Order **13**: 175–205, 1996.

Boolean Products of Lattices

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Communicated by H. A. Priestley

(Received: 30 July 1994; accepted: 1 December 1995)

Abstract. A study is made of Boolean product representations of bounded lattices over the Stone space of their centres. Special emphasis is placed on relating topological properties such as clopen or regular open equalizers to their equivalent lattice theoretic counterparts. Results are also presented connecting various properties of a lattice with properties of its individual stalks.

Mathematics Subject Classifications (1991). 06B15, 06E15.

Key words: Boolean products, Pierce sheaf, Stone space.

1. Introduction

By cardinality considerations alone, it is generally impossible to represent an algebra as a direct product of directly irreducible algebras. The Pierce sheaf representation, in well behaved cases, is a method to represent an algebra as a subalgebra of a direct product of directly irreducible algebras. The Pierce sheaf was first used to give such representations of commutative rings [29], and has been extensively studied by many authors [5], [14]. It was realized early on that a similar construction can be applied to nearly every algebra found in practice [12], [13], [15]. The focus of this paper is the Pierce sheaf representation of a bounded lattice. Before proceeding, we mention that the paper assumes a basic knowledge of lattice theory. In general, any unfamiliar lattice theoretic terminology can be found in [28].

There are limitations to the usefulness of the Pierce sheaf representation. For an irreducible algebra the representation is entirely useless. Also, in the general situation, one has no guarantee that the building blocks used in the representation will be directly irreducible, or simpler in any way than the original algebra.

* Research supported by the Natural Sciences and Engineering Research Council of Canada.
** Research supported by ONR Grant N00014-90-J-1008.

However, it is often found that natural conditions lead to a well behaved representation. For instance, commutativity is sufficient to ensure the representation of a ring produces a subalgebra of a product of directly irreducible rings. If one is interested in a class of algebras where this representation is well behaved and the directly irreducible members are well understood, then the sheaf representation can be tremendously useful.

Our aim is to give a detailed account of conditions on a bounded lattice L which force the sheaf representation of L to be well behaved. For the reader with a knowledge of sheaf representations, we give lattice theoretic conditions which ensure the stalks are irreducible, and conditions which imply that equalizers are clopen, or regular open. The conditions we describe seem quite natural and include a wide variety of lattices. For example, we show that the Z-lattices introduced by F. Maeda [27] and studied in [28] have clopen equalizers and irreducible stalks, as do the P-algebras introduced by Epstein and Horn [18]. It is interesting that some of the conditions used in this paper arose long ago in seemingly unrelated ventures – the work of Epstein and Horn was motivated by their study of multi-valued logic, while F. Maeda introduced Z-lattices in his study of dimension lattices.

The paper is organized in the following manner. The necessary background on the Pierce sheaf representation of a lattice is given in Section 2. We prefer to work with the terminology of weak Boolean products as given in [6]. The two notions are entirely equivalent, and the reader familiar with sheaves will find the transition to weak Boolean products very easy. Also introduced in this section are the fundamental concepts of Hausdorff and weakly Hausdorff lattices. While many equivalent characterizations will be given in later sections, a Hausdorff lattice is one in which equalizers are clopen and a weakly Hausdorff lattice is one in which equalizers are regular open. These concepts are completely described in Section 2.

In the third section we present a sequence of technical lemmas which provide the basic link between properties of the Pierce sheaf of a lattice and lattice theoretic properties of its centre. We also make the observation that a complete lattice satisfies a stronger version of the patchwork property which applies to infinite families of disjoint clopen sets. We call this the extended patchwork property (also known in the literature [8] as the extension property).

In the fourth section we characterize Hausdorff lattices; i.e., lattices with clopen equalizers. Several different characterizations are given. A lattice L is Hausdorff iff for every $s \in L$ the map $\phi_s\colon Z \to L$ from the centre Z of L to L defined by $\phi_s(e) = e \wedge s$ is residuated. This is shown to be equivalent to the set $\mathrm{eq}(s,t) = \{e \in Z\colon\ e \wedge s = e \wedge t\}$ being a principal ideal of the centre of L for each $s, t \in L$. In particular, a complete lattice is Hausdorff iff it is a Z-lattice [28], and a bounded distributive lattice is Hausdorff iff it is a B-algebra in the sense of [18]. Cignoli [11] proved that a weak Boolean product of chains is Hausdorff iff it is a P-algebra.

The fifth section parallels the fourth. We give several characterizations of weakly Hausdorff lattices; i.e., those lattices whose equalizers are regular open. A lattice L is weakly Hausdorff iff the maps ϕ_s described above preserve all existing joins in the centre of L which is equivalent to $\mathrm{eq}(s,t)$ being a normal ideal of the centre for each $s, t \in L$. These conditions can also be phrased in terms of a distributivity condition L must satisfy. L is weakly Hausdorff iff for any element a of L and any subset T of the centre of L whose join in the centre is 1, we have that $a = \bigvee_{t \in T}(a \wedge t)$.

In Section 6 we describe the connection between weakly Hausdorff lattices and Hausdorff lattices. We show that a lattice L is weakly Hausdorff iff L can be double-densely embedded into a Hausdorff lattice M with the extended patchwork property. We use the term double-dense embedding to mean a map $\varphi\colon L \to M$ such that every element of M is the join and meet of images of elements of L and every element of the centre of M is the join and meet of images of elements of the centre of L.

The seventh section is devoted to studying Hausdorff lattices which have the extended patchwork property. Here we make extensive use of results of Carson [8, 9] which connect the first order theory (and even a fragment of second order theory!) of the stalks to that of the lattice. We show that the stalks of a Hausdorff lattice with the extended patchwork property are irreducible, and characterize those Hausdorff lattices with the extended patchwork property that have the relative centre property. Though the general question of determining when a lattice has irreducible stalks is still open, we were able to show that any complete lattice with a countable centre does have irreducible stalks. In concluding, we show that a Hausdorff lattice with the extended patchwork property and complete stalks is necessarily complete (see also [8], [20]). This result has certain consequences in studying MacNeille completions [20].

The eighth and final section of the paper is devoted to giving examples which illustrate the scope of our results. Among these, we show that any complete lattice which is either (i) continuous, (ii) relatively complemented, (iii) uniquely complemented, or (iv) atomistic and dual atomistic is a Hausdorff lattice with the extended patchwork property. Using the fact that P-algebras are Hausdorff, it follows that any Post algebra and any Lukasiewicz algebra is Hausdorff. We also show that an SSC lattice with central covers is weakly Hausdorff. Several examples are included to show that certain extensions to our results are not possible.

Throughout the paper $\wedge$ and $\vee$ are used to denote lattice meet and join. All lattices considered will be bounded lattices with 0 and 1 their bounds. The operations of set union and set intersection will be denoted by $\cup$ and $\cap$. The set complement of the set A will be denoted by A^c (the universal set will be clear from the context).

2. Weak Boolean Products and Sheaves

In this section we give the necessary background on Boolean products, and their relationship to sheaves. We begin with a short review of the Stone space [30] of a Boolean algebra. For a more detailed treatment of Stone duality and Boolean products, the reader should consult [6].

Let B be a Boolean algebra and let $\beta(B)$ be the set of all maximal ideals of B endowed with the Stone topology. Recall that the Stone topology on $\beta(B)$ has as a basis sets of the form: $\beta(e) = \{m \in \beta(B)\colon\ e \notin m\}$, where $e \in B$. For each $e \in B$, the set $\beta(e)$ is both open and closed, or *clopen*, and all clopen sets of $\beta(B)$ are of this form. The space $\beta(B)$ with the Stone topology is called the *Stone space* of B. These spaces are compact, Hausdorff and totally disconnected. Conversely, any topological space X with these properties is called a Stone space, and is homeomorphic to $\beta(B)$, where B is the Boolean algebra of clopen subsets of X.

Subdirect product representations of algebras were introduced in [4]. Let $\{L_m\colon\ m \in \beta(B)\}$ be a family of lattices indexed over the elements of the Stone space $\beta(B)$. A subdirect product $L \subseteq \prod_{m\in\beta(B)} L_m$ is called a *weak Boolean product* if it satisfies:

Equalizers are open. For each $s, t \in L$, $[\![s = t]\!] = \{m \in \beta(B)\colon\ s_m = t_m\}$ is an open subset of $\beta(B)$.

Patchwork property. For each $s, t \in L, e \in B$ there is a $u \in L$ such that $\beta(e) \subseteq [\![u = s]\!]$ and $\beta(e') \subseteq [\![u = t]\!]$.
L is said to be a Boolean product if it also satisfies:

Clopen equalizers. For each $s, t \in L$ we have $[\![s = t]\!]$ is a clopen subset of $\beta(B)$.

If L is a weak Boolean product, then the elements of L will be called *sections* and the factors L_m will be called *stalks.*

The relationship between weak Boolean products and sheaves is discussed in [7]. For completeness, we describe the sheaf associated with a weak Boolean product. For $L \subseteq \prod_{m\in\beta(B)} L_m$ a weak Boolean product, let S be the disjoint union of the sets L_m. Define $\pi\colon\ S \to \beta(B)$ by $\pi(x) = m$ if $x \in L_m$ and endow S with the topology generated by sets of the form $S(s, e) = \{s_m\colon\ m \in \beta(e)\}$ where $s \in L, e \in B$. Then π is a local homeomorphism; moreover, the elements $s \in L$ are global sections of the sheaf (S, π). As noted in [25] (see also [20], p. 287), the sheaf space S is Hausdorff if and only if equalizers are clopen.

The Pierce sheaf for a ring was described in [29]. The construction for a lattice is similar, but will be described here for completeness. We will employ the less cumbersome terminology of Boolean products. Let L be a lattice with centre Z. Let $X = \beta(Z)$. Then, for each $m \in X$, define a congruence relation $\equiv_m$ on L by $s \equiv_m t$ if there is an $e \in Z - m$ such that $e \wedge s = e \wedge t$. Set

$L_m = L/\equiv_m$. For each $s \in L$, define $\widehat{s} \in \prod_{m \in X} L_m$ by $\widehat{s}_m = [s]_{\equiv_m}$. Define $\widehat{L} = \{\widehat{s}\colon s \in L\}$. It is easily seen that $\widehat{L}$ is a weak Boolean product of the family of lattices $\{L_m\colon m \in X\}$. Further, the mapping $\xi\colon L \to \widehat{L}$ defined by $\xi(s) = \widehat{s}$ is a lattice isomorphism. Thus, L is represented as a weak Boolean product over Z. We will refer to this as the *usual representation of* L over the centre Z of L. Finally, we note in passing that this construction can also be employed to represent L as a weak Boolean product over the Stone space of any Boolean subalgebra of the centre of L.

Throughout the paper, we will identify elements in L with sections in the usual weak Boolean product representation of L. Thus, when we speak of the stalks of L, we refer to the stalks of the usual weak Boolean product representation of L, and when we refer to the equalizer $[\![f = g]\!]$ of two elements $f, g \in L$, we mean the equalizer of f and g in the usual weak Boolean product representation of L.

We will say that a lattice L is *Hausdorff* if the usual representation of L is a Boolean product. As in [20], we define a lattice L to be *weakly Hausdorff* if for each $s, t \in L$, such that $[\![s = t]\!]$ is a dense open set, we have $s = t$.

A number of natural questions now arise.

(1) Which lattices are Hausdorff or weakly Hausdorff?

(2) When are the stalks of a lattice irreducible?

(3) What happens when the lattice itself is complete?

