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# FREE CENTRAL EXTENSIONS

JOHN HARDING
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ABSTRACT. We show that every lattice $L$ can be embedded into a lattice $M$ in such a way that the neutral elements of $L$ are all central in $M$. Moreover, among all such embeddings there is one which is universal. We call this the free central extension of $L$.

A simple internal characterization of free central extensions is given. An extension $E$ of a lattice $L$ is a free central extension of $L$ iff each neutral element of $L$ is central in $E$ and $E$ is generated by $L \cup B$, where $B$ is the Boolean sublattice of the centre of $E$ generated by the neutral elements of $L$. We further show that the free central extension of a lattice $L$ is an essential extension of $L$ which lies in the variety generated by $L$. These results are proved without the use of the axiom of choice.

In the special case of a distributive lattice, the free central extension of $L$ is what is known in the literature as the free Boolean extension of $L$. This topic has been thoroughly investigated by Chen, Grätzer and Schmidt, and Peremans.

If one allows the axiom of choice, the free central extension of a bounded lattice $L$ has a particularly simple description. It is the algebra of global sections of the Pierce sheaf of $L$ over the Stone space of the Boolean sublattice of $Con(L)$ generated by all congruences of the form $\theta_{a,b}$ where $a, b$ are neutral elements of $L$.

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## 1. Introduction.

Central elements play a significant role in lattice theory. An element $a$ in a bounded lattice $L$ is central if $a$ is complemented and the sublattice generated by $a, b, c$ is distributive for every $b, c$ in $L$. Central elements are significant largely because they correspond to direct decompositions of a bounded lattice. A related notion is that of a neutral element in a lattice $L$. An element $a$ is said to be neutral if the sublattice of $L$ generated by $a, b, c$ is distributive for each $b, c$ in $L$. One is clearly tempted to say that neutral elements are simply central elements without complements. Of course, what is really meant by this type of statement is that any lattice $L$ can be extended to a bounded lattice $M$ so that every element which is neutral in $L$ is central in $M$. It is not obvious that this is the case.

On a related note, a lattice in which every element is neutral is clearly a distributive lattice. It is well known that every distributive lattice can be embedded into a Boolean lattice, i.e. a lattice in which every element is central. This dates back to Stone's representation of a distributive lattice as a ring of sets (clearly a ring of sets can be embedded into a field of sets, i.e. a Boolean algebra). Unfortunately, Stone's representation theorem requires the axiom of choice, or more precisely, the prime ideal theorem for distributive lattices. A first attempt at embedding a distributive lattice into a Boolean algebra without using the axiom of choice was made by MacNeille [7]. There was a gap in his proof which was filled by Peremans [10]. Later Grätzer and Schmidt [5], then Chen [2], also provided such embeddings. A particularly simple embedding provided by Grätzer and Schmidt will play a prominent role in this paper, albeit in a modified form. For a bounded distributive lattice $D$, the sublattice of $Con(D)$ generated by all congruences of the form $\theta_{a,b}$ is a Boolean lattice which contains a sublattice isomorphic to $D$.

For a lattice $L$ we will construct a lattice $M$ so that $L$ is embedded in $M$ and for each neutral element $a$ in $L$, the image of $a$ is central in $M$. This construction is free of the axiom of choice, and we shall see that the lattice $M$ lies in the variety generated by $L$. Therefore, this paper can be regarded as a generalization of [2], [5], [10]. The fundamental observation is as follows. For a bounded lattice $L$, the sublattice $B$ of $Con(L)$ generated by all congruences of the form $\theta_{a,b}$, where $a, b$ are neutral in $L$, is a Boolean lattice. Then for a bounded lattice $L$, the extension $M$ is the algebra of global sections of the Pierce sheaf of $L$ over the Stone space of $B$. Of course, some modifications must be made to the presentation of our construction if we are to do without the axiom of choice, but this is only a small obstacle.

This paper is organized in the following fashion. In the second section, we give some preliminary definitions and results which will be needed later. In the third section, we define the notion of a free central extension of a lattice. This is an extension which is universal among all mappings $f : L \longrightarrow M$ for which the neutral elements of $L$ are mapped to central elements of $M$. We also prove some preliminary results, namely that up to a unique isomorphism, each lattice can have at most one free central extension.

In the fourth section, we give some conditions which are sufficient to guarantee that an extension of a lattice $L$ is a free central extension of $L$. Later we shall see that these conditions are also necessary. In the fifth section we construct the free central extension of a bounded lattice. This construction is essentially the Pierce sheaf construction mentioned above, only phrased in a manner which does not use the axiom of choice. In the final section we combine our earlier results. We show that every lattice has a free central extension, give a simple internal characterization of free central extensions, and show that the free central extension of a lattice $L$ is an essential extension of $L$ which lies in the variety generated by $L$.

## 2. Preliminaries.

Unless explicitly stated, we shall not assume that a lattice has either a largest or a least element. However, if a lattice $L$ has a largest element (a unit) we shall refer to this largest element as $1_L$. Similarly, if $L$ has a least element (a zero) we shall refer to this least element as $0_L$. If $L$ has both a largest and a least element, we say that $L$ is a bounded lattice. The notation $a + b$ is used to indicate the join of $a, b$, while $\sum a_i$ is used to denote the join of a finite family $(a_i)_I$. The notation $a \cdot b$ or simply $ab$ is used to denote the meet of $a, b$. We shall never have occasion to consider infinite joins or meets. By 'map' we mean lattice homomorphism.

**Definition 2.1.** Let $L$ be a lattice. We say that an element $x$ in $L$ is *neutral* if for every $y, z$ in $L$ the sublattice generated by $x, y, z$ is distributive. We denote the set of all neutral elements of $L$ by $N(L)$. Note that if $L$ has a largest element, then this element must be neutral, and if $L$ has a least element, then this element must be neutral.

**Definition 2.2.** Let $L$ be a bounded lattice. We say that an element $x$ in $L$ is *central* if $x$ is neutral and $x$ has a complement in $L$. We denote the set of all central elements of $L$ by $Z(L)$. Note that the bounds $0_L$ and $1_L$ are both central.

The following propositions are well known. Proofs can be found in [8].

**Proposition 2.3.** *Let L be a lattice. The set $N(L)$ of all neutral elements of L is a distributive sublattice of L.*

**Proposition 2.4.** *Let L be a bounded lattice. The set $Z(L)$ of all central elements of L is a Boolean sublattice of L.*

**Definition 2.5.** Let $f : L \longrightarrow M$. We say that $f$ *preserves existing bounds* if it satisfies the following conditions

(i) If $L$ has a unit $1_L$, then $M$ has a unit $1_M$ and $f(1_L) = 1_M$.
(ii) If $L$ has a zero $0_L$, then $M$ has a zero $0_M$ and $f(0_L) = 0_M$.

**Definition 2.6.** Let $f : L \longrightarrow M$. We say that $f$ is a *neutral map* if $f$ preserves existing bounds and $f[N(L)] \subseteq N(M)$. A neutral map which happens to be a lattice embedding will be called a *neutral embedding.*

**Definition 2.7.** Let $f : L \longrightarrow M$ where $M$ is a bounded lattice. We say that $f$ is a *central map* if $f$ preserves existing bounds and $f[N(L)] \subseteq Z(M)$. A central map which happens to be a lattice embedding will be called a *central embedding.*

The following observations are easily established.

