B_0/\simeq \,\in U(I)$ and $\tilde{B}_0 = \tilde{I}$ and $C_0 \in M$ with $C_0/\simeq \,\in U(J)$ and $\tilde{C}_0 = \tilde{J}$. In particular $\tilde{I}$ and $\tilde{J}$ are finite.

Proof. By Lemma 4 we may choose $B_0 \in M$ with $B_0/\simeq \,\in U(I)$ so that the cardinality of $\tilde{B}_0$ is minimal among all such possible choices. As $B_0/\simeq$ is an upper bound of I, $\tilde{B}_0$ contains $\tilde{I}$. If $X^* \in M$ and $X^*/\simeq$ is not an element of I, then as I is a normal ideal there is some $B \in M$ with $B/\simeq \,\in U(I)$ and $X^*/\simeq$ not an element of $\tilde{B}$. Then for $D \equiv B \cap B_0$, by Lemma 1 $D \in M$ and $D/\simeq \,\in U(I)$. As $B_0 \supseteq D$ we have $\tilde{D}$ is contained in $\tilde{B}_0$, so by the minimality of B_0 we have $\tilde{D} = \tilde{B}_0$. So $X^*/\simeq$ is not an element of $\tilde{B}_0$. Therefore $\tilde{B}_0 = \tilde{I}$ and by Lemma 4 $\tilde{I}$ is finite.

Lemma 6. If $U(I \smile J) = \{u/\simeq\}$, then there is $X^* \in M$ with $X^*/\simeq \,\in \tilde{I} \smile \tilde{J}$ so that for each $B, C \in M$ with $B/\simeq \,\in U(I)$ and $C/\simeq \,\in U(J)$ we have $B \cup C \supseteq X$.

Proof. Let $B_0, C_0 \in M$ be the operator polynomials given by Lemma 5. By Lemma 2 $B_0 \cup C_0$ is singular, so by Lemma 3 there is some $X^* \in M$ with $B_0 \cup C_0 \supseteq X^*$. Then by *Theorem 2.7* either $B_0 \supseteq X^*$ or $C_0 \supseteq X^*$, so in either case $\tilde{I} \smile \tilde{J}$ is nonempty. By Lemma 5 $\tilde{I} \smile \tilde{J}$ is finite, so we may choose $X_1^*, \ldots, X_n^* \in M$ so that $\{X_i^*/\simeq \,: 1 \le i \le n\} = \tilde{I} \smile \tilde{J}$. Suppose the conclusion of the lemma does not hold. Then for each $1 \le i \le n$ we can find $B_i, C_i \in M$ so that $B_i/\simeq \,\in U(I)$ and $C_i/\simeq \,\in U(J)$ but $B_i \cup C_i \sim\supseteq X_i$ ($\sim\supseteq$ means "does not contain"). Set $B \equiv (((\ldots(B_0 \cap B_1) \cap B_2) \ldots) \cap B_n$ and $C \equiv (((\ldots(C_0 \cap C_1) \cap C_2) \ldots) \cap C_n$. A simple induction using Lemma 1 shows that $B, C \in M$ and $B/\simeq \,\in U(I)$, $C/\simeq \,\in U(J)$. Further $B_i \supseteq B$ and $C_i \supseteq C$ for each $0 \le i \le n$. By Lemma 2 and Lemma 3 there is some $X^* \in M$ with $B \cup C \supseteq X, X^*$. Then by *Theorem 2.7* either $B \supseteq X^*$ or $C \supseteq X^*$. But $B_0 \supseteq B$ and $C_0 \supseteq C$ so in either case $X^* \simeq X_i^*$ for some $1 \le i \le n$. Then by *Theorem 2.5* $X \simeq X_i$ for some $1 \le i \le n$, so $B_i \cup C_i \supseteq X_i$ contrary to our choice of B_i and C_i.

Lemma 7. If $A \in M$ and $B \cup C \supseteq A$ for each $B, C \in M$ with $B/\simeq \in U(I)$ and $C/\simeq \in U(J)$, then $A/\simeq$ is in the ideal of $M/\simeq$ generated by $I \smile J$.

Proof. The proof is by induction on the rank of A *(Definition 1.2)*. If $A \in P$ satisfies the conditions of the lemma, then $A/\simeq$ is an element of $I \smile J$. Indeed, if $A/\simeq$ is not in $I \smile J$, then as I and J are normal ideals there are $B, C \in M$ with $B/\simeq \in U(I)$ and $C/\simeq \in U(J)$ so that $B \sim \supseteq A$ and $C \sim \supseteq A$. Then by Theorem 1, $B \cup C \sim \supseteq A$. Similarly if $A \equiv A_1^*$ satisfies these conditions, then $A/\simeq$ is an element of $I \smile J$. For $A \equiv A_1 \cup A_2$ the conclusion follows from the inductive hypothesis. We have only to verify the claim for $A \equiv A_1 \cap A_2$. Consider four cases.

(1) For all $B, C \in M$ with $B/\simeq \in U(I)$ and $C/\simeq \in U(J)$, $B \cup C \supseteq A_1$.

(2) For all $B, C \in M$ with $B/\simeq \in U(I)$ and $C/\simeq \in U(J)$, $B \cup C \supseteq A_2$.