(4) If a lattice L has complete stalks, does this imply that L is complete?

In the sequel, these and similar questions will be addressed.

We conclude this section with an example of a class of lattices which have easily described Boolean product representations. These lattices will be a rich source of examples in later sections.

EXAMPLE 2.1 Let X be a Stone space and M be a bounded lattice which is directly irreducible. If we equip M with the discrete topology, then the set L of continuous functions from X to M is a sublattice of the power M^X. The lattice L is usually referred to as the Boolean power of M by X [6].

Let Z denote the centre of L. It is easily verified that $Z = \{f \in L\colon f(x) \in \{0, 1\}$ for all $x \in X\}$. So Z is just the continuous functions from X to the two element lattice 2. It follows that the map $\sim\colon X \to \beta(Z)$ defined by $\tilde{x} = \{f \in Z\colon f(x) = 0\}$ is a homeomorphism. Then for $g, h \in L$ and $x \in X$, we have that $g \equiv_{\tilde{x}} h$ if and only if $g(x) = h(x)$. Therefore, the stalk $L_{\tilde{x}}$ of the usual weak Boolean product representation of L is isomorphic to M. Also, if we set $K = \{x \in X\colon g(x) = h(x)\}$, it is easily verified that K is clopen in X (X is compact so all members of L have finite range). From the remarks above, the equalizer $[\![g = h]\!]$ in the usual weak Boolean product representation of L is equal to $\tilde{K}$ which is clopen.

In summary, L is a Hausdorff lattice and the stalks of L are all isomorphic to M. The reader should note that the irreducibility of M was only required to ensure an easily described Boolean product representation.

3. Central Equalizers

In what follows L denotes a lattice with centre Z. Represent L as a weak Boolean product over the Stone space X of Z. Let $s, t \in L$. Then the *central equalizer of* s, t is defined by $\text{eq}(s,t) = \{e \in Z\colon\ e \wedge s = e \wedge t\}$.

LEMMA 3.1 *Let* $s, t \in L$, *and* $e \in Z$. *Then* $[\![s = t]\!] \subseteq \beta(e)$ *iff* e *is an upper bound in* Z *for* $\text{eq}(s,t)$.

Proof. Assume $[\![s = t]\!] \subseteq \beta(e)$. We are to prove that $f \in \text{eq}(s,t)$ implies $f \leqslant e$. If not, then $e \vee f' < 1$, so by Zorn there is a prime ideal n such that $e \vee f' \in n$. Then $f \notin n$ implies $s_n = t_n$, since $f \wedge s = f \wedge t$. But now, $n \in [\![s = t]\!] \subseteq \beta(e)$, contrary to $e \in n$.

Assume now that $f \leqslant e$ for all $f \in \text{eq}(s,t)$. We must show that $[\![s = t]\!] \subseteq \beta(e)$. Let $m \in [\![s = t]\!]$. Then $f \wedge s = f \wedge t$ for some $f \notin m$. Then $f \in \text{eq}(s,t)$, so $f \leqslant e$. This forces $e \notin m$, so $m \in \beta(e)$, as claimed. □

LEMMA 3.2 *Let* $s, t \in L$. *Then* $\beta(f) \subseteq [\![s = t]\!]$ *iff* $f \in \text{eq}(s,t)$.

Proof. If $f \in \text{eq}(s,t)$ and $m \in \beta(f)$, then clearly $s_m = t_m$, so $m \in [\![s = t]\!]$. If conversely $\beta(f) \subseteq [\![s = t]\!]$ and $f \notin \text{eq}(s,t)$, there exists $n \in X$ such that $\text{eq}(s,t) \subseteq n$ and $f \notin n$. But then $n \in \beta(f) \subseteq [\![s = t]\!]$ implies $s_n = t_n$. Hence for some $g \in Z - n, g \wedge s = g \wedge t$. But now $g \in \text{eq}(s,t) \subseteq n$ produces a contradiction. □

LEMMA 3.3 $[\![s = t]\!] = \bigcup\{\beta(f)\colon\ f \in \text{eq}(s,t)\}$.

Proof. Follows from $[\![s = t]\!]$ open and Lemma 3.2. □

LEMMA 3.4 $\overline{[\![s = t]\!]} = \bigcap\{\beta(e)\colon$ *e is an upper bound for* $\text{eq}(s,t)\}$.

Proof. Using Lemma 3.1, this follows from $\overline{[\![s = t]\!]}$ being closed and $[\![s = t]\!] \subseteq \beta(e)$ iff $\overline{[\![s = t]\!]} \subseteq \beta(e)$. □

Recall that a *regular open set* is a set which is equal to the interior of its closure.

LEMMA 3.5 $[\![s = t]\!]$ *is a regular open set iff* $\text{eq}(s,t)$ *is a normal ideal in the sense that it is the set of lower bounds of its set of upper bounds.*

Proof. Clear from Lemmas 3.1 and 3.2. □

We say that L satisfies the *extended patchwork property* if for any family $(K_i)_{i \in I}$ of pairwise disjoint clopen subsets of the Stone space of Z and any family $(a_i)_{i \in I}$ of elements of L, there is an element $a \in L$ so that for each $i \in I, a$ agrees with

a_i on K_i. Carson [8] refers to the extended patchwork property as the extension property. We say that a lattice L is *orthogonally complete* if for any pairwise disjoint family $(c_i)_{i\in I}$ of central elements and every family $(a_i)_{i\in I}$ of elements of L (over the same indexing set I) the family $(a_i \wedge c_i)_{i\in I}$ has a least upper bound in L.

LEMMA 3.6 *Let L be a lattice.*
(i) *If L is orthogonally complete, then L has the extended patchwork property.*
(ii) *If L is weakly Hausdorff and has the extended patchwork property, then L is orthogonally complete.*

Proof. (i) Let $(K_i)_{i\in I}$ be a family of pairwise disjoint clopen sets, and $(c_i)_{i\in I}$ be the associated family of pairwise disjoint central elements. For any family $(a_i)_{i\in I}$ of elements of L set $a = \bigvee_{i\in I}(a_i \wedge c_i)$. Then

$$a \wedge c_j = c_j \wedge \bigvee_{i\in I}(a_i \wedge c_i) = \bigvee_{i\in I}(a_i \wedge c_i \wedge c_j) = a_j \wedge c_j.$$

(Note that central elements always distribute over infinite joins in this manner.) The extended patchwork property follows.

(ii) Let $(c_i)_{i\in I}$ be a pairwise disjoint family of central elements and $(a_i)_{i\in I}$ be a family of elements of L. We may assume that $(c_i)_{i\in I}$ is a maximal set of pairwise disjoint central elements, as otherwise we could extend it to such putting the new elements a_j equal to 0. For $(K_i)_{i\in I}$ the associated family of clopen sets, we have that the union of the sets K_i is a dense open set. By applying the extended patchwork property we obtain an element a of L which agrees with a_i on K_i for each $i \in I$. Clearly a is an upper bound of $(a_i \wedge c_i)_{i\in I}$, but if $u \leqslant a$ is an upper bound of $(a_i \wedge c_i)_{i\in I}$, then u would agree with a on each K_i. Therefore u and a would agree on a dense open set, and since L is weakly Hausdorff, $u = a$. It follows that $a = \bigvee_{i\in I}(a_i \wedge c_i)$. □

COROLLARY 3.7 *Any complete lattice satisfies the extended patchwork property.*

4. Representation by Boolean Products

In this section we give several characterizations of Hausdorff lattices. A general theme will involve discovering the meaning of assertions of the form $e \wedge s = e \wedge t$ for e a central element of the lattice L. It will be useful at the outset to mention a connection with congruences on L. Accordingly, we agree to let $\mathrm{Con}(L)$ denote the lattice of congruences of L. If $\theta \in \mathrm{Con}(L)$, then θ^* denotes the pseudocomplement of θ. For $a, b \in L$, we agree to let $\theta_{a,b}$ be the minimum congruence which identifies a and b, while θ_0 denotes the smallest congruence on L.

LEMMA 4.1 *Let L be a lattice with centre Z. Let $e \in Z$, $s, t \in L$. Then the following conditions are equivalent:*

(i) $e \wedge s = e \wedge t$.

(ii) $\theta_{e,0} \leqslant \theta^*_{s,t}$.

(iii) $e \equiv 0 \ (\theta^*_{s,t})$.

Thus, $\{e \in Z\colon\ e \wedge s = e \wedge t\} = \{e \in Z\colon\ e \equiv 0\,(\theta^*_{s,t})\} = \{e \in Z\colon\ \theta_{e,0} \leqslant \theta^*_{s,t}\}$.

Proof. (i) $\Rightarrow$ (ii) If $e \wedge s = e \wedge t$, then $s \equiv t\ (\theta_{e,1})$, so $\theta_{s,t} \leqslant \theta_{e,1}$. But this implies that $\theta_{s,t} \cap \theta_{e,0} = \theta_0$, so $\theta_{e,0} \leqslant \theta^*_{s,t}$.

(ii) $\Rightarrow$ (i) If $\theta_{e,0} \leqslant \theta^*_{s,t}$, then $\theta_{s,t} \leqslant \theta^{**}_{s,t} \leqslant \theta_{e,1}$ and $e \wedge s = e \wedge t$.

The equivalence of (ii) and (iii) is clear. □

We now consider a class of lattices which we will see includes the class of Hausdorff lattices. A *central cover lattice* is a lattice L with a function $e\colon\ L \to Z$ such that for each $s \in L$, $e(s)$ is the smallest element in Z such that $s \leqslant e(s)$. It will be convenient to say that Z is *meet regular* if whenever a subset T of Z has a meet e in Z, then e is also the meet of T in L. We call Z *meet complete* if Z is closed under existing meets in L, and note that there are dual notions of *join complete* and *join regular.*

LEMMA 4.2 *If L is a central cover lattice, then Z is both meet regular and meet complete.*

Proof. Assume $T \subseteq Z$ and $z = \bigwedge T$ in Z. If x is a lower bound for T in L, then $e(x)$ is a lower bound for T in Z. Hence $x \leqslant e(x) \leqslant z$, thus showing that Z is meet regular. To show that Z is meet complete, let $x = \bigwedge T$ in L. Since $e(x)$ is also a lower bound for T, we have $e(x) \leqslant x$. Since $x \leqslant e(x)$ is always true, it follows that $x = e(x)$, so $x \in Z$. □

THEOREM 4.3 *For a lattice L, the following conditions are equivalent:*

(i) *L is a central cover lattice.*

(ii) *For each $s \in L, \mathrm{eq}(s, 0)$ is a principal ideal in Z.*

(iii) *For each $s \in L, [\![s = 0]\!]$ is clopen.*

(iv) *For each $s \in L, \{e \in Z\colon\ e \equiv 0\,(\theta^*_{s,0})\}$ is a principal ideal of Z.*

(v) *For each $s \in L, \{e \in Z\colon\ \theta_{e,0} \leqslant \theta^*_{s,0}\}$ is a principal ideal of Z.*

Proof. (i) $\Rightarrow$ (ii) Let $e(s)$ denote the minimum central element above s. Note that $e \in \mathrm{eq}(s, 0)$ if and only if $s \leqslant e'$. Let $f = e(s)$. Then $\mathrm{eq}(s, 0)$ is a principal ideal with generator f'.

(ii) $\Rightarrow$(iii) Let $\mathrm{eq}(s, 0)$ be the principal ideal in Z generated by f. It follows from Lemma 3.2 that $[\![s = 0]\!] = \beta(f)$ which is clopen.

