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Local Radon-Nikodym Derivatives of Set Functions

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1. Introduction

In the last twenty years non-additive set functions have played a major role in several research areas, including Artificial Intelligence, Mathematical Economics, and Bayesian Statistics, particularly in the area of upper and lower probabilities (see, e.g., Grabisch et al. (1994), Schmeidler (1989), and Walley (1991) for an introduction to their use in these areas). The study of non-additive set functions is also useful in *interval computations* where *interval probabilities* represent uncertainty.

Abandoning additivity is a very important departure from the classical case, and it is natural to expect that several standard results will no longer hold in this more general setting. In particular, this is the case for the classical Radon-Nikodym theorem, a basic result in measure theory with very important applications in probability theory. Several papers explored to which extent this failure occurs, and provided conditions under which non-additive counterparts of this famous result hold (see Graf (1981), Greco (1981a), and Nguyen et al. (1997)). In this paper we study a version of the Radon-Nikodym Theorem, which is equivalent to the original one in the classical setting, but different in the non-additive case. More precisely, the classical result says that given any two countably additive set functions ν and

μ defined on a σ-algebra $\mathcal{U}$, there exists a $\mathcal{U}$-measurable function $f : U \longrightarrow [0,\infty)$ such that

$$\mu(A) = \int_A f d\nu \quad \text{for all} \quad A \in \mathcal{U} \tag{1}$$

provided that $\mu << \nu$ (see next section). In the classical case this is equivalent to saying that for all finite subalgebras $\mathcal{U}'$ of $\mathcal{U}$ there exists a $\mathcal{U}'$-measurable function $f_{\mathcal{U}'} : U \longrightarrow [0,\infty)$ such that

$$\mu(A) = \int_A f_{\mathcal{U}'} d\nu \quad \text{for all} \quad A \in \mathcal{U}'. \tag{2}$$

However, these two conditions are no longer equivalent for non-additive set functions, as will be seen later. In the paper we focus on this second version of the Radon-Nikodym theorem, which we call the finite Radon-Nikodym property (abbreviated FRNP), and we explore its validity for non-additive set functions. Of course, the absolute continuity condition $\mu << \nu$ is no longer sufficient for (2). However, our main result contains an interesting characterization of the FRNP, and provides a simple condition that on top of absolute continuity is equivalent to (2). Therefore, our result allows one to check easily when two set functions μ and ν have the FRNP.

A useful secondary contribution of the paper is to show that, if two non-additive set functions μ and ν are such that either of (1) or (2) holds, then μ is alternating of infinite order whenever ν is alternating of infinite order. This provides a very simple way to generate new alternating of infinite order set functions from old ones. Moreover, the same result applies to set functions monotone of infinite order, i.e. belief functions. As an application, we show that all maxitive set functions* are alternating of infinite order.

The paper is organized as follows. In the second section we give the necessary background to make the paper reasonably self contained. In the third section we prove an approximation result for the Choquet integral that we need later, but which also seems of interest in itself. In the fourth section we prove that if two set functions (μ, ν) have the Radon-Nikodym property (abbreviated RNP), i.e. (1) above holds, and ν is alternating of infinite order, then μ must also be alternating of infinite order. In the fifth section we make an observation which allows us to extend the results of Section 4 to pairs of functions (μ, ν) which satisfy a weaker condition than the RNP. This leads us to the definition of the local Radon-Nikodym property and the finite Radon-Nikodym property. We then prove a simple characterization of when two functions (μ, ν) have the finite Radon-Nikodym property. This is our main result. We use this characterization in Section 6 to prove that any maxitive function is alternating of infinite order. Finally, Section 7 details the relationships existing between the three Radon-Nikodym properties that we consider in the paper.

*A set function $\nu : \mathcal{U} \longrightarrow [0,\infty)$ is maxitive if $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ for every $A, B \in \mathcal{U}$. Maxitive set functions are important in fuzzy measure theory.

2. Preliminaries

We begin with definitions of the various notions which will appear in the paper. For a set U, we say that $\mathcal{U}$ is an algebra over U if $\mathcal{U}$ is a collection of subsets of U which is closed under complementation and finite unions and intersections. Given an algebra $\mathcal{U}$ over U, we say that a map $\nu : \mathcal{U} \longrightarrow [0, \infty)$ is a capacity[†]if

(i) $\nu(\emptyset) = 0$.