**Lemma 2.8.**

(i) *Any central map is neutral.*
(ii) *The identity map on $M$ is central iff $M$ is bounded and $N(M) = Z(M)$.*
(iii) *If $f : L \longrightarrow M$ is neutral and $g : M \longrightarrow N$ is neutral, then $g \circ f$ is neutral.*
(iv) *If $f : L \longrightarrow M$ is neutral and $g : M \longrightarrow N$ is central, then $g \circ f$ is central.*

**Definition 2.9.** Let $L$ be a lattice. Construct a bounded lattice $L^*$ as follows: *if $L$ has a zero and a unit then $L^*$ is equal to $L$, if $L$ has a zero but no unit then $L^*$ is $L$ with a unit adjoined, if $L$ has a unit but no zero then $L^*$ is $L$ with a zero adjoined, and if $L$ has neither a zero nor a unit then $L^*$ is $L$ with a zero and a unit adjoined.* Let $*_L : L \longrightarrow L^*$ be the identical embedding. Note that $*_L$ preserves existing bounds.

**Definition 2.10.** Let $f : L \longrightarrow M$ be a lattice embedding. We say that $f : L \longrightarrow M$ is an *essential extension* of $L$ if for every $g : M \longrightarrow K$ we have that $g \circ f$ is an embedding only if $g$ is an embedding.

**Lemma 2.11.**

(i) *The map $*_L : L \longrightarrow L^*$ is an essential extension.*
(ii) *The map $*_L : L \longrightarrow L^*$ is a neutral embedding.*
(iii) *If $g : L \longrightarrow K$ is a central map, then there exists exactly one central map $g^* : L^* \longrightarrow K$ with $g^* \circ *_L = g$.*

*Proof.* (i) Assume that $g : L^* \longrightarrow K$ is not an embedding. We must show that $g \circ *_L$ is not an embedding. This is obvious if $L$ has a zero and a unit, since this implies that $L^* = L$. Assume that $L$ has no unit and that $g(1_{L^*}) = g(x)$ for some $x$ in $L$. Then $g(y) = g(x)$ for all $y > x$ in $L$, showing that $g \circ *_L$ is not an embedding. The other cases are obviously similar.

(ii) This is a consequence of the following general fact. If $x, y, z$ are elements of a lattice $M$ and $z$ is a bound of $M$, then the sublattice generated by $x, y, z$ is distributive.

(iii) We extend $g$ to a map $g^* : L^* \longrightarrow K$ by mapping the bounds of $L^*$ to the bounds of $K$. Clearly this is the only possible bound preserving extension of $g$. We must only show that $g^*$ is a central map. But this follows easily as $N(L^*) = N(L) \cup \{0_{L^*}, 1_{L^*}\}$. □

**Lemma 2.12.** *$L^*$ is in the variety generated by $L$.*

*Proof.* It is well known that the ideal lattice $\Im L$ of $L$ is in the variety generated by $L$ ([3], pg. 69), and the proof of this result depends in no way on the axiom of choice. But $L^*$ is clearly isomorphic to a sublattice of $\Im L$. □

## 3. Free central extensions.

**Definition 3.1.** Let $L$ be a lattice. *A free central extension* of $L$ is a pair $(E, f)$ such that

(i) The map $f : L \longrightarrow E$ is a central embedding.
(ii) $N(E) = Z(E)$.
(iii) For any central map $g : L \longrightarrow K$, there is exactly one central map $h : E \longrightarrow K$ with $h \circ f = g$.

Later, in Theorem 6.2, we will show that the second condition is a consequence of the other two.

**Proposition 3.2.** *Let $L$ be a lattice. If $(E, f)$ and $(E', f')$ are free central extensions of $L$ then there is exactly one isomorphism $i : E \longrightarrow E'$ such that $i \circ f = i'$.*

*Proof.* As $(E, f)$ is a free central extension and $f' : L \longrightarrow E'$ is central, there is exactly one central map $h : E \longrightarrow E'$ with $h \circ f = f'$. Dually, there is exactly one central map $h' : E' \longrightarrow E$ with $h' \circ f' = f$. Then $h' \circ h : E \longrightarrow E$ is a central map, by Lemma 2.8 (iv), and $(h' \circ h) \circ f = h' \circ f' = f$. But the identity map $id_E : E \longrightarrow E$ is central, by Lemma 2.8 (ii), and $id_E \circ f = f$. The uniqueness condition of free central extensions then gives us that $h' \circ h = id_E$. Dually, $h \circ h' = id_{E'}$. So $h : E \longrightarrow E'$ is an isomorphism with $h \circ f = f'$. Uniqueness again follows as $(E, f)$ is a free central extension. $\square$

**Proposition 3.3.** *Let $L$ be a lattice. If $(E, f)$ is a free central extension of $L^*$, then $(E, f \circ *_L)$ is a free central extension of $L$.*

*Proof.* We must verify conditions (i) through (iii) of Definition 3.1.

(i) We must show that $f \circ *_L : L \longrightarrow E$ is central. But by Lemma 2.11 (ii) the map $*_L : L \longrightarrow L^*$ is neutral, and as $(E, f)$ is a free central extension we have that $f : L^* \longrightarrow E$ is central. Our result then follows by Lemma 2.8 (iv).

(ii) That $N(E) = Z(E)$ follows immediately as $(E, f)$ is a free central extension of $L^*$.

(iii) We must show that if $g : L \longrightarrow K$ is central then there is exactly one central map $h : E \longrightarrow K$ with $h \circ (f \circ *_L) = g$. But by Lemma 2.11 (iii) there is a central map $g^* : L^* \longrightarrow K$ with $g^* \circ *_L = g$. As $(E, f)$ is a free central extension of $L^*$, there is a central map $h : E \longrightarrow K$ with $h \circ f = g^*$. It follows that $h \circ (f \circ *_L) = g$. We have only to show that $h$ is the only such map. Assume that $h' : E \longrightarrow K$ is a central map with $h' \circ (f \circ *_L) = g$. Then by Lemma 2.8 (iv) $h' \circ f$ is central, and by the uniqueness clause in Lemma 2.11 (iii) we have that $h' \circ f = g^*$. Then by the uniqueness of maps in free central extensions we have that $h' = h$. □

## 4. The structure of free central extensions.

**Definition 4.1.** Let $C$ be a Boolean algebra and let $X$ be a finite subset of $C$. We say that $X$ is a *partition* of $C$ if

$$\sum X = 1_C \text{ and } x \cdot y = 0 \text{ for all } x \neq y \text{ in } X.$$

Let $D$ be a sublattice of $C$ such that $C$ is generated as a Boolean algebra by $D$, and $D$ contains $0_C$, $1_C$. We say that a partition $X$ of $C$ is *basic over* $D$ if each $x \in X$ can be expressed in the form $x = c \cdot d'$ where $d < c \in D$. If the choice of sublattice $D$ is clear from the context, we will simply say that $X$ is a basic partition of $C$. Finally, if $X$ and $Y$ are partitions, we say that $Y$ refines $X$ if for each non-zero $y \in Y$ there is $x \in X$ with $y \leq x$.