(iii) $\Rightarrow$ (i) Let $\beta(f) = [\![s = 0]\!]$. Then $f' = e(s)$.

The remaining equivalences are clear. □

The next Theorem verifies our claim that a central cover lattice is a generalization of the notion of a Hausdorff lattice. In preparation for this Theorem we pause to mention that for ordered sets P, Q, a mapping $\varphi\colon P \to Q$ is said to be *residuated* if the preimage under φ of any principal ideal of Q is a principal ideal of P. The dual to this notion is called a *residual* mapping. For each residuated mapping $\varphi\colon P \to Q$ there is an associated residual mapping $\varphi^+\colon Q \to P$. The mapping φ^+ is defined by taking $\varphi^+(q)$ to be the generator in P of the preimage of the principal ideal $[\leftarrow, q]$. We shall make frequent use of the fact that residuated mappings preserve arbitrary existing joins, while residual maps preserve arbitrary existing meets. A detailed treatment of residuated and residual mappings can be found in [2].

THEOREM 4.4 *For a lattice L, the following conditions are equivalent:*

(i) *L is a Hausdorff lattice in the sense that the usual representation of L represents it as a Boolean product.*

(ii) *For each $s, t \in L, \mathrm{eq}(s,t) = \{e \in Z\colon\ e \wedge s = e \wedge t\}$ is a principal ideal of Z.*

(iii) *For each $s, t \in L$, there exists a central element $t\colon s$ having the property that for $e \in Z$, $e \wedge s \leqslant t$ iff $e \leqslant t\colon s$.*

(iv) *For each $s \in L$, the mapping $\phi_s\colon Z \longrightarrow L$ defined by $\phi_s(e) = e \wedge s$, is residuated.*

(v) *For each $s \in L$, the mapping $\eta_s\colon Z \longrightarrow L$ defined by $\eta_s(e) = e \vee s$, is a residual mapping.*

(vi) *For each $s, t \in L, \{e \in Z\colon\ \theta_{e,0} \leqslant \theta^*_{s,t}\}$ is a principal ideal of Z.*

(vii) *For each $s, t \in L, \{e \in Z\colon\ e \equiv 0\,(\theta^*_{s,t})\}$ is a principal ideal of Z.*

Proof. (i) $\Rightarrow$ (ii). Let $s, t \in L$. As $[\![s = t]\!]$ is clopen there is some $e \in Z$ with $[\![s = t]\!] = \beta(e)$. The result then follows from Lemmas 3.1 and 3.2.

(ii) $\Rightarrow$ (iii) Observe that $e \wedge s \leqslant t$ iff $e \wedge s = e \wedge s \wedge t$. Now take $t\colon s$ to be the largest element of $\mathrm{eq}(s, s \wedge t)$.

(iii) $\Rightarrow$ (iv) Define $\phi_s^+\colon L \to Z$ by $\phi_s^+(t) = t\colon s$. Then ϕ_s is residuated with ϕ_s^+ its associated residual mapping.

(iv) $\Rightarrow$ (i) For each $s \in L$, let ϕ_s^+ denote the residual mapping associated with ϕ_s. Then $\phi_s^+(t) = \max\{e \in Z\colon\ e \wedge s \leqslant t\}$. It follows for $s, t \in L$ that $e = \phi_s^+(t) \wedge \phi_t^+(s)$ is the largest element in $\mathrm{eq}(s,t) = \{e \in Z\colon\ e \wedge s = e \wedge t\}$. By Lemma 3.3, $[\![s = t]\!] = \beta(e)$ and is therefore clopen.

Clearly, (iv) and (v) are dual, hence equivalent. The remaining equivalences follow from Lemma 4.1. □

Note that one consequence of Theorem 4.4 is that a lattice L is Hausdorff if and only if its dual L^* is Hausdorff. This is immediate from the fact that, for each $s, t \in L$, and for every $e \in Z$, $e \wedge s = e \wedge t$ if and only if $e' \vee s = e' \vee t$. It should also be noted that for weak Boolean products of bounded chains, the equivalence of (i) and (iv) in Theorem 4.4 was established by Cignoli [11].

COROLLARY 4.5 *Let L be a Hausdorff lattice with centre Z. Then both L and its dual are central cover lattices.*

Proof. By Theorem 4.4, $[\![s = 0]\!]$ is clopen for all $s \in L$. Hence by Theorem 4.3, L is a central cover lattice. As the dual L^* is also Hausdorff, it follows that L^* is also a central cover lattice. □

DEFINITION 4.6 The centre Z of L is called *regular* if it is both meet and join regular; similarly, it is called *complete* if it is both meet and join complete. In particular, if Z is a complete sublattice of a complete lattice L, then Z is a regular sublattice of L.

COROLLARY 4.7 *Let L be a Hausdorff lattice with centre Z. Then Z is a regular sublattice and a complete sublattice of L.*

Proof. This is an immediate consequence of Corollary 4.5 and Lemma 4.2. □

Remark 4.8 A Z-lattice (see [28]) is a complete lattice L which satisfies the following conditions:

(i) The centre $Z(L)$ is a complete sublattice.
(ii) If $z_\alpha \in Z(L)$ for $\alpha \in I$, then $s \wedge (\bigvee_{\alpha \in I} z_\alpha) = \bigvee_{\alpha \in I}(s \wedge z_\alpha)$ for each $s \in L$.

We note that a complete lattice L is a Z-lattice if and only if for every $s \in L$ the mapping ϕ_s is residuated. Examples of Z-lattices will be presented in Section 8. The following two results are now an immediate consequence of Theorem 4.4.

COROLLARY 4.9 *Let L be a complete lattice. Then L is Hausdorff if and only if L is a Z-lattice.*

COROLLARY 4.10 *The dual of a Z-lattice is a Z-lattice.*

5. Weakly Hausdorff Lattices

In Section 4 we gave several conditions which were equivalent to a lattice being Hausdorff. In this section we will see that the theory of weakly Hausdorff lattices has many parallels with the theory of Hausdorff lattices. We begin with two technical lemmas. In what follows, Z_a will denote the principal ideal in Z generated by the central element a.

LEMMA 5.1 *Let I be an ideal of Z. Then x is an upper bound for I iff $I \cap Z_{x'} = \{0\}$.*

Proof. Clearly x an upper bound for I in Z implies $I \cap Z_{x'} = \{0\}$. Suppose conversely that $I \cap Z_{x'} = \{0\}$. Then if $b \in I$, $b \wedge x' = 0$ implies that $b \leqslant x$. Hence x is an upper bound for I. □

LEMMA 5.2 *An ideal I of Z is closed under the formation of existing suprema in Z if and only if it is normal in the sense that it consists of the set of lower bounds of its set of upper bounds.*

Proof. Let I be closed under the formation of existing suprema in Z, and suppose $a \leqslant$ all upper bounds of I. Assume $a \notin I$. Then a cannot be the join of $Z_a \cap I$, since that would force $a \in I$. Thus $Z_a \cap I$ must have an upper bound $b < a$. It is immediate that $Z_a \cap Z_{b'} \cap I = Z_{a \wedge b'} \cap I = \{0\}$. Hence by Lemma 5.1, $a' \vee b$ is an upper bound for I. It follows that $a \leqslant a' \vee b$, so $a = a \wedge (a' \vee b) = a \wedge b \leqslant b$, contrary to $b < a$. We deduce after all that a must have been a member of I, thus showing that I consists of the set of lower bounds of its set of upper bounds. The converse is trivially true. □

Before beginning our discussion of equivalent characterizations of weakly Hausdorff lattices, we present a weakening of the notion of central cover lattices. This serves nicely as a preview of what is to come.

THEOREM 5.3 *For a lattice L with centre Z, the following conditions are equivalent:*

(i) *Z is meet regular.*

(ii) *For each $s \in L, \mathrm{eq}(s, 0)$ is closed under existing suprema in Z.*

(iii) *For each $s \in L$, $[\![s = 0]\!]$ is regular open.*

(iv) *For each $s \in L, \{e \in Z\colon\ e \equiv 0\,(\theta^*_{s,0})\}$ is closed under existing suprema in Z.*

(v) *For each $s \in L, \{e \in Z\colon\ \theta_{e,0} \leqslant \theta_{s,0}\}$ is closed under existing suprema in Z.*

(vi) *For each $s \in L$, if $[\![s = 0]\!]$ is dense, then $s = 0$.*

(vii) *For any subset T of Z whose join in Z equals 1, if $s \wedge e = 0$ for all $e \in T$, then $s = 0$.*

Proof. (i) ⇒ (ii) This follows from the fact that for $e \in Z, e \wedge s = 0$ if and only if $s \leqslant e'$.

(ii) ⇔ (iii) is a direct consequence of Lemma 5.2.

(ii) ⇒ (i) Let $z = \bigwedge_i z_i$ exist in Z, and let s be any lower bound for $\{z_i\}$ in L. Then $z_i' \in \mathrm{eq}(s, 0)$ for all i implies that $z' \in \mathrm{eq}(s, 0)$, whence $s \leqslant z$.

(iv) and (v) are equivalent to (ii) by Lemma 4.1.

(iii) ⇒ (vi) is trivial.

(vi) ⇒ (vii) For such a set T we have $\bigcup\{\beta(e)\colon\ e \in T\}$ is a dense set contained in $[\![s = 0]\!]$.

(vii) ⇒ (i) Let $z = \bigwedge_i z_i$ exist in Z, and let s be any lower bound for $\{z_i\}$ in L. Setting $y = z' \wedge s$ we have $y \wedge z = 0$ and $y \wedge z_i' = 0$ for each i. As the join of $\{z_i'\colon\ i \in I\} \cup \{z\}$ in Z equals 1, we have that $y = 0$, whence $s \leqslant z$. □

We now show that the lattices described in the above Theorem are related to weakly Hausdorff lattices in the same manner in which central cover lattices are related to Hausdorff lattices.

THEOREM 5.4 *Let L be a lattice with centre Z. Then, the following conditions are equivalent:*

(i) *L is weakly Hausdorff.*

(ii) *For any subset T of Z whose join in Z equals 1, if $s \wedge e = t \wedge e$ for all $e \in T$, then $s = t$.*

(iii) *If T is a subset of Z whose join in Z equals 1, then for any $s \in L, s$ is the least upper bound in L of $\{s \wedge f\colon f \in T\}$.*

(iv) *If T is a subset of Z whose join in Z exists and is equal to e, then for any $s \in L, s \wedge e$ is the least upper bound in L of $\{s \wedge f\colon f \in T\}$.*

(v) *For each $s \in L$ the mapping $\phi_s\colon Z \to L$ defined by $\phi_s(e) = e \wedge s$ preserves all existing suprema in Z.*

(vi) *For each $s, t \in L$ the set $\mathrm{eq}(s,t)$ is closed under existing suprema in Z.*

(vii) *For each $s, t \in L, \{e \in Z\colon \theta_{e,0} \leqslant \theta^*_{s,t}\}$ is closed under existing suprema in Z.*

(viii) *For each $s, t \in L, \{e \in Z\colon e \equiv 0\,(\theta^*_{s,t})\}$ is closed under existing suprema in Z.*

(ix) *For each $s, t \in L, [\![s = t]\!]$ is a regular open set.*

Proof. (i) $\Rightarrow$ (ii) If $T \subseteq Z$ is a subset with join 1, then $D = \bigcup_{e \in T} \beta(e)$ is a dense open subset of X. Since s, t agree on D, $s = t$.