(ii) $\nu(A) \leq \nu(B)$ if $A \subseteq B$.

A function $f : U \longrightarrow [0, \infty)$ is said to be measurable with respect to $\mathcal{U}$ if for every real number $t \geq 0$ the set $f^{-1}[t, \infty)$ is in $\mathcal{U}$. For more general definitions of measurability we refer the interested reader to Greco (1981b) and Denneberg (1994).

Given an algebra $\mathcal{U}$ over U, a capacity ν defined on $\mathcal{U}$, and a function f which is measurable with respect to $\mathcal{U}$ we define

$$\int_U f d\nu = \int_0^\infty \nu(\{u \in U : f(u) \geq t\}) dt.$$

This notion of integral is due to Choquet (1953). One should note that the function $g(t) = \nu(\{u \in U : f(u) \geq t\})$ is well defined as f is measurable with respect to $\mathcal{U}$. Further, as ν is monotone, the function g is nonincreasing. As any nonincreasing function has an extended Riemann integral, the definition is valid. If $\int_U f d\nu < \infty$, we say that f is integrable.

Let μ, ν be capacities defined on an algebra $\mathcal{U}$ over a set U. We say that the ordered pair (μ, ν) has the Radon-Nikodym property (abbreviated RNP) if there is a function $f : U \longrightarrow [0, \infty)$, measurable with respect to $\mathcal{U}$, such that

$$\mu(A) = \int_A f d\nu \quad \text{for all} \quad A \in \mathcal{U}.$$

We will further say that μ is absolutely continuous with respect to ν over $\mathcal{U}$, written $\mu << \nu$, if $\nu(A) = 0$ impies $\mu(A) = 0$ for every $A \in \mathcal{U}$. In the classicalal setting the Radon-Nikodym property is linked to absolute continuity by the following well known result.

Theorem 2.1 *Let $\mathcal{U}$ be a σ-algebra over the set U and let μ, ν be countably additive set functions on $\mathcal{U}$. Then (μ, ν) has the RNP iff $\mu << \nu$.*

Given a capacity μ on an algebra $\mathcal{U}$, we say μ is alternating of infinite order if

$$\mu\left(\bigcap_{i=1}^n A_i\right) \leq \sum_{I \in \Phi} (-1)^{|I|+1} \mu\left(\bigcup_I A_i\right)$$

for every $A_1, \ldots, A_n \in \mathcal{U}$. Here Φ denotes the collection of all non-empty subsets of $\{1, \ldots, n\}$ and $|I|$ denotes the cardinality of the set I. This notion is important in

[†]This is also the usual definition of a fuzzy measure. Of course, even though we stick to the original mathematical terminology, all our results hold for fuzzy measures as well.

the theory of capacities and much of this paper shall deal with connections between the RNP and set functions which are alternating of infinite order.

3. A Lemma

Here we establish, for the Choquet integral, a version of the classical result that the integral of a function f may be approximated to an arbitrary degree of accuracy by a simple function $f' \leq f$. We think that this result is of some interest in itself.

Lemma 3.1 *Let $\mathcal{U}$ be an algebra over a set U, ν be a capacity on $\mathcal{U}$, and f be a function on U which is measurable with respect to $\mathcal{U}$ and satisfies $\int_U f d\nu < \infty$. Then for any $A_1, \ldots, A_n \in \mathcal{U}$ and any $\epsilon > 0$ there exists a map f' such that*

(a) *f' is measurable with respect to $\mathcal{U}$.*

(b) *f' is simple.*

(c) *$f' \leq f$.*

(d) *$\int_{A_i} f' d\nu \leq \int_{A_i} f d\nu \leq \int_{A_i} f' d\nu + \epsilon$ for each $i \leq n$.*