(3) For all $B, C \in M$ with $B/\simeq \in U(I)$ and $C/\simeq \in U(J)$, $B \supseteq A_1 \cap A_2$.

(4) For all $B, C \in M$ with $B/\simeq \in U(I)$ and $C/\simeq \in U(J)$, $C \supseteq A_1 \cap A_2$.

First we show that one of these cases must apply. If B_i, C_i for $i = 1, \ldots, 4$ witness a failure of case i, then set $B \equiv ((B_1 \cap B_2) \cap B_3) \cap B_4$ and $C \equiv ((C_1 \cap C_2) \cap C_3) \cap C_4$. By Lemma 1 $B, C \in M$ with $B/\simeq \in U(I)$ and $C/\simeq \in U(J)$. By Theorem 1 either $B \cup C \supseteq A_1$, $B \cup C \supseteq A_2$, $B \supseteq A_1 \cap A_2$ or $C \supseteq A_1 \cap A_2$. But $B_i \supseteq B$ and $C_i \supseteq C$ for each $1 \leq i \leq 4$ contradicting our choices of B_i, C_i. Therefore one of the four cases must apply.

If the first case applies, then by the inductive hypothesis $A_1/\simeq$ is in the ideal generated by $I \smile J$, so $A/\simeq$ is also in the ideal generated by $I \smile J$. The second case is obviously similar. The third case implies that $A/\simeq$ is an element of I since I is a normal ideal, and the fourth case implies that $A/\simeq$ is an element of J.

Theorem 3. The MacNeille completion of $M/\simeq$ is a sublattice of the ideal lattice of $M/\simeq$.

Proof. Let I and J be normal ideals of $M/\simeq$. The meet of I and J in the MacNeille completion of $M/\simeq$ is $I \frown J$ which agrees with the meet of I and J in the ideal lattice of $M/\simeq$. The join of I and J in the MacNeille completion of $M/\simeq$ is $LU(I \smile J)$ which is an ideal containing I and J. We must show that $LU(I \smile J)$ is contained in the ideal generated by $I \smile J$. It will do no harm to assume that I and J are distinct from $\{z/\simeq\}$ and neither contains $u/\simeq$. We consider two cases.

If $U(I \smile J) = \{u/\simeq\}$, then by Lemma 6 there is $X^* \in M$ with $X^*/\simeq \in \tilde{I} \smile \tilde{J}$ so that for each $B, C \in M$ with $B/\simeq \in U(I)$ and $C/\simeq \in U(J)$ we have $B \cup C \supseteq X$. So by Lemma 7 $X/\simeq$ is in the ideal generated by $I \smile J$. But $X^*/\simeq$ is also in the ideal generated by $I \smile J$ and $u/\simeq$ is the join of $X/\simeq$ and $X^*/\simeq$. Therefore the ideal generated by $I \smile J$ is all of $M/\simeq$ and hence contains $LU(I \smile J)$.

Let $D \in M$ with $D/\simeq \in U(I \smile J)$. Then for $B, C \in M$ with $B/\simeq \in U(I)$ and $C/\simeq \in U(J)$, by Lemma 1 $B \cap D \in M$ and $C \cap D \in M$. Further $(B \cap D)/\simeq \in U(I)$ and $(C \cap D)/\simeq \in U(J)$, and since D is nonsingular $(B \cap D) \cup (C \cap D) \in M$. For $A \in M$ with $A/\simeq \in LU(I \smile J)$ and $B, C \in M$ with $B/\simeq \in U(I)$ and $C/\simeq \in U(J)$, we have that $(B \cap D) \cup (C \cap D) \supseteq A$. So $B \cup C \supseteq A$ for each $B, C \in M$ with $B/\simeq \in U(I)$ and $C/\simeq \in U(J)$, then by Lemma 7 $A/\simeq$ is in the ideal generated by $I \smile J$.

Theorem 4. The complemented elements of the MacNeille completion of $M/\simeq$ are exactly the principal ideals of $M/\simeq$.

Proof. This follows from Theorem 3 since each element of $M/\simeq$ has only one complement.

Theorem 5. If the generating set P has more than one element, then the MacNeille completion of $M/\simeq$ is not complemented.

Proof. Note that if P has only one element, then $M/\simeq$ is a four element Boolean algebra. Assume that P has at least two elements. By Theorem 4 it is enough to show that $M/\simeq$ has a normal ideal which is not principal, this is equivalent to showing that $M/\simeq$ is not complete. By Theorem 1, $O/\simeq$ is freely generated as a lattice [7, 8] by $P/\simeq \smile \{A^*/\simeq \, : \, A \in O\}$. So by [4] each chain in $O/\simeq$ is at most countable. As $M/\simeq$ is a sub-poset of $O/\simeq$, each chain in $M/\simeq$ is also at most countable. Noting that the sublattice of $M/\simeq$ generated by $P/\simeq$ is freely generated by $P/\simeq$, by *Theorem 4.7* $M/\simeq$ contains a sublattice freely generated by a countable set. Therefore $M/\simeq$ contains a chain isomorphic to the rationals. Any complete lattice containing a chain isomorphic to the rationals must contain a chain isomorphic to the MacNeille completion of the rationals, that is, the extended reals. As each chain in $M/\simeq$ is at most countable, $M/\simeq$ is not complete.

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