(ii) $\Rightarrow$ (iii) Let T be a subset of Z whose join in Z equals 1, and let $t \leqslant s$ be an upper bound of $\{s \wedge f\colon f \in T\}$. Then for all $f \in T, t \wedge f = t \wedge s \wedge f = s \wedge f$, and therefore $s = t$.

(iii) $\Rightarrow$ (iv) Let T be a subset of Z whose join in Z equals e. Define $S = T \cup \{e'\}$. Since the join in Z of S equals 1, we have $s = \bigvee_{f \in S}(s \wedge f)$. Therefore $s \wedge e = \bigvee_{f \in S}(s \wedge e \wedge f) = \bigvee_{f \in T}(s \wedge f)$.

(iv) $\Rightarrow$ (i) Let $s, t \in L$ be such that s, t agree on a dense open set $U \subseteq X$. Then there is a family $\{e_i\colon i \in I\} \subseteq Z$ such that $U = \bigcup_{i \in I} \beta(e_i)$. Hence, $1 = \bigvee_{i \in I} e_i$. Now

$$s = s \wedge 1 = \bigvee_{i \in I}(s \wedge e_i) = \bigvee_{i \in I}(t \wedge e_i) = t \wedge 1 = t.$$

(iv) $\Leftrightarrow$ (v) is trivial.

(v) $\Rightarrow$ (vi) Let $T \subseteq \mathrm{eq}(s,t)$ with the join of T in Z equal to e. As ϕ_s preserves existing suprema, $e \wedge s$ is the least upper bound in L of $\{s \wedge f\colon f \in T\}$. Similarly, $e \wedge t$ is the least upper bound in L of the set $\{t \wedge f\colon f \in T\}$. But $s \wedge f = t \wedge f$ for all $f \in T$, so $e \wedge s = e \wedge t$. Thus $e \in \mathrm{eq}(s,t)$.

(vi) $\Rightarrow$ (ii) Let T be a subset of Z whose join in Z equals 1. If $s \wedge e = t \wedge e$ for all $e \in T$, then $T \subseteq \mathrm{eq}(s,t)$. As $\mathrm{eq}(s,t)$ is closed under existing suprema, 1 is in $\mathrm{eq}(s,t)$. Thus $s = t$.

The equivalence of (vii) and (viii) with the previous items follows by Lemma 4.1.

Finally (ix) $\Leftrightarrow$ (vi) follows by Lemmas 5.2 and 3.5. □

COROLLARY 5.5 *If L is a weakly Hausdorff lattice with centre Z, then*
(i) *The dual of L is weakly Hausdorff.*
(ii) *Z is a regular sublattice of L.*

Proof. (i) This follows from condition (ii) of the above Theorem once we note that for $e \in Z$ and $s, t \in L$ we have $s \wedge e = t \wedge e$ if and only if $s \vee e' = t \vee e'$.

(ii) Clearly the conditions of the previous Theorem all imply the conditions of Theorem 5.3. Therefore Z is meet regular, and as the dual of L is weakly Hausdorff, Z is also join regular. □

The following theorem illustrates the relationship between weakly Hausdorff lattices and Z-lattices (see Remark 4.8).

THEOREM 5.6 *Let L be a lattice with centre Z. Then, the following are equivalent.*
(i) *L is weakly Hausdorff.*
(ii) *Z is a join regular sublattice of L and if $(z_\alpha)_{\alpha \in I}$ is a subset of Z whose join exists in L, then for any $s \in L$, $s \wedge (\bigvee_{\alpha \in I} z_\alpha) = \bigvee_{\alpha \in I}(s \wedge z_\alpha)$. (All joins being taken in L.)*

Proof. (i) $\Rightarrow$ (ii) That Z is join regular is given by the above Corollary. Let $(z_\alpha)_{\alpha \in I}$ be a subset of Z whose join in Z is equal to z. We must show that for any $s \in L$, $s \wedge z$ is the least upper bound in L of $\{s \wedge z_\alpha\colon \alpha \in I\}$. Suppose that $t \in L$ is is an upper bound of $\{s \wedge z_\alpha\colon \alpha \in I\}$. Then $s \wedge z \wedge z_\alpha = s \wedge t \wedge z \wedge z_\alpha$ for each $\alpha \in I$. So

$$\mathrm{eq}(s \wedge z, s \wedge t \wedge z) \supseteq \{z_\alpha\colon \alpha \in I\} \cup \{z'\}.$$

As L is weakly Hausdorff, we have that $s \wedge z = s \wedge t \wedge z$, whence $s \wedge z \leqslant t$. So $s \wedge z$ is the least upper bound in L of $\{s \wedge z_\alpha\colon \alpha \in I\}$.

(ii) $\Rightarrow$ (i) We will verify condition (iv) of Theorem 5.4. Suppose that T is a subset of Z whose join in Z exists and is equal to e. We must show that for any $s \in L$, $s \wedge e$ is the least upper bound in L of $\{s \wedge f\colon f \in T\}$. This would follow directly from our assumption if we knew that e was the join of T in L. But this is precisely the information given by Z being a join regular sublattice of L. □

Remark 5.7 Next we recall some of the parallels which we have seen. In the case of Hausdorff lattices: $[\![s = t]\!]$ is clopen, $\mathrm{eq}(s, t)$ is a principal ideal in Z, and ϕ_s is residuated. In the case of weakly Hausdorff lattices: $[\![s = t]\!]$ is a regular open set, $\mathrm{eq}(s, t)$ is closed under existing suprema in Z, and ϕ_s preserves existing suprema in Z. The reader might also note that in [20], p. 287, the second author showed that an algebra A was weakly Hausdorff iff the Boolean algebra B of factor congruences of A was a meet regular sublattice of $\mathrm{Con}(A)$.

The reader will recall the definition of complete sublattices given in Definition 4.6.

THEOREM 5.8 *Let L be a weakly Hausdorff lattice with centre Z. If L is orthogonally complete or complemented, then Z is a complete sublattice of L.*

Proof. Let T be a subset of Z whose join in L equals a, and set $A = \bigcup_{z \in T} [\![z = 1]\!]$ and B equal to the interior of A^c. Then $A \subseteq [\![a = 1]\!]$ and $B \subseteq [\![a = 0]\!]$. So a is neutral on the dense open set $A \cup B$, and as L is weakly Hausdorff, it follows that a is neutral in L. To show that a is central, it is sufficient to show that a has a complement. This is trivial if L is complemented. If L is orthogonally complete, an application of the extended patchwork property produces an element b which is complementary to a on a dense open subset of X. Then as L is weakly Hausdorff, b is a complement of a. The argument to show that Z is closed under existing meets in L is dual to the above. □

THEOREM 5.9 *Every weakly Hausdorff lattice L whose centre is a complete lattice is necessarily Hausdorff; in particular, any orthogonally complete weakly Hausdorff lattice is Hausdorff.*

Proof. Assume that the centre Z of L is complete. As Z is a regular sublattice of L, it follows immediately from Theorems 5.4 and 4.4 that L is Hausdorff. If L is orthogonally complete, then by Theorem 5.8 Z is a complete sublattice of L. But all orthogonal subsets of Z have joins in L, and hence also in Z. It is a simple exercise to show that a Boolean algebra is complete iff all orthogonal subsets have joins. So Z is complete, which implies that L is Hausdorff. □

EXAMPLE 5.10 Let D be the distributive lattice consisting of all subsets of $\mathbb{N}$ which are either (i) finite (ii) contain all but finitely many odd numbers and only finitely many even numbers or (iii) contain all but finitely many numbers. It is easy to verify that D is closed under finite unions and intersections and is therefore a sublattice of the power set of $\mathbb{N}$. To verify that D is weakly Hausdorff, it is enough to verify Theorem 5.4(iii) for the set T consisting of all finite subsets of $\mathbb{N}$. However, $\{\{2n + 1\}\colon n \in \mathbb{N}\}$ is a subset of the centre of D whose join is not central. Therefore the centre of D is not a complete sublattice of D. In view of Corollary 4.7, D is a weakly Hausdorff lattice which is not Hausdorff.

EXAMPLE 5.11 Here is an example of a Hausdorff lattice having a complete centre, but which is not orthogonally complete. Let 3 denote the three element chain $0 < a < 1$, and X an infinite set. We then take L to be the set of all $f \in 3^X$ such that $f^{-1}[\{a\}]$ is finite. We leave to the reader the routine verification that L is Hausdorff, has a complete centre, but is not orthogonally complete.

6. From Weakly Hausdorff to Hausdorff

In this section we describe the realtionship between weakly Hausdorff lattices and Hausdorff lattices. As we shall see, a lattice is weakly Hausdorff if and only if it can be embedded in a well behaved fashion into a Hausdorff lattice.

DEFINITION 6.1 Let φ be an embedding of a bounded lattice L into a bounded lattice M. We say that φ is a *dense embedding* if every element in M is the join and meet of images of elements of L, and we say that φ is a regular embedding if it preserves all existing joins and meets of elements of L. If L can be embedded into M, we will denote this by writing $L \leqslant M$.

LEMMA 6.2 *Let* φ: $L \longrightarrow M$ *be a dense embedding. Then*
(i) φ *is a regular embedding.*
(ii) *If* $z \in Z(L)$, *then* $\varphi(z) \in Z(M)$.

Proof. (i) It is obvious that dense embeddings are regular embeddings.

(ii) By duality, it is enough to show that for each $m \in M, m \leqslant (m \wedge \varphi(z)) \vee (m \wedge \varphi(z'))$. As φ is a dense embedding, there is a subset $A \subseteq L$ so that $m = \bigvee\{\varphi(a)\colon a \in A\}$. As $z \in Z(L)$ $a = (a \wedge z) \vee (a \wedge z')$ for each $a \in A$. Therefore $\varphi(a) \leqslant (m \wedge \varphi(z)) \vee (m \wedge \varphi(z'))$ for each $a \in A$, so $m \leqslant (m \wedge \varphi(z)) \vee (m \wedge \varphi(z'))$. □

DEFINITION 6.3 Let φ: $L \to M$ be a dense embedding. We have seen in the above Lemma that the restriction $\varphi_{|Z(L)}$ of φ to the centre of L is a mapping into the centre of M. We shall say that the dense embedding φ is *double-dense* if $\varphi_{|Z(L)}$: $Z(L) \to Z(M)$ is also a dense embedding.

Our aim is to show that a lattice L is weakly Hausdorff if and only if there is a double-dense embedding from L into an orthogonally complete Hausdorff lattice.

LEMMA 6.4 *If* L *can be double-densely embedded into a weakly Hausdorff lattice, then* L *is weakly Hausdorff.*

Proof. Let φ: $L \to M$ be a double-dense embedding of L into a weakly Hausdorff lattice M. Suppose that $T \subseteq Z(L)$ with $\bigvee T = 1$ (this join being taken in $Z(L)$) and $s, t \in L$ with $s \wedge e = t \wedge e$ for all $e \in T$. We must show that $s = t$. As φ is double-dense, we have that the restriction $\varphi_{|Z(L)}$ is dense, and hence also regular. Thus, $\bigvee \varphi[T] = 1$ (this join being taken in $Z(M)$). But $\varphi(s) \wedge \varphi(e) = \varphi(t) \wedge \varphi(e)$ for all $e \in T$ and as M is weakly Hausdorff it follows that $\varphi(s) = \varphi(t)$. Since φ is an embedding, $s = t$. □

One might hope that any lattice which could be densely emebedded into a weakly Hausdorff lattice would be weakly Hausdorff. The following example shows this is not the case.