Proof. We first prove the result in the case that our family $A_1, \ldots, A_n$ consists only of a single set A. For each t define

$$A_t = \{a \in A : f(a) \geq t\} \quad \text{and} \quad g(t) = \nu(A_t).$$

Then as $A_t = A \cap f^{-1}[t, \infty)$ we have that A_t is in $\mathcal{U}$. Note that g is a decreasing function defined on the interval $[0, \infty)$, and hence is bounded above by $g(0) = \nu(A)$. Further, we have directly from the definition of g that

$$\int_0^\infty g(t) dt = \int_A f d\nu < \infty.$$

Then, it is possible to find a number λ such that

$$\int_0^\infty g dt < \int_0^\lambda g dt + \epsilon/4.$$

We may further assume that λ has been chosen so that $g(t) > 0$ for all $t \leq \lambda$. Then it is possible[‡] to find a function g' so that (i) $g' \leq g$, (ii) g' is strictly decreasing on $[0, \lambda]$, (iii) g' is continuous on $[0, \lambda]$, (iv) $g(\lambda) = 0$ and (v)

$$\int_0^\infty g dt < \int_0^\lambda g dt + \epsilon/4 < \int_0^\lambda g' dt + \epsilon/2.$$

[‡] First find an appropriate simple function beneath g, then smooth the jump discontinuities with nearly vertical line segments to obtain a continuous function. Finally, subtract an appropriate linear function to ensure that we have a strictly decreasing function.

As g' is strictly decreasing and continuous on $[0, \lambda]$ and $g(0) = \nu(A)$, $g(\lambda) = 0$ there must be a function $h : [0, \nu(A)] \longrightarrow [0, \lambda]$ which is the inverse of g. Clearly h is strictly decreasing, its range is all of $[0, \lambda]$ and therefore h is also continuous. But

$$\int_0^{\lambda} g'(t)dt = \int_0^{\nu(A)} h(x)dx.$$

As with any Riemann integrable function, we can find a simple function $h' \leq h$ such that the integral of h' over $[0, \nu(A)]$ is within $\epsilon/2$ of the integral of h. Assume that $0 = x_0 < \cdots < x_n = \nu(A)$ is the partition associated with the simple function h'. We may clearly assume that h' is as large as possible with respect to this partition. This means that

$$h(x) = h(x_i) \quad \text{for all} \quad x \in [x_{i-1}, x_i].$$

Therefore

$$\int_0^{\nu(A)} h'dx = \sum_{i=1}^{n-1} (x_i - x_{i-1})h(x_i) > \int_0^{\nu(A)} hdx - \epsilon/2.$$

Rearranging this sum we have that

$$\int_0^{\nu(A)} h'(x)dx = \sum_{i=1}^{n-1} x_i(h(x_i) - h(x_{i+1})) > \int_0^{\nu(A)} hdx - \epsilon/2. \tag{3}$$

Note that $0 = h(x_n) < \cdots < h(x_1)$, so $U = A_{h(x_n)} \supseteq \cdots \supseteq A_{h(x_1)}$. Next we define $f' : U \longrightarrow [0, \infty)$ by setting

$$f'(u) = \begin{cases} h(x_1) & \text{if } u \in A_{h(x_1)} \\ h(x_2) & \text{if } u \in A_{h(x_2)} - A_{h(x_1)} \\ \vdots & \\ h(x_n) & \text{if } u \in A_{h(x_n)} - A_{h(x_{n-1})} \end{cases}$$

Then by the definition of A_t we have that $f' \leq f$. Also, the inverse image under f' of any interval $[t, \infty)$ is of the form $A_{h(x_i)}$ and therefore f' is measurable with respect to $\mathcal{U}$. Surely f' is simple. We have only to verify condition (d). But

$$\int_A f'd\nu = \int_0^{\infty} \nu(\{a \in A : f'(a) \geq t\})dt.$$

And as $0 = h(x_n) < \cdots < h(x_1)$ we have this integral equal to

$$\sum_{i=1}^{n-1} \int_{h(x_{i+1})}^{h(x_i)} \nu(\{a \in A : f'(a) \geq t\})dt.$$

Which in turn is equal to

$$\sum_{i=1}^{n-1} (h(x_i) - h(x_{i+1}))\nu(A_{h(x_i)}).$$

Then as $\nu(A_{h(x_i)}) = g(h(x_i)) = x_i$ we have

$$\int_A f' d\nu = \sum_{i=1}^{n-1} (h(x_i) - h(x_{i+1}))x_i.$$

Comparing this with (3) gives that

$$\int_A f' d\nu \leq \int_A f d\nu \leq \int_A f' d\nu + \epsilon.$$

Having established our result in the case that our family consists of only a single set, we now consider the general case of a finite family of sets $A_1, \ldots, A_n$. From what we have shown we know that for each $i \leq n$ we can find a simple function f_i' with

$$\int_{A_i} f_i' d\nu \leq \int_{A_i} f d\nu \leq \int_{A_i} f_i' d\nu + \epsilon.$$