**Lemma 4.2.** *Let $C$ be a Boolean algebra, $D$ be a sublattice of $C$ which generates $C$ as a Boolean algebra and contains $0_C$ and $1_C$, and $X, Y, Z$ be partitions of $C$.*

(i) *If $Z$ refines $Y$ and $Y$ refines $X$, then $Z$ refines $X$.*
(ii) *Any two partitions $X$,$Y$ of $C$ have a common refinement.*
(iii) *Any $b \in C$ can be expressed as $b = \sum c_i d_i'$ where $d_i < c_i \in D$ and $c_i d_i' \cdot c_j d_j' = 0$ for $i \neq j$.*
(iv) *Any partition $X$ of $C$ can be refined by a partition which is basic over $D$.*

*Proof.* (i) Obvious.

(ii) $\{x \cdot y : x \in X, y \in Y\}$ is a refinement of $X$ and $Y$.

(iii) As $C$ is generated as a Boolean algebra by $D$, it follows that each $b \in C$ can be represented as

$$b = \sum x_i \text{ where } x_i = \prod x_{ij} \text{ with } x_{ij} \text{ or } x_{ij}' \in D \text{ for each } i, j.$$

As $D$ is a sublattice of $C$ which contains both $0_C$ and $1_C$ we can find families $c_i, d_i$ in $D$ with $x_i = c_i d_i'$ for each $i \in I$. Further, as $c_i d_i' = (c_i + d_i)d_i'$, we may assume that $c_i, d_i$ have been chosen so that $d_i < c_i$. The statement we are to prove follows easily if $C$ is a finite Boolean algebra, as each atom in $C$ will have a representation in the form $cd'$ for some $d < c \in D$.

For an infinite Boolean algebra $C$, we have noted that any $b \in C$ has a representation $b = \sum c_i d_i'$ where $c_i, d_i$ are a finite family from $D$. Let $D'$ be the sublattice of $D$ generated by $\{c_i, d_i : i \in I\}$ and $C'$ to be the Boolean sublattice of $C$ generated by $D'$. As finitely generated distributive lattices and finitely generated Boolean algebras are finite, we have that $C'$ and $D'$ are finite. But $b \in C'$. Using the above remarks about finite Boolean algebras, we can express $b$ in the form

$$b = \sum c_i d_i' \text{ where } d_i < c_i \in D' \text{ and } c_i d_i' \cdot c_j d_j' = 0 \text{ for } i \neq j.$$

As $D'$ is contained in $D$ our result is established.

(iv) Let $X$ be a partition of $C$. Applying the third part of this Lemma to each $x \in X$ we obtain a partition which is basic over $D$ and refines $X$. □

**Proposition 4.3.** *Let $C$ be a Boolean algebra and $D$ be a sublattice of $C$ which generates $C$ as a Boolean algebra and contains $0_C$, $1_C$. If $K$ is a Boolean algebra and $\alpha : D \longrightarrow K$ is a bound preserving map, then there is exactly one map $\beta : C \longrightarrow K$ which extends $\alpha$. This map $\beta$ satisfies $\beta(\sum c_i d_i') = \sum \alpha(c_i)\alpha(d_i)'$.*

*Proof.* This is a well known result, however many common proofs use the axiom of choice. This is not necessary. We have only one possible candidate for this mapping, and it is only a matter of verifying that this map is well defined and is a Boolean algebra homomorphism. The details are left to the reader. □

*Assumptions*: Throughout the remainder of this section we assume that $L$ is a sublattice of a bounded lattice $E$, so that $L$ contains $0_E$, $1_E$ and satisfies

(i) $N(L)$ is a sublattice of $Z(E)$.
(ii) $E$ is generated by $L \cup B$, where $B$ is the Boolean sublattice of $Z(E)$ generated by $N(L)$.

**Lemma 4.4.**

(i) *Let $e, f \in E$ and $X$ be a partition of $B$. If $e \cdot x = f \cdot x$ for all non-zero $x \in X$, then $e = f$.*

(ii) *Let $(x_i)_I$ and $(y_j)_J$ be partitions of $B$ with $(y_j)_J$ refining $(x_i)_I$. Then for any family $(p_i)_I$ in $L$ there is a family $(q_j)_J$ in $L$ with $\sum p_i x_i = \sum q_j y_j$.*

(iii) *Let $(x_i)_I$ be a partition of $B$ and let $(p_i)_I$ and $(q_i)_I$ be families in $L$. Then $(\sum p_i x_i) + (\sum q_i x_i) = \sum (p_i + q_i) x_i$.*

(iv) *Let $(x_i)_I$ be a partition of $B$ and let $(p_i)_I$ and $(q_i)_I$ be families in $L$. Then $(\sum p_i x_i) \cdot (\sum q_i x_i) = \sum (p_i \cdot q_i) x_i$.*

*Proof.* (i) As $X$ is contained in the centre of $E$ we may distribute freely, so

$$e = e \cdot 1 = e \cdot \sum X = \sum_X e \cdot x = \sum_X f \cdot x = f.$$

(ii) For each $j \in J$ with $y_j \neq 0$, set $q_j = p_i$ if $y_j \leq x_i$. If $y_j = 0$, make an arbitrary choice for $q_j$. As $y_k$ is central and $y_k \cdot y_j = 0$ for $j \neq k$, it follows that $y_k \cdot \sum q_j y_j$ is equal to $y_k \cdot \sum p_i x_i$ for all $k \in J$. Our result then follows from the first part of this Lemma.

(iii) Again by the first part of this Lemma, it is enough to show that for each $k \in J$

$$x_k \cdot [(\textstyle\sum p_i x_i) + (\sum q_i x_i)] = x_k \cdot [\sum (p_i + q_i) x_i].$$

But $x_k$ is central and $x_k \cdot x_i = 0$ for $i \neq k$, so the left side of this expression is equal to $(p_k \cdot x_k) + (q_k \cdot x_k)$ and the right side of this expression is equal to $(p_k + q_k) \cdot x_k$. Using once more the fact that $x_k$ is central, we have our result.

(iv) This follows along similar lines to (iii). $\square$

**Proposition 4.5.**

(i) *$E = \{\sum p_i x_i : (x_i)_I$ is a basic partition of $B$ and each $p_i \in L\}$.*

(ii) *If $a^1, \dots, a^n \in E$, then there is a basic partition $(x_i)_I$ and elements $a_i^j \in L$ so that $a^j = \sum a_i^j x_i$ for each $j \leq n$.*

(iii) *$E$ is in the variety generated by $L$.*

*Proof.* (i) Let

$$Q = \{\textstyle\sum p_i x_i : (x_i)_I \text{ is a basic partition of } B \text{ and each } p_i \in L\} .$$

Then $Q$ contains $L$ since $p = p(1 \cdot 0')$ for any $p \in L$. For any $b \in B$ we have by Lemma 4.2 (iii) that

$$b = \textstyle\sum c_i d_i' \text{ where } d_i < c_i \in D \text{ and } c_i d_i' \cdot c_j d_j' = 0 \text{ for } i \neq j \text{ and}$$
$$b' = \textstyle\sum e_k f_k' \text{ where } f_k < e_k \in D \text{ and } e_k f_k' \cdot e_l f_l' = 0 \text{ for } k \neq l.$$

So

$$b = \textstyle\sum 1 c_i d_i' + \sum 0 e_k f_k',$$

giving that $b \in Q$.