EXAMPLE 6.5 Consider the family of all maps f from the reals into the real unit interval which satisfy the following condition: if $f(0) \in \{0, 1\}$ then f is constant on some open interval containing 0. It is easily seen that the collection of all such maps is a sublattice L of the family of all maps from the reals into the real unit interval, which we shall denote by D. As L contains all functions which vanish except at a single point and take on a value strictly less than one at that singleton, it follows that L is join dense in D, and by duality L is meet dense in D. It is not difficult to verify that D is Hausdorff (it is even a Z-lattice). However, L is not weakly Hausdorff. To see this, let $T = \{e \in Z(L)\colon\ e(0) = 0\}$, and note the join of T in $Z(L)$ equals 1. Then taking s, t to be the elements of L which take the values 1/3 and 2/3 at zero and vanish elsewhere, we have $s \wedge e = t \wedge e$ for all $e \in T$. Thus contradicting part (ii) of Theorem 5.4.

DEFINITION 6.6 Let $L \leqslant \prod_{m\in X} L_m$ be the usual weak Boolean product representation of L over the Stone space of its centre. We say that a function $f \in \prod_{m\in X} L_m$ is a *dense open section* if there is a family $(K_i)_{i\in I}$ of pairwise disjoint clopen subsets of X and a family $(a_i)_{i\in I}$ of elements of L such that:
(i) $\bigcup_{i\in I} K_i$ is a dense open subset of X.
(ii) f agrees with a_i on the clopen set K_i for each $i \in I$.
In terms of the Pierce sheaf of L, a dense open section is one which is continuous on a dense open set. We then define $\mathcal{D}L$ to be the collection of all dense open sections of L and define a relation θ on $\mathcal{D}L$ by setting $f\theta g$ if f and g agree on a dense open subset of X.

LEMMA 6.7 *Let $L \leqslant \prod_{m\in X} L_m$ be the usual weak Boolean product representation of L.*
(i) $\mathcal{D}L \leqslant \prod_{m\in X} L_m$.
(ii) *θ is a congruence on $\mathcal{D}L$.*

Proof. Both these assertions follow as the intersection of two dense open subsets of X is a dense open subset of X. □

In the following, we let $\Re L$ denote the lattice $\mathcal{D}L/\theta$ and we denote the θ equivalence class of an element $f \in \mathcal{D}L$ by $[f]$. We also define a map $\alpha\colon\ L \to \mathcal{D}L/\theta$ by setting $\alpha(a) = [a]$. It is a simple matter of checking the definitions of *weakly Hausdorff* and the *extended patchwork property* to verify the following Lemma.

LEMMA 6.8 *Let $L \leqslant \prod_{m\in X} L_m$ be the weak Boolean product representation of L.*
(i) *$\alpha\colon\ L \longrightarrow \Re L$ is an embedding iff L is weakly Hausdorff.*
(ii) *$\alpha\colon\ L \longrightarrow \Re L$ is surjective iff L is has the extended patchwork property.*

LEMMA 6.9 *Let L be a bounded lattice.*
(i) *For $f, g \in \mathcal{D}L$, $[f] \geqslant [g]$ iff $[\![f \geqslant g]\!]$ contains a dense open subset of X.*
(ii) *If $\alpha\colon\ L \longrightarrow \Re L$ is an embedding it is a dense embedding.*

Proof. (i) $[f] \geqslant [g]$ iff $[f \wedge g] = [g]$ iff $f \wedge g$ and g agree on a dense open set, which in turn is equivalent to saying that $[\![f \geqslant g]\!]$ contains a dense open set.

(ii) For f a dense open section of L, let $(K_i)_{i\in I}$ be a family of clopen sets and $(a_i)_{i\in I}$ be a family of elements of L so that f agrees with a_i on K_i and $\bigcup_{i\in I} K_i$ is dense in X. Define $(g_i)_{i\in I}$ by setting g_i equal to a_i on K_i and equal to 0 on the complement of K_i. Then g_i is an element of L, and we clearly have that $[f]$ is an upper bound of $([g_i])_{i\in I}$. Suppose that $h \in \mathcal{D}L$ and that $[h]$ is an upper bound of $([g_i])_{i\in I}$. By the first part of the Lemma we have that $[\![h \geqslant g_i]\!]$ contains a dense open set, and therefore $[\![h \geqslant f]\!]$ contains a dense open subset of K_i for each $i \in I$. This implies that $[\![h \geqslant f]\!]$ contains a dense open subset of X and therefore $[h] \geqslant [f]$. So $[f]$ is the least upper bound of the family $([g_i])_{i\in I}$. The argument to show that $[f]$ is the meet of images of elements of L is dual to the above. Therefore, if α is an embedding, it is a dense embedding. □

For $R \subseteq X$, it will be convenient to define $\chi_R \in \prod_{m\in X} L_m$ by $\chi_R(m) = 1_m$ if $m \in R$ and 0_m otherwise. The following theorem is due to Carson [8]. As Carson's notation differs greatly from our own, we have included a proof of this theorem for the convenience of the reader.

THEOREM 6.10 *Let L be a weakly Hausdorff lattice. Then:*
(i) $Z(\Re L) = \{[\chi_R]\colon$ *R is a regular open subset of X*$\}$.
(ii) *$\Re L$ is Hausdorff.*
(iii) *$\Re L$ is orthogonally complete.*
(iv) *$\alpha\colon L \to \Re L$ is a double-dense embedding.*

Proof. We note first that for any open subset $U \subseteq X$, there is a smallest regular open subset $R \subseteq X$ containing U. This set R is the interior of the closure of U. It is also worthwhile to note that U is dense in R.

(i) It is a routine matter to verify that for an open set U, χ_U is a dense open section of L and $[\chi_U]$ is central in $\Re L$. The difficulty is in showing that every central element of $\Re L$ is of the form $[\chi_R]$ for some regular open set R.

Suppose that $f \in \mathcal{D}L$ is such that $[f]$ is central in $\Re L$. Let $g \in \mathcal{D}L$ be such that $[g]$ is the complement of $[f]$ in $\Re L$. In particular, f and g must be complements on a dense open subset of X. As f, g are dense open sections, we can find a family of pairwise disjoint clopen sets $(K_i)_{i\in I}$ and families $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ of elements of L such that f agrees with a_i on K_i for each $i \in I, g$ agrees with b_i on K_i for each $i \in I$, and $\bigcup_{i\in I} K_i$ is dense in X. By choosing the families $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ wisely, we may also assume that a_i agrees with 0 and b_i agrees with 1 on the complement K_i^c of K_i. In other words $a_i = f \wedge \chi_{K_i}$ and $b_i = g \vee \chi_{K_i^c}$ for each $i \in I$.

As f and g are complements on a dense open subset of X, it follows that for each $i \in I, a_i$ and b_i are also complements on a dense open subset of X. As L is weakly Hausdorff, we have that a_i and b_i are complements for each $i \in I$. Further, as $[f]$ and $[\chi_{K_i}]$ are central in $\Re L$, it follows that $[f] \wedge [\chi_{K_i}] = [f \wedge \chi_{K_i}]$ must also be central. But $[f \wedge \chi_{K_i}] = \alpha(a_i)$. In particular, $\alpha(a_i)$ is neutral in $\Re L$, and as α is an embedding, a_i must be neutral in L. As a_i is both complemented and neutral, it must be central in L.

As a_i is central in L, it follows that $a_i = \chi_{\beta(a_i)}$. Setting $U = \bigcup_{i\in I} \beta(a_i)$, we then have that f agrees with χ_U on the dense open set $\bigcup_{i\in I} K_i$. If R is the smallest regular open set containing U, it is then a simple matter to verify that f agrees with χ_R on a dense open set, and hence $[f] = [\chi_R]$.

(ii) To see that $\Re L$ is Hausdorff, suppose that $f, g \in \mathcal{D}L$. Choose families $(K_i)_{i\in I}, (a_i)_{i\in I}$ and $(b_i)_{i\in I}$ as above and set $D = \bigcup_{i\in I} K_i$. Then $[\![f = g]\!] \cap D = \bigcup_{i\in I} [\![a_i = b_i]\!]$ is an open subset of D. Call this open set U, and let R be the smallest regular open set containing U. It is then a routine matter to check that $[f] \wedge [\chi_R] = [g] \wedge [\chi_R]$ and that $[\chi_R]$ is the largest central element of $\Re L$ with this property. By Theorem 4.4(ii), we have that $\Re L$ is Hausdorff.

(iii) To see that $\Re L$ is orthogonally complete, it is sufficient to show by Lemma 3.6 that $\Re L$ has the extended patchwork property. Let $(R_i)_{i\in I}$ be a family of pairwise disjoint regular open subsets of X and let $(f_i)_{i\in I}$ be a family of elements of $\mathcal{D}L$. Then define f so that f agrees with f_i on R_i and set f to be 0 elsewhere. It is a routine matter to verify that f is a dense open section of L. Then as $[f] \wedge [\chi_{R_i}] = [f_i] \wedge [\chi_{R_i}]$ for each $i \in I$, we have that $[f]$ agrees with $[f_i]$ on the clopen set associated with $[\chi_{R_i}]$.

(iv) As L is weakly Hausdorff, α is an embedding. Therefore, by the previous Lemma, α is a dense embedding. To see that α is double-dense, we have only to show that every central element of $\Re L$ is the join of images of central elements of L. This is a consequence of the first part of this Theorem using the fact that every regular open set in X is the union of clopen subsets of X. □

We are now in a position to prove our result.

THEOREM 6.11 *Let L be a bounded lattice. The following are equivalent:*

(i) *L is weakly Hausdorff.*
(ii) *L can be double-densely embedded into a weakly Hausdorff lattice.*
(iii) *L can be double-densely embedded into a Hausdorff lattice.*
(iv) *L can be double-densely embedded into an orthogonally complete Hausdorff lattice.*

Proof. (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are obvious.

(ii) $\Rightarrow$ (i) was established in Lemma 6.4.

(i) $\Rightarrow$ (iv) By the previous Theorem the map α: $L \to \Re L$ has the desired properties. □

7. Orthogonally Complete Lattices

In this section, we study some of the consequences of orthogonal completeness on the weak Boolean product representation of a lattice. In particular, we will see that orthogonally complete weakly Hausdorff lattices (which are necessarily Hausdorff by Theorem 5.9) have very well behaved Boolean product representations.

The key result on orthogonally complete weakly Hausdorff lattices is due to Carson, namely that the equalizer of any first order formula $[\![\varphi(x)]\!]$ is clopen.

This follows by a simple induction on the length of the formula, making use of the extended patchwork property to handle existential quantifiers [8, Proposition 2.17, p. 16] Carson. We will not need the full generality of this result, and in fact will be concerned only with the first order formula $\varphi(x, x')$ which says that x and x' are complementary central elements of L, and a related formula. Specifically, $\varphi(x, x')$ is the first order formula given by

$$(\forall y \in L)[(y \vee x) \wedge (y \vee x') = (y \wedge x) \vee (y \wedge x')].$$

The equalizer $[\![\varphi(x, x')]\!]$ is defined by

$$\begin{aligned}&[\![\varphi(x, x')]\!]\\ &\quad = \{m \in X\colon\ (\forall y \in L)[(y \vee x) \wedge (y \vee x') \equiv_m (y \wedge x) \vee (y \wedge x')]\}.\end{aligned}$$

Note that $x, x' \in L$ are complementary central elements of L iff $X = [\![\varphi(x, x')]\!]$.