Then take f' to be the pointwise supremum of the f_i'. □

4. The RNP and alternating functions

In this section we shall prove that if a pair of capacities (μ, ν) have the RNP, then μ is alternating of infinite order whenever ν is alternating of infinite order. This provides a very simple way to generate new alternating capacities from old ones. It is important to note that the function f which realizes a pair (μ, ν) having the RNP need only be integrable and not bounded.

Theorem 4.1 *Suppose that the pair (μ, ν) has the RNP on $\mathcal{U}$. Then μ is alternating of infinite order whenever ν is alternating of infinite order.*

Proof. We first establish the result under the assumption that $\mathcal{U}$ is finite. Let f be the $\mathcal{U}$-measurable function from U to the interval $[0, \infty)$ with $\mu(A) = \int_A f d\nu$ for each $A \in \mathcal{U}$, or equivalently

$$\mu(A) = \int_0^\infty \nu(\{a \in A : f(a) \geq t\}) dt.$$

As the algebra $\mathcal{U}$ is finite, the function f can take only finitely many values, say $\beta_1 \leq \cdots \leq \beta_n$. If we set $\beta_0 = 0$ we may write the above integral as

$$\mu(A) = \sum_{k=1}^{n} \int_{\beta_{k-1}}^{\beta_k} \nu(\{a \in A : f(a) \geq t\}) dt.$$

Defining $B_k = \{u \in U : f(u) \geq \beta_k\}$ for each $k \leq n$ we then have

$$\mu(A) = \sum_{k=1}^{n} (\beta_k - \beta_{k-1}) \nu(B_k \cap A). \tag{4}$$

As this equation is valid for all $A \in \mathcal{U}$ we have in particular

$$\mu\left(\bigcap_{i=1}^{n} A_i\right) = \sum_{k=1}^{n}(\beta_k - \beta_{k-1})\nu\left(B_k \cap \bigcap_{i=1}^{n} A_i\right). \tag{5}$$

But $B_k \cap \bigcap_I A_i = \bigcap_I (B_k \cap A_i)$, and as ν is alternating of infinite order we have

$$\nu\left(B_k \cap \bigcap_{i=1}^{n} A_i\right) \leq \sum_{I\in\Phi}(-1)^{|I|+1}\nu\left(\bigcup_I (B_k \cap A_i)\right). \tag{6}$$

Here Φ is used to denote all non-empty subsets of $\{1,\ldots,n\}$. Then as $\bigcup_I (B_k \cap A_i) = B_k \cap \bigcup_I A_i$ we may substitute (6) into (5) to obtain

$$\mu\left(\bigcap_{i=1}^{n} A_i\right) \leq \sum_{k=1}^{n}(\beta_k - \beta_{k-1})\sum_{I\in\Phi}(-1)^{|I|+1}\nu\left(B_k \cap \bigcup_I A_i\right).$$

Rearranging this sum we have

$$\mu\left(\bigcap_{i=1}^{n} A_i\right) \leq \sum_{I\in\Phi}(-1)^{|I|+1}\sum_{k=1}^{n}(\beta_k - \beta_{k-1})\nu\left(B_k \cap \bigcup_I A_i\right).$$

Which by (4) gives

$$\mu\left(\bigcap_{i=1}^{n} A_i\right) \leq \sum_{I\in\Phi}(-1)^{|I|+1}\mu\left(\bigcup_I A_i\right).$$

This establishes our result in the case that $\mathcal{U}$ is finite.