Once we have established that $Q$ is closed under joins and meets, we will have that $E = Q$, since we assumed that $E$ was generated by $L \cup B$. Taking $a = \sum p_i x_i$ and $b = \sum q_j y_j$, by Lemma 4.2 we can find a basic partition $(z_k)_K$ which refines both $(x_i)_I$ and $(y_j)_J$. Then by Lemma 4.4 (ii) there are families $(r_k)_K$ and $(s_k)_K$ in $L$ with $a = \sum r_k z_k$ and $b = \sum s_k z_k$. It then follows from parts (iii) and (iv) of Lemma 4.4 that $a \cdot b$ and $a + b$ are both elements of $Q$.

(ii) Using the first part of this Lemma, each $a^j$ can be expressed as a sum over some partition of $B$. A simple induction using Lemma 4.2 (i) and (ii) shows that we can find a common refinement $(z_k)_K$ of these partitions. Then by Lemma 4.2 (i) and (iv) we can find a common refinement of these partitions by a basic partition $(x_i)_I$. Our result then follows by Lemma 4.4 (ii).

(iii) Let $u$ and $v$ be n-ary lattice terms such that $L$ satisfies the identity $u(\vec{x}) \approx v(\vec{x})$. We must show that $E$ also satisfies this identity. Let $a^1, \dots, a^n$ be elements of $E$. Using the second part of this Lemma, we can find a partition $(x_k)_K$ and elements $p_k^i \in L$ so that $a^i = \sum p_k^i x_k$. It follows from parts (iii) and (iv) of Lemma 4.4 that

$$u(a^1, \dots, a^n) = \textstyle\sum u(p_k^1, \dots, p_k^n) x_k, \text{ and}$$
$$v(a^1, \dots, a^n) = \textstyle\sum v(p_k^1, \dots, p_k^n) x_k.$$

Then the equality $u(a^1, \dots, a^n) = v(a^1, \dots, a^n)$ follows as $u(p_k^1, \dots, p_k^n)$ is equal to $v(p_k^1, \dots, p_k^n)$ for each $k \in K$. $\square$

**Lemma 4.6.** *Let $(c_i d_i')_I$ and $(e_j f_j')_J$ be basic partitions of $B$ and $(p_i)_I$ and $(q_j)_J$ be families in $L$. Then for any central map $\wedge : L \longrightarrow K$*

(i) *$(\hat{c}_i \hat{d}_i{}')_I$ is a partition of $Z(K)$.*

(ii) *If $(e_j f_j')_J$ refines $(c_i d_i')_I$ and $\sum p_i c_i d_i' = \sum q_j e_j f_j'$, then $\sum \hat{p}_i \hat{c}_i \hat{d}_i{}' = \sum \hat{q}_j \hat{e}_j \hat{f}_j{}'$.*

(iii) *We may define a map $h : E \longrightarrow K$ by setting $h(a) = \sum \hat{p}_i \hat{c}_i \hat{d}_i{}'$ where $a = \sum p_i c_i d_i'$.*

(iv) *There is exactly one map $h : E \longrightarrow K$ which extends $\wedge$.*

*Proof.* (i) This follows from Proposition 4.3 as the map $\beta : B \longrightarrow Z(K)$ which extends $\wedge : N(L) \longrightarrow K$ must satisfy $\beta(cd') = \hat{c} \cdot \hat{d}{}'$.

(ii) As $(\hat{e}_j \hat{f}_j{}')_J$ is a partition, by Lemma 4.4 (i) it is enough to show that for each $j \in J$

$$\hat{e}_j \hat{f}_j{}' \cdot \textstyle\sum \hat{p}_i \hat{c}_i \hat{d}_i{}' = \hat{e}_j \hat{f}_j{}' \cdot \sum \hat{q}_j \hat{e}_j \hat{f}_j{}'.$$

Choosing $i$ such that $e_j f_j' \leq c_i d_i'$ we have that

$$\hat{e}_j \hat{f}_j{}' \cdot \textstyle\sum \hat{p}_i \hat{c}_i \hat{d}_i{}' = \hat{p}_i \hat{e}_j \hat{f}_j{}' \quad \text{and} \quad \hat{e}_j \hat{f}_j{}' \cdot \sum \hat{q}_j \hat{e}_j \hat{f}_j{}' = \hat{q}_j \hat{e}_j \hat{f}_j{}'.$$

However, we have assumed that $\sum p_i c_i d_i' = \sum q_j e_j f_j'$. Taking the meet of both sides with $e_j f_j'$ we have that $p_i e_j f_j' = q_j e_j f_j'$. It follows that $p_i e_j + f_j$ is equal to $q_j e_j + f_j$, and therefore $\hat{p}_i \hat{e}_j + \hat{f}_j = \hat{q}_j \hat{e}_j + \hat{f}_j$. Taking the meet with $\hat{f}_j{}'$ gives $\hat{p}_i \hat{e}_j \hat{f}_j{}' = \hat{q}_j \hat{e}_j \hat{f}_j{}'$. This yields the desired equality.

(iii) Let $a$ be an element of $E$ and suppose that

$$a = \textstyle\sum p_i c_i d_i' \text{ and } a = \sum q_j e_j f_j'$$

are two representations of $a$ with $(c_i d_i')_I$ and $(e_j f_j')_J$ basic partitions. By Lemma 4.2 there is a basic partition $(r_k s_k')_K$ refining both of these partitions. Then by part (ii) of Lemma 4.4 there is a family $(t_k)_K$ so that $a = \sum t_k r_k s_k'$. Using the second part of this Lemma, we have

$$\textstyle\sum \hat{p}_i \hat{c}_i \hat{d}_i{}' = \sum \hat{t}_k \hat{r}_k \hat{s}_k{}' = \sum \hat{q}_j \hat{e}_j \hat{f}_j{}'.$$

Therefore the value of $h(a)$ is independent of the particular representation of $a$.

We have only to show that $h$ is a lattice homomorphism. Let $a, b$ be elements of $E$. By Proposition 4.5 (ii) there is a basic partition $(c_i d_i')_I$ and families $(p_i)_I$ and $(q_i)_I$ in $L$ with $a = \sum p_i c_i d_i'$ and $b = \sum q_i c_i d_i'$. Then by Lemma 4.4 (iii) we have that

$$a + b = \sum (p_i + q_i) c_i d_i'.$$

To show that $h(a) + h(b)$ is equal to $h(a+b)$, it is sufficient to show that for each $i \in I$

$$[h(a) + h(b)] \cdot \hat{c}_i \hat{d}_i{}' = h(a+b) \cdot \hat{c}_i \hat{d}_i{}'.$$

But it is easy to verify that both of these quantities are $(\hat{p}_i + \hat{q}_i) \cdot \hat{c}_i \hat{d}_i{}'$. A similar argument shows that $h$ preserves meets.