THEOREM 7.1 *If L is an orthogonally complete Hausdorff lattice, then the stalks of L are directly irreducible.*

Proof. If $x \in L$ is such that x_m is central in L_m, then there is some element $x' \in L$ such that $m \in [\![\varphi(x, x')]\!] = K$. Define $z = x \wedge \chi_K$ and $z' = x' \vee \chi_{K^c}$. Then $[\![\varphi(z, z')]\!] = X$ and therefore z, z' are complementary central elements of L. But $z \equiv_m x$ and therefore $x_m = 0_m$ or $x_m = 1_m$. □

Recall that a lattice L satisfies the *relative centre property* if for every element $a \in L$ and every element c which is central in $[0, a]$ there is a $z \in Z$ such that $c = z \wedge a$. For elements $a, x, x' \in L$ we define $\varphi(a, x, x')$ to be the first order formula which states that $x \wedge a$ and $x' \wedge a$ are complementary central elements of the section $[0, a]$ of L. Specifically, $\varphi(a, x, x')$ is given by

$$\begin{aligned}&(\forall y \in L)[((y \wedge a) \vee (x \wedge a)) \wedge ((y \wedge a) \vee (x' \wedge a))\\ &\quad = ((y \wedge a) \wedge (x \wedge a)) \vee ((y \wedge a) \wedge (x' \wedge a))].\end{aligned}$$

Note that $x \wedge a$ and $x' \wedge a$ are complementary central elements of the section $[0, a]$ iff $X = [\![\varphi(a, x, x')]\!]$.

THEOREM 7.2 *Let L be an orthogonally complete Hausdorff lattice. Then L has the relative centre property iff every section $[0, a_m]$ of every stalk L_m of L is directly irreducible.*

Proof. Assume L has the relative centre property. If $a, x, x' \in L$ are such that $x_m \wedge a_m$ and $x'_m \wedge a_m$ are complementary central elements of $[0, a_m]$, then $m \in [\![\varphi(a, x, x')]\!] = K$. Define $z = x \wedge \chi_K$ and $z' = x' \vee \chi_{K^c}$. Then $[\![\varphi(a, z, z')]\!] = X$, and so $z \wedge a$ and $z' \wedge a$ are complementary central elements of $[0, a]$. As L has the relative centre property, there is some $c \in Z$ with $z = c \wedge a$. Then $x_m = c_m \wedge a_m$ which in turn is equal to either a_m or 0_m. Hence the section $[0, a_m]$ is directly irreducible.

Conversely, assume that the sections of the stalks of L are directly irreducible. If $a, x \in L$ are such that x is central in $[0, a]$ then for each $m \in X$ we have that x_m is central in $[0, a_m]$. Therefore x_m is equal to either a_m or 0_m for each $m \in X$. Let K denote the clopen set $[\![x = a]\!]$. Then χ_K is central in L and $x = a \wedge \chi_K$. □

Having seen that orthogonally complete weakly Hausdorff lattices have irreducible stalks, we should point out that there are other conditions sufficient to guarantee irreducible stalks.

THEOREM 7.3 *If the complemented elements of L are all central then the stalks of L are directly irreducible. In particular, any distributive lattice has irreducible stalks.*

Proof. If $x \in L$ and x_m is central in L_m then there is some $x' \in L$ with x_m and x'_m complements in L_m. Setting $P = [\![x \wedge x' = 0]\!]$ and $Q = [\![x \vee x' = 1]\!]$ we have that $P \cap Q$ is an open set containing the point m. As X has a basis of clopen sets, there is a clopen set K containing m on which x and x' are complements. Set $z = x \wedge \chi_K$ and $z' = x' \vee \chi_{K^c}$. Then z, z' are complements in L and therefore central in L. So $x_m = z_m$ is equal to either 0_m or 1_m. □

THEOREM 7.4 *If L is orthogonally complete and the centre of L is at most countable, then the stalks of L are irreducible.*

Proof. Assume that $a, b \in L$ are such that a_p and b_p are complementary central elements of L_p. By our standard techniques, we may assume that a and b are complements in L. Defining $T = \{m \in X\colon\ a_m \text{ is not neutral in } L_m\}$, it is enough to show that p is not in the closure of T, since this would imply there is a clopen neighbourhood K of p on which a was neutral and hence central. This conclusion is obvious if p is a principal prime ideal, as this would imply that $\{p\}$ was clopen, so we assume that p is non-principal.

Enumerate the elements of $Z - p$ as $e_1, e_2, \ldots$, with $e_1 = 1$. Setting $f_n = \bigwedge_{i=1}^{n} e_i$, we have a decreasing chain $f_1 \geqslant f_2 \geqslant \cdots$ of elements of $Z - p$ such that for every $e \in Z - p$ there is some $f_n \leqslant e$. In other words $\beta(f_1) \supseteq \beta(f_2) \supseteq \cdots$ is a neighbourhood base of p. As a Stone space is Hausdorff, it follows that $\bigcap_n \beta(f_n) = \{p\}$. Defining $d_n = f_n \wedge f'_{n+1}$, we have that $(\beta(d_n))_{n\in\mathbb{N}}$ is a family of pairwise disjoint clopen sets and

$$\bigcup_{n\in\mathbb{N}} \beta(d_n) = \beta(f_1) - \bigcap_{n\in\mathbb{N}} \beta(f_n) = X - \{p\}.$$

For each $n \in \mathbb{N}$ with $\beta(d_n) \cap T \neq \emptyset$, choose $x(n), y(n) \in L$ and $m(n) \in X$ such that $a_{m(n)}, x(n)_{m(n)}, y(n)_{m(n)}$ are not a distributive triple in $L_{m(n)}$. Apply the extended patchwork property to get $x, y \in L$ so that x agrees with $x(n)$ on $\beta(d_n)$ and y agrees $y(n)$ on $\beta(d_n)$ for each $n \in \mathbb{N}$ with $\beta(d_n) \cap T \neq \emptyset$.

As a_p is neutral in L_p, there is a clopen neighbourhood K of p on which a, x, y are a distributive triple. As $K \supseteq \beta(f_{n_0})$ for some $n_0 \in \mathbb{N}$, we have that $K \supseteq \beta(d_n) = \beta(f_n \wedge f'_{n+1})$ for all $n \geqslant n_0$. It follows that $\beta(f_{n_0})$ is a clopen neighbourhood of p which is disjoint from T, as required. □

In particular, the above Theorem shows that the ideal lattice of any countable lattice has irreducible stalks. It is not known whether the assumption of a countable centre can be removed from Theorem 7.4. However, the following example shows that Theorems 7.1, 7.3 and 7.4 cannot be generalized in some other directions.

EXAMPLE 7.5 Let X be the one point compactification of the natural numbers with the discrete topology, say $X = \mathbb{N} \cup \{\infty\}$. Then points in $\mathbb{N}$ are clopen. A set $U \subseteq X$ such that $\infty \in U$ is open iff U^c is finite iff U is clopen. Then X is compact, Hausdorff, and totally disconnected, hence a Stone space (in fact X is the Stone space of the Boolean algebra consisting of all finite subsets of $\mathbb{N}$ and their complements). Let M be the five element modular lattice with the three atoms a, b, c, and let L be the Boolean power of M over the Stone space X. As we have seen in Example 2.1, the stalks of the Boolean product representation of L are all isomorphic to M. Define

$$L' = \{x \in L\colon\ x_\infty \neq c\}.$$

It is easily seen that L' is a sublattice of L and that the centre of L' is equal to the centre of L. It follows that L' is a modular, Hausdorff lattice with a countable centre. But the stalk L'_∞ is a four element Boolean algebra, and hence reducible.

We also take this opportunity to show that there are complete lattices which are not weakly Hausdorff.

EXAMPLE 7.6 Let B be an infinite Boolean algebra and L be the ideal lattice of B. It is easily seen that the centre of L consists of exactly the principal ideals of B. It follows that the centre of L is not a regular sublattice of L, and therefore L is not weakly Hausdorff. Note that if B is not complete, the centre of L is not complete, even though L is a complete, even upper continuous [1], distributive lattice.

We say that an ideal A of a lattice L is an *orthogonally closed ideal* if A is closed under all existing orthogonal joins of L. For such an ideal $A \subseteq L$ we define for each $m \in X, A_m = \{d_m\colon\ d \in A\}$.

LEMMA 7.7 *Let L be an orthogonally complete Hausdorff lattice and A be an orthogonally closed ideal of L.*

(i) *For $w \in L, \{m \in X\colon\ w_m \geqslant d_m$ for all $d \in A\}$ is a clopen set which we denote by $[\![w \geqslant A]\!]$.*

(ii) *For $l \in L$, if l_n is the supremum of A_n, then there is a clopen neighbourhood K of n such that l_m is the supremum of A_m for all $m \in K$.*

(iii) *$\{m \in X\colon\ A_m$ has a supremum in $L_m\}$ is clopen.*

(iv) *A has a supremum in L iff A_m has a supremum in L_m for all $m \in X$. Further, if A has a supremum, it is the componentwise supremum.*

Proof. (i) For each $w \in L$ and each $d \in A$, we have that $[\![d \nleqslant w]\!]$ is clopen. So $P = \bigcup\{[\![d \nleqslant w]\!]\colon\ d \in A\}$ is an open set. We claim that P is clopen, which will establish our claim since $[\![w \geqslant A]\!]$ is the complement of P. Let $(K_i)_{i\in I}$ be a maximal family of pairwise disjoint clopen subsets of P such that for each $i \in I$ there is some $d_i \in A$ so that $d_i \nleqslant w$ on K_i. Let d be the element given by the extended patchwork property such that d agrees with d_i on K_i and d is 0 on the clopen set $\overline{P}^c$ (the closure of an open set is clopen since the centre is complete). Note that d is an orthogonal join of elements of A, and therefore $d \in A$. But $[\![d \nleqslant w]\!]$ is a clopen set which contains $\bigcup_i K_i$, and therefore $[\![d \nleqslant w]\!]$ contains P. However $[\![d \nleqslant w]\!]$ is clearly contained in P. Therefore $P = [\![d \nleqslant w]\!]$ is clopen.

(ii) Let $(K_i)_{i\in I}$ be a maximal family of pairwise disjoint clopen sets such that for each $i \in I$ there is an element $w_i \in L$ with $K_i \subseteq [\![w_i < l]\!] \cap [\![w_i \geqslant A]\!]$. Let Q denote the closure of $\bigcup_i K_i$. By the extended patchwork property there is an element $w \in L$ agreeing with w_i on K_i for each $i \in I$. As $\bigcup_i K_i \subseteq [\![w < l]\!] \cap [\![w \geqslant A]\!]$, it follows that $Q \subseteq [\![w < l]\!] \cap [\![w \geqslant A]\!]$. Since l_n is the least upper bound of A_n, it follows that n is not in Q.

As n is not in the closure of $\bigcup_i K_i$, we can find a clopen set K which is contained in $[\![l \geqslant A]\!]$, contains n, and is disjoint from each K_i. If $m \in K$ and l_m is not the supremum of A_m, then there would be some element $v \in L$ with $m \in [\![v \geqslant A]\!] \cap [\![v < l]\!]$. This would contradict the maximality of the family $(K_i)_{i\in I}$. Therefore l_m is the supremum of A_m for each $m \in K$ and this establishes our claim.