We now consider the general case in which $\mathcal{U}$ may be infinite. Given $A_1,\ldots,A_n \in \mathcal{U}$, the subalgebra $\mathcal{F}$ of $\mathcal{U}$ generated by $A_1,\ldots,A_n$ is finite. So by Lemma 3.1, for each $\epsilon > 0$ we can find a simple function $f' \leq f$, measurable with respect to $\mathcal{U}$, such that

$$\int_A f' d\nu \leq \int_A f d\nu \leq \int_A f' d\nu + \epsilon \quad \text{for all} \quad A \in \mathcal{F}.$$

Then as f' is simple we can find a finite subalgebra $\mathcal{G}$ of $\mathcal{U}$ such that (i) $\mathcal{G}$ contains $\mathcal{F}$ and (ii) f' is measurable with respect to $\mathcal{G}$. Define $\sigma : \mathcal{G} \longrightarrow [0,\infty)$ by setting

$$\sigma(A) = \int_A f' d\nu.$$

Unraveling the definitions

$$\sigma(A) \leq \mu(A) \leq \sigma(A) + \epsilon \quad \text{for all} \quad A \in \mathcal{F}. \tag{7}$$

But by definition (σ,ν) has the RNP on the finite algebra $\mathcal{G}$. And as ν is alternating of infinite order on $\mathcal{U}$ it is also alternating of infinite order on $\mathcal{G}$. As we have established our result for the finite case, we have that σ is alternating of infinite order on $\mathcal{G}$. Hence

$$\sigma\left(\bigcap_{i=1}^{n} A_i\right) \leq \sum_{I\in\Phi}(-1)^{|I|+1}\sigma\left(\bigcup_{I} A_i\right). \tag{8}$$

Here Φ is used to denote the collection of all non-empty subsets of $\{1,\ldots,n\}$. But by (7)

$$\sigma\left(\bigcup_{i=1}^{n} A_i\right) \leq \mu\left(\bigcup_{i=1}^{n} A_i\right) \leq \sigma\left(\bigcup_{i=1}^{n} A_i\right) + \epsilon$$

and as there are $2^n - 1$ sets in Φ

$$\sum_{I\in\Phi}(-1)^{|I|+1}\sigma\left(\bigcup_{I} A_i\right) \leq \sum_{I\in\Phi}(-1)^{|I|+1}\mu\left(\bigcup_{I} A_i\right) + (2^n-1)\epsilon.$$

Together with (7) and (8), this implies

$$\mu\left(\bigcap_{i=1}^{n} A_i\right) \leq \sum_{I\in\Phi}(-1)^{|I|+1}\mu\left(\bigcup_{I} A_i\right) + 2^n\epsilon.$$

As this holds for any $\epsilon > 0$ our result follows. □

Remark. Theorem 4.1 clearly holds for k-alternating set functions as well, where k is any natural number. Moreover, Theorem 4.1 also holds for capacities which are monotone of infinite order[§](i.e., belief functions). Indeed, proceeding as in the last proof it can be proved that if (μ,ν) has the RNP on $\mathcal{U}$, then μ is monotone of infinite order (k order) whenever ν is monotone of infinite order (k order). Again, this is a very simple way to generate new k-monotone capacities (e.g. belief functions) from old ones.

5. The Local Radon-Nikodym property

In this section we define the local Radon-Nikodym property and the finite Radon-Nikodym property and show that the result of the previous section applies to a pair of capacities (μ,ν) which satisfies the local or finite Radon-Nikodym property. We also derive a simple characterization of the finite Radon-Nikodym property, which will be used in the next section to show that any maxitive function is alternating of infinite order.

[§]A capacity is monotone of infinite order if

$$\mu\Big(\bigcup_{i=1}^{n} A_i\Big) \geq \sum_{I\in\Phi}(-1)^{|I|+1}\mu\Big(\bigcap_{I} A_i\Big)$$

for every $A_1,\ldots,A_n \in \mathcal{U}$. It is easy to check that a capacity ν is alternating of infinite order if and only if its dual capacity $\bar{\nu}$, defined by $\bar{\nu}(A) = 1 - \nu(A^c)$ for all $A \in \mathcal{U}$, is monotone of infinite order.

Definition *Let μ, ν be capacities on an algebra $\mathcal{U}$. For a subalgebra $\mathcal{V}$ of $\mathcal{U}$ we say that (μ, ν) have the RNP on $\mathcal{V}$ if the restrictions of μ, ν to $\mathcal{V}$ have the RNP. This means that there is a function f which is measurable with respect to $\mathcal{V}$ with*

$$\mu(A) = \int_A f d\nu \text{ for all } A \in \mathcal{V}.$$

(i) We say that (μ, ν) have the local Radon-Nikodym property (abbreviated LRNP) if every finite collection of subsets of $\mathcal{U}$ is contained in a subalgebra on which (μ, ν) have the RNP.