(iv) The map from the third part of this Lemma extends $\wedge$, as any $p \in L$ can be represented as $p = p(1 \cdot 0')$. If $k : E \longrightarrow K$ is another map extending $\wedge$, then for any $d \in N(L)$ we have that $k(d) = \hat{d}$ is in $Z(K)$, and therefore $k(d')$ must be the unique complement $\hat{d}'$ of $\hat{d}$. Therefore $k$ agrees with $h$ on $L \cup \{d' : d \in N(L)\}$. As this is a generating set of $E$, we have that $k = h$. □

**Lemma 4.7.** $N(E) = Z(E) = B$.

*Proof.* Clearly $N(E) \supseteq Z(E) \supseteq B$. Let $a \in N(E)$. By Proposition 4.5 (i) we can find a basic partition $(c_i d_i')_I$ and a family $(p_i)_I$ in $L$ so that

$$a = \sum p_i c_i d_i'.$$

Taking the meet of $a$ with $c_i d_i'$ we have that $p_i c_i d_i'$ is neutral (recall that $N(E)$ is a sublattice of $E$). Taking the join of this element with $d_i$ we have that $(p_i c_i) + d_i$ is neutral. But this is an element of $L$ and $N(L) \subseteq B$, so $(p_i c_i) + d_i$ is in $B$. Taking the meet of this element with $d_i'$ we have that $p_i c_i d_i'$ is in $B$ and therefore $a = \sum p_i c_i d_i'$ is in $B$. □

**Proposition 4.8.** *$(E, id_L)$ is a free central extension of $L$.*

*Proof.* We must show that $id_L : L \longrightarrow E$ is central, that $N(E) = Z(E)$, and that for any central map $g : L \longrightarrow K$ there is exactly one central map

$h : E \longrightarrow K$ with $h \circ id_L = g$. The first condition we have assumed. The second is Lemma 4.7. If $g : L \to K$ is a central map, then by Lemma 4.6 (iv) there is exactly one map $h : E \to K$ which extends $g$. We must show that $h$ is central. As $h$ extends $g$ we have that $h[N(L)] \subseteq Z(K)$. But by Lemma 4.7, $Z(E)$ is generated as a Boolean algebra by $N(L)$. It follows that $h[Z(E)] \subseteq Z(K)$. As $N(E) = Z(E)$, our result follows. $\square$

**Proposition 4.9.** *$(E, id_L)$ is an essential extension of $L$.*

*Proof.* Assume that $g : E \longrightarrow K$ and that the restriction of $g$ to $L$ is an embedding. We must show that $g$ is an embedding. Let $a, b \in E$. By Proposition 4.5 (ii) there is a basic partition $(c_i d_i')_I$ and families $(p_i)_I$ and $(q_i)_I$ in $L$ with

$$a = \sum p_i c_i d_i' \text{ and } b = \sum q_i c_i d_i'.$$

If $g(a) = g(b)$, then $g(ac_i d_i')$ is equal to $g(bc_i d_i')$, which implies that $g(p_i c_i d_i')$ is equal to $g(q_i c_i d_i')$. Taking the joins of these elements with $g(d_i)$ we have that

$$g((p_i c_i) + d_i) = g((q_i c_i) + d_i).$$

But we have assumed that the restriction of $g$ to $L$ is an embedding, which implies that $(p_i c_i) + d_i$ is equal to $(q_i c_i) + d_i$. Taking the meets of these elements with $d_i'$ we have that $p_i c_i d_i'$ is equal to $q_i c_i d_i'$, and therefore that $a = b$. $\square$

**Proposition 4.10.** *Let $L$ be a lattice and suppose $f : L \longrightarrow E$ satisfies*

(i) *$f$ is a central embedding.*
(ii) *$E$ is generated by $f[L] \cup B$ where $B$ is the Boolean sublattice of $Z(E)$ generated by $f[N(L)]$.*

*Then $(E, f)$ is a free central extension of $L$, the centre of $E$ is equal to $B$, $(E, f)$ is an essential extension of $L$, and $E$ is in the variety generated by $L$.*

*Proof.* By Lemma 2.11 (ii) there is a central map $f^* : L^* \longrightarrow E$ with $f^* \circ *_L = f$ and this map $f^*$ is clearly an embedding. It follows that the bounded sublattice $f^*[L^*]$ of $E$ satisfies the assumptions of this section. Then by Propositions 4.8, 4.9 and 4.5 (iii) we have that $(E, id_{f^*[L^*]})$ is a free central extension of $f^*[L^*]$, an essential extension of $f^*[L^*]$ and that

$E$ is in the variety generated by $f^*[L^*]$. Further, by Lemma 4.7, the centre of $E$ is equal to $B$.

As $f^* : L^* \longrightarrow f^*[L^*]$ is an isomorphism, $(E, f^*)$ is both a free central extension and an essential extension of $L^*$. That $(E, f)$ is a free central extension of $L$ follows by Proposition 3.3. That $(E, f)$ is an essential extension of $L$ follows from Lemma 2.11 (i) and the fact that the composition of essential extensions is an essential extension. That $E$ is in the variety generated by $L$ follows from the fact that $E$ is in the variety generated by $f^*[L^*]$, that $f^*[L^*]$ is isomorphic to $L^*$ and that $L^*$ is in the variety generated by $L$, the latter being provided by Lemma 2.12. □

## 5. The existence of free central extensions.

**Definition 5.1.** Let $L$ be a lattice. For each $a \in N(L)$ define

$$\Delta_a = \{(x,y) \in L^2 : x + a = y + a\}, \quad \nabla_a = \{(x,y) \in L^2 : x \cdot a = y \cdot a\}.$$

**Lemma 5.2.** *Let $L$ be a lattice and $a, b \in N(L)$.*

(i) *$\Delta_a$ and $\nabla_a$ are congruences on $L$.*
(ii) *If $a \leq b$, then $\nabla_a \cap \Delta_b = \theta_{a,b}$.*
(iii) *$\nabla_a$ and $\Delta_a$ are complementary in $Con(L)$.*
(iv) *$\Delta_a \cap \Delta_b = \Delta_{ab}$ and $\Delta_a + \Delta_b = \Delta_{a+b}$.*

*We are using $\theta_{a,b}$ to denote the congruence generated by $(a, b)$ and $Con(L)$ to denote the congruence lattice of $L$.*

*Proof.* (i) It is clear that $\Delta_a$ is an equivalence relation on $L$. Assume that $(x, y)$ and $(x', y')$ are in $\Delta_a$. Then

$$(x + x') + a = (x + a) + (x' + a) = (y + a) + (y' + a) = (y + y') + a$$

which implies that $(x + x', y + y')$ is in $\Delta_a$. Also, as $a$ is neutral

$$(xx') + a = (x + a)(x' + a) = (y + a)(y' + a) = (yy') + a$$

which implies that $(xx', yy')$ is in $\Delta_a$. Thus $\Delta_a$ is a congruence. A similar argument shows that $\nabla_a$ is also a congruence.

(ii) It is clear that if $a \leq b$ then $(a, b) \in \nabla_a \cap \Delta_b$, so $\theta_{a,b} \subseteq \nabla_a \cap \Delta_b$. If $(x, y) \in \nabla_a \cap \Delta_b$, then

$$\begin{aligned} x &= x(x+b) = x(y+b), \\ y &= y(y+b) = y(x+b), \text{ and} \\ (xy) + (xa) &= (xy) + (ya). \end{aligned}$$

Using the last of these identities and the fact that $a$ is neutral, we have

$$x(y+a) = (xy) + (xa) = (xy) + (ya) = y(x+a).$$

Therefore

$$x = x(y+b)\theta_{a,b}\; x(y+a) = y(x+a)\; \theta_{a,b}\; y(x+b) = y$$

which implies that $(x, y) \in \theta_{a,b}$.