(iii) We know from (ii) that if l_m is the supremum of A_m in L_m, then there is a clopen neighbourhood K of m such that l_n is the supremum of A_n for all $n \in K$. Let $(K_i)_{i\in I}$ be a maximal family of pairwise disjoint clopen sets, having the property that for each $i \in I$ there is some $l^i \in L$ that is the supremum of A on K_i. Let K be the closure of $\bigcup_i K_i$, and note that $K = \beta(e)$ is clopen (since Z is complete). We now use the extended patchwork property to construct an element l such that l agrees with l^i on K_i. Hence l_m is the supremum of A_m for all $m \in \bigcup_i K_i$. We would be done if we could show that if A_n did not have a supremum, then $n \notin K$. Suppose to the contrary that $n \in K$. We know from (i) that $[\![l \geqslant A]\!]$ is a clopen set that contains $\bigcup K_i$. Hence it contains K, so l_n is an upper bound for A_n in L_n. Since A_n does not have a join in L_n, there is an element $c \in L$ such that $c_n < l_n$, and c_n is an upper bound of A_n. By (i), $[\![c \geqslant A]\!]$ is clopen and contains n. Hence $J = [\![c < l]\!] \cap [\![c \geqslant A]\!]$ is a clopen set containing n. Since n is in the closure of $\bigcup K_i$, J must have a nonvoid intersection with some K_i. But if $m \in J \cap K_i$, then $c_m < l_m$ is an upper bound for A_m, contrary to l_m being the supremum of A_m in L_m.

(iv) Assume that w is the supremum of A in L. We will show that w_m is the supremum of A_m for each $m \in X$. Assume that d_n is an upper bound of A_n and that $d_n < w_n$. Then $K = [\![d < w]\!] \cap [\![d \geqslant A]\!]$ is a clopen set containing n. By defining e to agree with d on K and to agree with w on K^c we would have that e is an upper bound of A strictly less than w, a contradiction.

Conversely, assume that A_m has a supremum in L_m for all $m \in X$. Let $(K_i)_{i\in I}$ be a maximal family of pairwise disjoint clopen sets such that for each $i \in I$ there is an element $l^i \in L$ so that l^i_m is the supremum of A_m for each $m \in K_i$. By (ii) and the maximality of $(K_i)_{i\in I}$ it follows that $\bigcup_i K_i$ is a dense open subset of X. By the extended patchwork property, there is an element $l \in L$ such that l agrees with l_i on K_i for each $i \in I$. Then $[\![l \geqslant A]\!]$ is a clopen set containing $\bigcup_i K_i$, so $[\![l \geqslant A]\!] = X$, and therefore l is an upper bound of A. But if $u \in L$ is an upper bound of A, then $[\![l \leqslant u]\!]$ is a clopen set containing $\bigcup K_i$, so $[\![l \leqslant u]\!] = X$. Therefore l is the least upper bound of A. □

THEOREM 7.8 *Let L be a weakly Hausdorff lattice having complete stalks on a dense subset U of X. Then L is complete iff it is orthogonally complete.*

Proof. Assume first that L is orthogonally complete, and recall that by Theorem 5.9, L is Hausdorff. Let $A \subseteq L$ be a nonempty subset of L. There is a smallest orthogonally closed ideal $\widehat{A}$ containing A, and a simple argument shows that the upper bounds of A are exactly the upper bounds of $\widehat{A}$. By Lemma 7.7(iii) $\{m \in X\colon\ \widehat{A}_m$ has a supremum in $L_m\}$ is a clopen set containing the dense set U, so $\widehat{A}_m$ has a supremum in L_m for each $m \in X$. Then by Lemma 7.7(iv), $\widehat{A}$ has a supremum in L, and therefore A also has a supremum in L.

The fact that L complete implies L orthogonally complete follows trivially from the definition of orthogonal completeness. □

A. Carson [9] has recently discovered that a fragment of the second order theory of the stalks is transferable to the full Boolean product. The above result on completeness is a special case of this more general theory. We next present two examples which show that Theorem 7.8 is reasonably sharp.

PROPOSITION 7.9 *There exists a complete, distributive, Hausdorff lattice L with an incomplete stalk.*

Proof. Let $D = \mathbb{N} \cup \{\infty\}$ be the natural numbers with a largest element ∞ added. Then D is a complete, irreducible, distributive lattice. In fact D is just a chain. Let L be $D^{\mathbb{N}}$; i.e., the lattice D raised to the power of the natural numbers. It is easily seen that L is a Hausdorff lattice and that $Z(L)$ is isomorphic to the power set of the natural numbers. It follows that for any prime ideal m of $Z(L)$ that L_m is just the ultrapower $D^{\mathbb{N}}/\mathcal{U}$ where $\mathcal{U}$ is the ultrafilter consisting of all elements of the power set of $\mathbb{N}$ which are not in m. Further, each ultrapower of D arises in this manner. It is well known that a non-principal ultrapower of D is not complete ([24], p. 88). □

EXAMPLE 7.10 By Theorem 7.3, the stalks of any Boolean algebra B are irreducible Boolean algebras, hence isomorphic to the two element Boolean algebra 2, and consequently all complete. For any $s, t \in B, \mathrm{eq}(s,t)$ is the principal ideal generated by $(s \wedge t) \vee (s' \wedge t')$, and so any Boolean algebra B is Hausdorff. Thus there are incomplete Hausdorff lattices, all of whose stalks are complete.

Before concluding this section, we give an application of Theorem 7.8 which has its origins in [20]. Let V be a variety of algebras such that each $A \in V$ has a lattice reduct and the central elements of A give rise to direct decompositions of the algebra A. Note that the weak Boolean product representation of A is then a weak Boolean product of algebras in V. We say that such a variety V is weakly Hausdorff if the lattice reduct of each $A \in V$ is weakly Hausdorff.

COROLLARY 7.11 *Let V be a weakly Hausdorff variety. If the directly irreducible members of V are all complete, then V is closed under MacNeille completions* [26]. *In fact, the MacNeille completion of $A \in V$ is $\Re A$.*

Proof. It is sufficient to show that each $A \in V$ can be densely embedded into a complete algebra in V [3]. By Lemma 6.9, the map $A \to \Re A$ is a dense embedding and $\Re A$ is both orthogonally complete and Hausdorff. So by Theorem 7.1, the stalks of $\Re A$ are directly irreducible, and hence complete. The result now follows by Theorem 7.8. □

Remark 7.12 If we make the mild assumption that the variety V in the above Theorem has only finitely many basic operations, the condition that every directly irreducible algebra in V is complete is equivalent to the condition that the class of directly irreducible members of V have a finite uniform upper bound on the lengths of their chains. To see the equivalence, note first that if the variety V has only finitely many basic operations, then there is a first order sentence φ such that $A \models \varphi$ iff A is directly irreducible. Loosely speaking, this sentence φ says there is no central element other than $0, 1$ whose associated lattice congruence is compatible with the basic operations of the algebra. This being said, we then have that an ultraproduct of directly irreducible members of V must again be directly irreducible. Therefore, if for each $n \geqslant 1$ there is a directly irreducible algebra A_n in V such that A_n has a chain of at least n elements, we may take an ultraproduct of the A_n to find a directly irreducible algebra $A \in V$ such that A has an infinite chain. Using standard techniques, one can construct an ultrapower of A which is not complete. Therefore, if the directly irreducible members of V are all complete, there is a finite uniform upper bound on the lengths of their chains. The converse follows trivially as any chain finite lattice is complete.

If V has only finitely many basic operations and the lattice reducts of members of V are all distributive we can proceed further. Under these conditions, the condition that every directly irreducible member of V is complete is equivalent to the condition that there are only finitely many directly irreducibles in V and these directly irreducibles are all finite. This equivalence follows from the above once one notices that a distributive lattice having at most n elements in any of its chains can have at most 2^{n-1} elements in total.

8. Examples

As a preparation for concluding the paper, it is appropriate that we illustrate the scope of our results by showing that they apply to an extremely broad class of lattices. Before doing so, it will prove useful to introduce some notation and terminology. When dealing with notions of *upper* and *lower-continuity*, we shall follow the notation and terminology of [28]. Also, a lattice L with 0 is called *section semicomplemented* (SSC) if $a, b \in L$ with $a < b$ implies the existence of $c \in L$ such that $0 < c \leqslant b$ with $c \wedge a = 0$. The dual notion is called *dual section semicomplemented* and referred to in symbols by DSSC.

We start by considering complete lattices, noting in doing so that by Corollary 4.9, a complete lattice is a Hausdorff lattice if and only if it is a Z-lattice in the sense of Remark 4.8.

EXAMPLE 8.1 The complete lattice L is a Z-lattice if it satisfies any one of the following conditions:

(i) Any continuous lattice L ([28], Remark 5.12, p. 24).

(ii) L is upper continuous and its centre is join closed. To see that such a lattice is Hausdorff, we note first that the centre Z is necessarily a complete lattice; moreover, by upper continuity, the mapping $e \to e \wedge s$ is residuated, so we may apply Theorem 4.4.

(iii) L is semicomplemented and upper continuous. By (ii), it suffices to show that the centre Z is join closed. To do this, it will suffice to show that if $(e_\delta)_{\delta \in D}$ is an upward directed family, then $e_\delta \uparrow e$ with each $e_\delta \in Z$ implies that $e \in Z$. Let $e'_\delta \downarrow e'$. Then for each $\delta \in D, e_\delta \wedge e' = 0$. By upper continuity, $e \wedge e' = 0$. If $e \vee e' < 1$, we could find a nonzero semicomplement s of $e \vee e'$. But then $s \wedge e = 0$ implies $s \wedge e_\delta = 0$ for all δ, so $s \leqslant e'_\delta$ for all δ. But then $s \leqslant e'$, contrary to $s \wedge e' = 0$. This shows that e and e' are complements. But an easy application of upper continuity shows e to be neutral; hence $e \in Z$, as desired.

(iv) Any lattice L which is SSC and DSSC ([28], Corollary 5.14, p. 25). Among these lattices we have: (a) any relatively complemented lattice; (b) any lattice having the property that all intervals of the form $[0, a]$ or $[b, 1]$ are complemented; (c) any lattice that is both atomistic and dual atomistic ([28], p. 30); (d) any uniquely complemented lattice (such lattices are of interest because of Dilworth's Theorem [16] that any lattice may be embedded in a uniquely complemented lattice).

(v) L is SSC and ∇–continuous ([28], Theorem 5.13, p. 24).

(vi) Any SSC-lattice whose centre is a complete sublattice ([28], Exercise 5.1, p. 25).
(vii) L is SSC, dual semicomplemented and satisfies Axiom (A) of [22] (see example 3, p. 178 of [22] and Theorem 2, p. 237 of [23]).
(viii) The trivial example where L has a finite centre. Of course completeness is not required here.
(ix) Any direct product of Z-lattices is a Z-lattice, as is the dual of any Z-lattice.

The reader should observe that Hausdorff lattices need not be complete. This was already illustrated by the class of lattices having a finite centre. But the next Theorem also describes an interesting class of lattices which includes any bounded relatively complemented lattice.

THEOREM 8.2 *A bounded section complemented lattice L is Hausdorff if and only if it is a central cover lattice.*

Proof. By Theorem 4.3 of the paper, L is a central cover lattice iff for every $s \in L, \mathrm{eq}(s,0)$ is a principal ideal of Z. We must show that this implies that $\mathrm{eq}(s,t)$ is a principal ideal of Z for any $s,t \in L$. Accordingly, let $s,t \in L$, and take w to be a complement of $s \wedge t$ in $[0, s \vee t]$. Using the fact that $e \wedge s = e \wedge t \iff e \wedge w = 0$, we see that $\mathrm{eq}(s,t) = \mathrm{eq}(w,0)$, thus proving L is Hausdorff. The converse implication is trivial. □

We turn next to some examples of weakly Hausdorff lattices.