(iii) By part (ii) we have that $\nabla_a \cap \Delta_a = \theta_{a,a}$ the zero of $Con(L)$. For any $x, y \in L$ we have that

$$x \; \Delta_a \;\; (x+a) \; \nabla_a \;\; (y+a) \, \Delta_a \;\; y.$$

So $\Delta_a + \nabla_a$ is the unit of $Con(L)$.

(iv) It is clear that if $a \leq b$, then $\Delta_a \subseteq \Delta_b$. It follows that $\Delta_{ab}$ is contained in $\Delta_a \cap \Delta_b$ and that $\Delta_a + \Delta_b$ is contained in $\Delta_{a+b}$. If $(x, y) \in \Delta_a \cap \Delta_b$, then as $a, b$ are neutral, it follows that

$$x + (ab) = (x+a)(x+b) = (y+a)(y+b) = y + (ab)$$

which implies that $(x, y) \in \Delta_{ab}$, and therefore $\Delta_{ab} = \Delta_a \cap \Delta_b$. We have only to show that $\Delta_{a+b}$ is contained in $\Delta_a + \Delta_b$. For any $x \in L$ we have that

$$x \, \Delta_a \;\; (x+a) \; \Delta_b \;\; (x+a+b)$$

and therefore $(x, x+a+b) \in \Delta_a + \Delta_b$. So if $(x, y) \in \Delta_{a+b}$, then $x + a + b = y + a + b$, and therefore $(x, y) \in \Delta_a + \Delta_b$. □

**Definition 5.3.** Let $L$ be a lattice, and define $C$ to be the sublattice of $Con(L)$ which is generated by $\{\Delta_a, \nabla_a : a \in N(L)\}$. Note that by Lemma 5.2 (iii), $C$ is a Boolean sublattice of $Con(L)$.

**Definition 5.4.** Define $F$ to be all formal sums $\sum p_i\theta_i$ where $(\theta_i)_I$ is a partition of $C$ and $(p_i)_I$ is a family of elements of $L$. We define a relation $\simeq$ on $F$ by setting

$$\sum p_i\theta_i \simeq \sum q_j\phi_j$$

if there is a partition $(\lambda_k)_K$ such that

(i) $(\lambda_k)_K$ refines $(\theta_i)_I$ and $(\phi_j)_J$.
(ii) $(p_i, q_j) \in \lambda'_k$ if $\lambda_k \leq \theta_i, \phi_j$.

We say that a partition such as $(\lambda_k)_K$ *witnesses the equivalence* $\sum p_i\theta_i \simeq \sum q_j\phi_j$. Once we have shown that $\simeq$ is an equivalence relation on $F$, we will denote the equivalence class of $\sum p_i\theta_i$ by $[\sum p_i\theta_i]$.

**Lemma 5.5.**

(i) *If $(\lambda_k)_K$ witnesses $\sum p_i\theta_i \simeq \sum q_j\phi_j$ and $(\delta_m)_M$ is a refinement of $(\lambda_k)_K$, then $(\delta_m)_M$ also witnesses $\sum p_i\theta_i \simeq \sum q_j\phi_j$.*
(ii) *$\simeq$ is an equivalence relation on $F$.*
(iii) *If $(\lambda_k)_K$ is a refinement of $(\theta_i)_I$, then for any family $(p_i)_I$ in $L$ there is a family $(q_k)_K$ in $L$ with $(\lambda_k)_K$ witnessing $\sum p_i\theta_i \simeq \sum q_k\lambda_k$.*
(iv) *For any $a^1, \dots, a^n \in F/\simeq$ there is a partition $(\theta_i)_I$ and elements $a_i^j \in L$ so that $a^j = [\sum a_i^j\theta_i]$.*
(v) *If $(\lambda_k)_K$ witnesses $\sum p_i\theta_i \simeq \sum r_j\phi_j$ and $\sum q_i\theta_i \simeq \sum s_j\phi_j$, then $(\lambda_k)_K$ also witnesses $\sum(p_i \cdot q_i)\theta_i \simeq \sum(r_j \cdot s_j)\phi_j$ and $\sum(p_i + q_i)\theta_i \simeq \sum(r_j + s_j)\phi_j$.*

*Proof.* (i) It is clear that $(\delta_m)_M$ refines both $(\theta_i)_I$ and $(\phi_j)_J$. Suppose that $\delta_m$ is non-zero and that $\delta_m \leq \theta_i, \phi_j$. Choosing $k$ so that $\delta_m \leq \lambda_k$ we have that $\lambda_k \leq \theta_i, \phi_j$. So $(p_i, q_j) \in \lambda'_k$, and as $\lambda'_k \subseteq \delta'_m$ we have $(p_i, q_j) \in \delta'_m$.

(ii) $\simeq$ is clearly symmetric and $(\theta_i)_I$ witnesses $\sum p_i\theta_i \simeq \sum p_i\theta_i$, so we need only show that $\simeq$ is transitive. Assume that

$$(\lambda_k)_K \text{ witnesses } \sum p_i\theta_i \simeq \sum q_j\phi_j \quad \text{and}$$
$$(\delta_m)_M \text{ witnesses } \sum q_j\phi_j \simeq \sum r_l\psi_l.$$

Choose a common refinement $(\mu_n)_N$ of $(\lambda_k)_K$ and $(\delta_m)_M$. By part (i) we have

$$(\mu_n)_N \text{ witnesses both } \sum p_i\theta_i \simeq \sum q_j\phi_j \text{ and } \sum q_j\phi_j \simeq \sum r_l\psi_l.$$

If $\mu_n \leq \theta_i, \psi_l$, then choosing $j$ so that $\mu_n \leq \phi_j$, we have $(p_i, q_j) \in \mu'_n$ and $(q_j, r_l) \in \mu'_n$. Then as $\mu'_n$ is transitive, $(p_i, r_l) \in \mu'_n$.

(iii) If $0 \neq \lambda_k$, set $q_k = p_i$ if $\lambda_k \leq \theta_i$. If $0 = \lambda_k$ any choice for $q_k$ will suffice.

(iv) Suppose that $a^j = [\sum q_k^j \phi_k^j]$. Choose $(\Theta_i)_I$ to be a common refinement of the $(\Phi_k^j)_{K_j}$ and then apply part (iii).

(v) We clearly have that $(\lambda_k)_K$ refines both $(\theta_i)_I$ and $(\phi_j)_J$. If $\lambda_k \leq \theta_i, \phi_j$, then

$$(p_i, r_j) \in \lambda'_k \text{ and } (q_i, s_j) \in \lambda'_k.$$

As $\lambda'_k$ is a congruence, it follows that

$$(p_i \cdot q_i, r_j \cdot s_j) \in \lambda'_k \text{ and } (p_i + q_i, r_j + s_j) \in \lambda'_k.$$

So $(\lambda_k)_K$ witnesses $\sum(p_i \cdot q_i)\theta_i \simeq \sum(r_j \cdot s_j)\phi_j$ and $\sum(p_i + q_i)\theta_i \simeq \sum(r_j + s_j)\phi_j$. □

**Definition 5.6.** Let $a, b \in F/\simeq$. By Lemma 5.5 (iv), we can find a partition $(\theta_i)_I$ and formal sums $\sum p_i\theta_i$ and $\sum q_i\theta_i$ so that $a = [\sum p_i\theta_i]$ and $b = [\sum q_i\theta_i]$. Define

$$[\textstyle\sum p_i\theta_i] + [\sum q_i\theta_i] = [\sum(p_i + q_i)\theta_i]$$
$$[\textstyle\sum p_i\theta_i] \cdot [\sum q_i\theta_i] = [\sum(p_i \cdot q_i)\theta_i].$$

Note that Lemma 5.5 (v) shows this definition is independent of the particular choice of $(\theta_i)_I$.

**Lemma 5.7.** *Let $a^1, \ldots, a^n \in F/\simeq$ and $t$ be any $n$-ary lattice term. If $(\theta_i)_I$ is a partition and $a_i^j \in L$ are such that $a^j = [\sum a_i^j\theta_i]$, then*

$$t(a^1, \ldots, a^n) = [\textstyle\sum t(a_i^1, \ldots, a_i^n)\theta_i].$$

*Proof.* This follows from Definition 5.6 by an obvious induction. □

**Lemma 5.8.** *Let $L$ be a bounded lattice and let $f : L \longrightarrow F/\simeq$ be defined by $f(p) = [p\Delta_1]$.*

(i) $(F/\simeq, +, \cdot)$ *is a lattice.*

(ii) $f$ *is a bound preserving lattice embedding.*

(iii) $f(a) = [1\Delta_a + 0\nabla_a]$ *for each* $a \in N(L)$.

(iv) $f$ *is a central embedding.*

(v) $f(p)f(a)f(b)' = [0\Delta_b + p(\Delta_a \cdot \nabla_b) + 0\nabla_a]$ *for* $b < a \in N(L)$.

(vi) *For any basic partition* $(\Delta_{a_i} \cdot \nabla_{b_i})_I$ *of* $C$ *and any family of elements* $p_i \in L$, $[\sum p_i(\Delta_{a_i} \cdot \nabla_{b_i})] = \sum[0\Delta_{b_i} + p_i(\Delta_{a_i} \cdot \nabla_{b_i}) + 0\nabla_{a_i}]$.

(vii) $F/\simeq$ *is generated by* $f[L] \cup B$ *where* $B$ *is the Boolean sublattice of* $Z(F/\simeq)$ *generated by* $f[N(L)]$.

*Proof.* (i) By Lemma 5.7, $F/\simeq$ satisfies all identities which hold in $L$. In particular, $F/\simeq$ satisfies the lattice identities.

(ii) It follows from Lemma 5.7 that for any $a \in F/\simeq$ that $a = [1\Delta_1]\cdot a$ and $a = [0\Delta_1] + a$, so $f$ preserves the bounds of $L$. Lemma 5.7 shows that

$$[p\Delta_1] + [q\Delta_1] = [(p+q)\Delta_1] \text{ and } [p\Delta_1]\cdot[q\Delta_1] = [(p\cdot q)\Delta_1]$$

so $f$ is a homomorphism. Finally, if $[p\Delta_1] = [q\Delta_1]$, then there is a partition $(\theta_i)_I$ witnessing $p\Delta_1 \simeq q\Delta_1$. So $(p,q) \in \theta_i'$ for all $i \in I$. Therefore $(p,q) \in \bigcap \theta_i'$ which implies that $p$ is equal to $q$. So $f$ is an embedding.

(iii) $\{\Delta_a, \nabla_a\}$ witnesses $1\Delta_a + 0\nabla_a \simeq a\Delta_1$.

(iv) Let $a$ be an element of $N(L)$, and $b = [\sum p_i\theta_i], c = [\sum q_i\theta_i]$ be elements of $F/\simeq$. Using Lemma 5.5 (iii), we may assume that $(\theta_i)_I$ refines $\{\Delta_a, \nabla_a\}$. Setting $r_i = 1$ if $\theta_i \leq \Delta_a$ and $r_i = 0$ if $\theta_i \leq \nabla_a$, we have that $(\theta_i)_I$ witnesses

$$1\Delta_a + 0\nabla_a \simeq \textstyle\sum r_i\theta_i.$$

Therefore, by the third part of this Lemma, $f(a) = [\sum r_i\theta_i]$. It then follows from Lemma 5.7 that the sublattice generated by $f(a), b, c$ is distributive, so $f(a)$ is neutral. Again using the third part of this Lemma, $[0\Delta_a + 1\nabla_a]$ is a complement of $f(a)$, and therefore $f(a)$ is central.

(v) Note that $\{\Delta_b, \Delta_a \cdot \nabla_b, \nabla_a\}$ is a refinement of $\{\Delta_a, \nabla_a\}$ and $\{\Delta_b, \nabla_b\}$. Using the third part of this Lemma,

$$\begin{aligned} f(a) &= [1\Delta_b + 1(\Delta_a \cdot \nabla_b) + 0\nabla_a] \\ f(b)' &= [0\Delta_b + 1(\Delta_a \cdot \nabla_b) + 1\nabla_a] \quad \text{and clearly} \\ f(p) &= [p\Delta_b + p(\Delta_a \cdot \nabla_b) + p\nabla_a]. \end{aligned}$$

Our result then follows by Lemma 5.7.

(vi) For each $j \in I$ set $q_i^j = 0$ for all $i \neq j$ and $q_j^j = p_j$. Then for each $j \in I$, $(\Delta_{a_i} \cdot \nabla_{b_i})_I$ witnesses

$$0\Delta_{b_j} + p_j(\Delta_{a_j} \cdot \nabla_{b_j}) + 0\nabla_{a_j} \simeq \sum q_i^j(\Delta_{a_i} \cdot \nabla_{b_i}).$$

Our result then follows from Lemma 5.7.

(vii) As $C$ is generated as a Boolean algebra by the sublattice $D = \{\Delta_a : a \in N(L)\}$, Lemma 4.2 (iv) gives that every partition of $C$ can be refined by a partition which is basic over $D$, i.e. a partition of the form $(\Delta_{a_i} \cdot \nabla_{b_i})_I$ where $b_i < a_i \in N(L)$. Thus by Lemma 5.5 (iii), for each $x \in F/\simeq$ there is a representation

$$x = [\sum p_i(\Delta_{a_i} \cdot \nabla_{b_i})]$$

where $(\Delta_{a_i} \cdot \nabla_{b_i})_I$ is a basic partition and each $p_i \in L$. So by parts (v) and (vi) of this Lemma,

$$x = \sum[0\Delta_{b_i} + p_i(\Delta_{a_i} \cdot \nabla_{b_i}) + 0\nabla_{a_i}] = \sum f(p_i)f(a_i)f(b_i)'.$$

But $f(p_i) \in f[L]$ and by the fourth part of this Lemma, $f(a) \cdot f(b)'$ is in the Boolean sublattice $B$ of $Z(F/\simeq)$ generated by $f[N(L)]$. Therefore $F/\simeq$ is generated by $f[L] \cup B$. $\square$

**Proposition 5.9.** *Let $L$ be a bounded lattice. Then $(F/\simeq, f)$ is a free central extension of $L$.*

*Proof.* This follows from Proposition 4.10 by Lemma 5.8 (iv) and (vii). $\square$

*Remark 5.10.* The reader familiar with the Pierce sheaf will recognize $F/\simeq$ as the algebra of global sections of the Pierce sheaf of the lattice $L$ over the Stone space of $C$, where $C$ is the Boolean sublattice of $Con(L)$ defined in Definition 5.3. The reason why the map $f$ is not an isomorphism, as one familiar with sheaves of rings might expect, is that the congruences in $C$ do not permute. If $L$ is not bounded, the global sections of this sheaf do not give the free central extension of $L$. In fact, the global sections of this sheaf need not be bounded. An example of this behaviour is provided by a lattice whose only neutral element is its unit. For background on the Pierce sheaf of a lattice, see [4], [6].