Remark 8.3 Recall that by Theorem 5.3, the following conditions are equivalent for any bounded lattice L:
(i) Z is meet-regular.
(ii) For each $s \in L, \mathrm{eq}(s,0)$ is closed under existing suprema in Z.

THEOREM 8.4 *Let L be a bounded SSC-lattice. Then:*
(i) *L is weakly Hausdorff iff it satisfies the conditions of Remark* 8.3.
(ii) *If the centre of L is complete, then L is Hausdorff iff it is a central cover lattice.*

Proof. (i) Suppose that L satisfies the conditions of Remark 8.3. Let $(e_i)_{i\in I}$ be a family of central elements and suppose $e = \bigvee_{i\in I} e_i$ in Z. Let $s \in L$. Suppose $e \wedge s$ were not the join of the family $(e_i \wedge s)_{i\in I}$ in L. Then there would exist an element $y < e \wedge s$ such that $y \geqslant e_i \wedge s$ for all $i \in I$. Use SSC to find an element w such that $0 < w \leqslant e \wedge s$ and $w \wedge y = 0$. Then $w \wedge e_i = 0$ for all i, so each $e_i \in \mathrm{eq}(w,0)$. But by Remark 8.3, $\mathrm{eq}(w,0)$ is closed under existing suprema in Z. Hence $e \in \mathrm{eq}(w,0)$, contrary to $0 < w \leqslant e$. The converse implication is clear.

(ii) Suppose first that L is a central cover lattice having a complete centre. By (i), L is weakly Hausdorff, and by Theorem 5.9, L is in fact Hausdorff. The converse is trivial. □

EXAMPLE 8.5 By [20], Proposition 5, p. 291, any orthomodular lattice in the variety generated by a set of orthomodular lattices having a finite uniform upper bound on the lengths of their chains is weakly Hausdorff. It is also shown in [20] that the directly irreducibles in such a variety are all complete. Therefore by Corollary 7.11, any such variety is closed under MacNeille completions.

EXAMPLE 8.6 Let B denote the lattice of all subsets of an infinite uncountable set X, and let L denote the subsets of X that are finite, countably infinite or have a finite complement. Then L is a bounded distributive lattice that is weakly Hausdorff but is not a central cover lattice. It follows that it is not Hausdorff.

EXAMPLE 8.7 Let B denote the Boolean algebra consisting of all finite subsets of $\mathbb{N}$ and their complements, and let X be the Stone space of B. Note that the elements of X are exactly the principal prime ideals $[\leftarrow, \{n\}^c]$, and the non-principal prime ideal ∞ consisting of all finite subsets of $\mathbb{N}$. A subset $U \subseteq X$ is clopen iff U consists only of a finite number of principal prime ideals, or if U^c consists only of a finite number of principal prime ideals. Let I be the ideal of B consisting of all finite subsets of odd elements of $\mathbb{N}$; i.e., I is the ideal of B generated by $\{\{2n+1\}\colon n \in \mathbb{N}\}$. We define $\beta(I) = \bigcup\{\beta(c)\colon c \in I\}$. Specifically, $\beta(I) = \{[\leftarrow, \{2n+1\}^c]\colon n \in \mathbb{N}\}$. Note that $\beta(I)$ is an open set and that there is no smallest clopen set containing $\beta(I)$ (as I has no join in B).

Let 3 represent the three element chain $0 < a < 1$, and define $f \in 3^X$ by setting

$$f(m) = \begin{cases} a & \text{if } m = [\leftarrow, \{2n+1\}^c] \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

We then set L to be the sublattice of 3^X generated by the set of all characteristic functions χ_K of clopen subsets K of X and the function f. As 3^X is distributive, it follows that

$$L = \{(f \wedge \chi_{K_1}) \vee \chi_{K_2}\colon K_1, K_2 \text{ are clopen subsets of } X\}.$$

Note that if $g \in L$ and $g(m) = a$ for some $m \in X$, then g has no complement in 3^X, and therefore g can have no complement in L. It follows that the centre of L consists exactly of the characteristic functions χ_K of clopen sets. Clearly χ_K is an upper bound of f iff $\beta(I) \subseteq K$, so f has no least central upper bound in L, and therefore L is not Hausdorff.

We claim that L is weakly Hausdorff. Let $s = (f \wedge \chi_{K_1}) \vee \chi_{K_2}$ be an element of L. It is sufficient to show that s is the least upper bound of $\{s \wedge \chi_{\{m\}}\colon m \text{ is a principal prime ideal of } B\}$, since any subset T of the centre of L which joins to 1 has the property that for each principal ideal $m \in X$ there is an element $e \in T$ with $\chi_{\{m\}} \leqslant e$. Now s is an upper bound of $\{s \wedge \chi_{\{m\}}\colon m \text{ is a principal prime ideal of } B\}$. Suppose that $t = (f \wedge \chi_{N_1}) \vee \chi_{N_2}$

is also an upper bound of this set. Then for any principal prime ideal m of B we have that $s \wedge \chi_{\{m\}} \leqslant t$, and therefore $s(m) \leqslant t(m)$. We have only to show that $s(\infty) \leqslant t(\infty)$ for the non-principal prime ideal ∞ of B. As $f(\infty) = 0$, it is enough to consider the case where $s(\infty) = 1$. In this case, $\infty \in K_2$. If $\infty \notin N_2$, then we would have that $K_2 \cap N_2^c$ is a clopen set containing ∞. but $K_2 \cap N_2^c$ must therefore contain some principal prime ideal m, which would contradict the assumption that $s \wedge \chi_{\{m\}} \leqslant t$.

An interesting feature of this example is that the equalizer $[\![f = 0]\!]$ appears to be the complement of the open set $\beta(I)$. But $\beta(I)$ is not clopen. Does this not imply that this is an equalizer which is not open? The point here is that the weak Boolean product representation of L is not exactly what one would at first guess. To simplify matters we first identify the Stone space of the centre of L with X. The stalks L_m for $m = [\leftarrow, \{2n+1\}^c]$ are three element chains, the stalks L_m for $m = [\leftarrow, \{2n\}^c]$ are two element chains and the stalk L_∞ is a three element chain. The subtle point is that in the weak Boolean product representation, we have $f \not\equiv_\infty 0$ even though as an element of 3^X we have $f(\infty) = 0$. It turns out that $[\![f = 0]\!] = \beta(I)^c - \{\infty\}$ and $[\![f = 1]\!] = \emptyset$. We are not led to the conclusion that $[\![f = a]\!]$ is the set $\beta(I) \cup \{\infty\}$ (which is not open) since a is not an element of L. □

The above example leads naturally into the work of Cignoli [11]. Before presenting the results, we need some terminology. A lattice L is called *B-completely normal* if (a) L is a bounded distributive lattice, (b) for any $x, y \in L$, there correspond elements $s, t \in Z$ such that $x \wedge s \leqslant y, y \wedge t \leqslant x$, and $s \vee t = 1$. L is a *dual B-completely normal lattice* in case its dual is a B-completely normal lattice.

Next we consider some conditions on a bounded distributive lattice D with centre Z:

(i) For each $x, y \in D$, there corresponds a greatest element $w = x \rightarrow y$ such that $x \wedge w \leqslant y$.
(ii) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.
(iii) For each $x, y \in D$, there corresponds a greatest $e \in Z$ such that $e \wedge x \leqslant y$; this e is denoted $e = x \Rightarrow y$.
(iv) $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$.

D is called a Heyting algebra if it satisfies (i); it is called an L-algebra if it satisfies (i) and (ii). Bounded distributive lattices satisfying (iii) are called B-algebras, and B-algebras that satisfy (iv) are called *BL*-algebras. Finally, D is called a P-algebra if both D and its dual satisfy all four conditions. Note that item (iii) above defines exactly the element of the centre that we denoted by $t{:}s$ in Theorem 4.4(iii). Thus by comparing Theorem 4.4(iii) to condition (iii) above, we see that a bounded distributive lattice is Hausdorff iff it is a B-algebra.

As indicated by [18], Theorem 3.4, p. 199, there is some redundancy in this definition of a P-algebra. For convenience of the reader we herewith restate this Theorem.

THEOREM 8.8 (Epstein and Horn). *For a bounded distributive lattice D, the following are equivalent:*
(i) *D is a P-algebra.*
(ii) *D is a BL-algebra.*
(iii) *Both D and its dual are L-algebras.*
(iv) *Both D and its dual are Heyting algebras, and the prime ideals of D lie in disjoint maximal chains.*
(v) *D is an L-algebra, and its dual is a Stone algebra.*
(vi) *D is an L-algebra whose dual is a central cover lattice; furthermore, if $!x$ denotes the largest central element under x, then $!(x \vee y) = !x \vee !y$.*

EXAMPLE 8.9 Cignoli [11] shows that a bounded distributive lattice is B-completely normal iff it is a weak Boolean product of bounded chains, and that such a lattice is Hausdorff if and only if it is a P-algebra. We have already provided an example of a lattice that is a weakly Hausdorff Boolean product of finite chains, but is not a P-algebra (Example 8.7).

EXAMPLE 8.10 As shown in [18] and [19], n-valued Post algebras and n-valued Lukasiewicz algebras are P-algebras, and therefore Hausdorff. As the directly irreducibles in the varieties of n-valued Post algebras and n-valued Lukasiewicz algebras are finite, and therefore complete, we may apply Corollary 7.11 to show that each of these varieties is closed under MacNeille completions. In fact, the MacNeille completion of an n-valued Post or Lukasiewicz algebra L is given by $\Re L$.

It is well known [11], [17] that any n-valued Post algebra is isomorphic to a Boolean power of an n-element chain C_n. Using the techniques of Section 6 we can develop a similar representation theory for complete n-valued Lukasiewicz algebras. Namely, a complete n-valued Lukasiewicz algebra L is isomorphic to a direct product $L_1 \times \cdots \times L_k$ where each L_i is a Boolean power of an n_i-element chain C_{n_i} with $n_i \leqslant n$.

Note first that if all of the stalks of L are n-element chains, then L is isomorphic to a Boolean power of C_n if and only if the usual Boolean product representation of L contains all constant functions. Next, suppose that L is a complete n-valued Lukasiewicz algebra and all the stalks of L are m-element chains. As L is orthogonally complete and Hausdorff, we may apply Carson's result that equalizers of first order formulas $[\![\varphi(x)]\!]$ are clopen. One can easily produce a formula $\varphi_i(x)$ saying that x is the i^{th} element from the bottom of a chain. Then using the compactness of the Stone space X and the patchwork property, we can show that all constant functions are in the usual Boolean product representation of L. So L is a Boolean power of the chain C_m, and is therefore an m-valued Post algebra.

Finally, assume that L is a complete n-valued Lukasiewicz algebra. Let ϕ_i be the first order formula characterizing an i-element chain. As $[\![\phi_i]\!]$ is clopen, we may partition the Stone space X by a finite number of pairwise disjoint non-empty clopen sets $[\![\phi_{n_1}]\!], \ldots, [\![\phi_{n_k}]\!]$. This gives a direct decomposition $L \cong L_1 \times \cdots \times L_k$, where each L_i is isomorphic to a Boolean power of an n_i-element chain. Note that our earlier remarks on MacNeille completions then imply that any n-valued Lukasiewicz algebra can be densely embedded into a product of Boolean powers of chains.

Finally, we would like to thank the referee for several helpful suggestions, including Theorem 5.6, Example 6.5 and Remark 7.12.

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