## 6. Summary.

**Theorem 6.1.** *Every lattice has a free central extension.*

*Proof.* This follows by Proposition 5.9 and Proposition 3.3. $\square$

**Theorem 6.2.** *Let $f : L \longrightarrow E$ be a central embedding. The following are equivalent.*

(i) *$(E, f)$ is a free central extension of $L$.*
(ii) *$E$ is generated by $f[L] \cup B$ where $B$ is the Boolean sublattice of $Z(E)$ generated by $f[N(L)]$.*
(iii) *For any central map $g : L \longrightarrow K$ there exists exactly one bound preserving map $h : E \longrightarrow K$ with $h \circ f = g$, and this map $h$ is central.*
(iv) *For any central map $g : L \longrightarrow K$ there is exactly one central map $h : E \longrightarrow K$ with $h \circ f = g$.*

*Further, if these conditions are satisfied, then $B$ is equal to the centre of $E$.*

*Proof.* (i) $\Rightarrow$ (ii) Assume that $(E, f)$ is a free central extension of $L$. Let $F/\simeq$ be the lattice constructed in Section 5 from the bounded lattice $L^*$ and let $g : L^* \longrightarrow F/\simeq$ be the embedding given in Section 5. Then by Proposition 5.9 we have that $(F/\simeq, g)$ is a free central extension of $L^*$. By Proposition 3.3 we have that $(F/\simeq, g \circ *_L)$ is a free central extension of $L$. But by the uniqueness of free central extensions given in Proposition 3.2 there is an isomorphism $i : F/\simeq \longrightarrow E$ so that $i \circ g \circ *_L = f$. By Lemma 5.8 (vii) we have that $F/\simeq$ is generated by $g[L^*] \cup C$, where $C$ is the Boolean sublattice of $Z(F/\simeq)$ generated by $g[N(L^*)]$. It follows that $E$ is generated by $i \circ g[L^*] \cup i[C]$. As $i \circ g \circ *_L = f$, we have that $E$ is generated by $f[L] \cup B$ where $B$ is the Boolean sublattice of $Z(E)$ generated by $f[N(L)]$.

(ii) $\Rightarrow$ (i) This follows by Proposition 4.10, which also provides that $B = Z(E)$.

(i) $\Rightarrow$ (iii) Let $g : L \longrightarrow K$ be central. As $E$ is generated by $f[L] \cup B$ there is at most one bound preserving map $h : E \longrightarrow K$ with $h \circ f = g$. But $(E, f)$ is a free central extension of $L$ so there is a central map $h : E \longrightarrow K$ such that $h \circ f = g$.

(iii) $\Rightarrow$ (iv) This is trivial.

(iv) $\Rightarrow$ (i). We must show that $N(E) = Z(E)$. Let $(A, \alpha)$ be a free central extension of $E$. We will first show that $(A, \alpha \circ f)$ is a free central extension of $L$. Clearly $\alpha \circ f$ is a central embedding. Let $g : L \longrightarrow K$ be a central map. By assumption there is exactly one central map $h : E \longrightarrow K$ with $h \circ f = g$. As $(A, \alpha)$ is a free central extension of $E$ there is a central map $q : A \longrightarrow K$ with $q \circ \alpha = h$. Then $q \circ (\alpha \circ f) = h \circ f = g$. Suppose $q' : A \longrightarrow K$ is a central map with $q' \circ (\alpha \circ f) = g$. Then $q' \circ \alpha : E \longrightarrow K$ is central and $(q' \circ \alpha) \circ f = h \circ f = g$ and hence $q' \circ \alpha = h$. As $(A, \alpha)$ is a free central extension and $q' \circ \alpha = q \circ \alpha = h$ we have that $q' = q$. Thus $(A, \alpha \circ f)$ is a free central extension of $L$.

As we have shown the equivalence of the first two parts of the Theorem, we have that $A$ is generated by $(\alpha \circ f)[L] \cup B$ where $B$ is the Boolean sublattice of $Z(A)$ generated by $(\alpha \circ f)[N(L)]$. Further $B = Z(A)$. But the map $f$ is central, so $f[N(L)] \subseteq Z(E)$. The map $\alpha$ is also central, so $(\alpha \circ f)[N(L)] \subseteq \alpha[Z(E)] \subseteq Z(A)$. So $\alpha[Z(E)]$ is a Boolean sublattice of $Z(A)$ containing $(\alpha \circ f)[N(L)]$ and therefore $\alpha[Z(E)] = B = Z(A)$. Suppose $n \in E$ is neutral. Then $\alpha(n) \in A$ is central. So $\alpha(n) \in \alpha[Z(E)]$. As $\alpha$ is an embedding, $n \in Z(E)$. Thus $N(E) = Z(E)$. □

**Theorem 6.3.** *The free central extension $(E, f)$ of a lattice $L$ is an essential extension of $L$ and lies in the variety generated by $L$.*

*Proof.* This follows by Theorem 6.2 and Proposition 4.10. □

By Lemma 2.8 the composition of neutral maps is a neutral map and the composition of central maps is a central map. Clearly the identity map of any lattice is neutral. The identity map of a lattice $L$ is central iff $N(L) = Z(L)$. We will call a lattice $L$ with $N(L) = Z(L)$ a central lattice. Clearly the class of all lattices with neutral maps forms a category $\mathcal{L}$ and the class of all central lattices and central maps forms a category $\mathcal{C}$. It is easy to see that $\mathcal{C}$ is a full subcategory of $\mathcal{L}$. We further have

**Theorem 6.4.** *The category $\mathcal{C}$ of central lattices and central maps is a reflective subcategory of the category $\mathcal{L}$ of all lattices and neutral maps.*

*Proof.* By general considerations of reflectors (see [1], Theorem 2, pg. 28) it is enough to show that there is a function which assigns to every lattice $A$ a central lattice $\mathcal{R}(A)$ and a function which assigns to every lattice $A$ a neutral map $\Phi_{\mathcal{R}}(A) : A \longrightarrow \mathcal{R}(A)$ such that for every central lattice $B$ and every neutral map $f : A \longrightarrow B$ there exists a unique central map $h : \mathcal{R}(A) \longrightarrow B$ such that $h \circ \Phi_{\mathcal{R}}(A) = f$. Let $\mathcal{R}(A)$ be some free central extension of $A$, perhaps the lattice $F/\simeq$ constructed in Section 5 from $A^*$, and let $\Phi_{\mathcal{R}}(A)$ be the central embedding of $A$ into this free central extension. Once we note that any neutral map $f : A \longrightarrow B$ into a central lattice is a central map, the above conditions follow at once from the definition of a free central extension. $\square$

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Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003, USA.

*E-mail address*: JHarding@NMSU.edu