(ii) We say that (μ, ν) have the finite Radon-Nikodym property (abbreviated FRNP) if (μ, ν) have the RNP on every finite subalgebra of $\mathcal{U}$.

For countably additive set functions defined on σ-algebras the properties RNP, LRNP, and FRNP are all equivalent, and by the classical Radon-Nikodym Theorem they hold if and only if $\mu << \nu$. If the set functions are finitely additive, then LRNP and FRNP are equivalent, and they hold if and only if $\mu << \nu$. However, they are no longer equivalent to the RNP, which is a stronger property in the finitely additive case (cf. Theorem 7.1). For capacities the only implications that hold are RNP $\Rightarrow$ LRNP and FRNP $\Rightarrow$ LRNP. In the last section we shall give examples which show that there are no other logical relationships between these notions. But first, we prove a version of Theorem 4.1 for the LRNP.

Theorem 5.1 *Suppose that the pair (μ, ν) has the LRNP on $\mathcal{U}$. Then μ is alternating of infinite order whenever ν is alternating of infinite order.*

Proof. Given $A_1, \ldots, A_n$ in $\mathcal{U}$ find a subalgebra $\mathcal{U}'$ on which (μ, ν) have the RNP. As ν is alternating of infinite order on $\mathcal{U}$ it is also alternating of infinite order on $\mathcal{U}'$. By Theorem 4.1 it follows that μ is also alternating of infinite order on $\mathcal{U}'$. Then as $A_1, \ldots, A_n$ are all elements of $\mathcal{U}'$ we have

$$\mu\left(\bigcap_{i=1}^{n} A_i\right) \leq \sum_{I \in \Phi} (-1)^{|I|+1} \mu\left(\bigcup_{I} A_i\right),$$

where Φ is the collection of all non-empty subsets of $\{1, \ldots, n\}$. As this is valid for any $A_1, \ldots, A_n$ in $\mathcal{U}$ we have that μ is alternating of infinite order on $\mathcal{U}$. □

If there is a simple characterization of the LRNP it has eluded us. However, we do have a simple characterization of the FRNP and hence a simple condition sufficient to guarantee the LRNP. Before describing this result, we introduce some notation. We say that a capacity ν, defined on an algebra $\mathcal{U}$, is null additive if $\nu(A) = 0$ implies $\nu(A \cup B) = \nu(B)$ for all $A, B \in \mathcal{U}$. Note that if ν is null additive, then for any integrable function f

$$\nu(A) = 0 \text{ implies } \int_B f d\nu = \int_{B \setminus A} f d\nu \quad \text{for all} \quad A, B \in \mathcal{U}.$$

Therefore, if f, f' are integrable functions such that $\{x : f(x) \neq f'(x)\}$ is contained in some $A \in \mathcal{U}$ with $\nu(A) = 0$, then $\int_B f d\nu = \int_B f' d\nu$ for all $B \in \mathcal{U}$.

If μ, ν are capacities on an algebra $\mathcal{U}$ we define $\gamma : \mathcal{U} \longrightarrow [0, \infty)$ by setting

$$\gamma(A) = \begin{cases} \mu(A)/\nu(A) & \text{if } \nu(A) \neq 0 \\ 0 & \text{if } \nu(A) = 0 \end{cases}$$

Note that if $\mu << \nu$ then $\mu(A) = \gamma(A)\nu(A)$ for all $A \in \mathcal{U}$.

Theorem 5.2 *Let μ, ν be capacities defined on an algebra $\mathcal{U}$. Then the first condition below implies the second. If we assume that ν is null additive, then the second condition implies the first (and hence that μ is null additive).*

(i) *$\mu << \nu$ and for all A, B disjoint sets in $\mathcal{U}$*

$$\gamma(A) \leq \gamma(B) \Rightarrow \mu(A \cup B) - \mu(B) = \gamma(A)[\nu(A \cup B) - \nu(B)].$$

(ii) *(μ, ν) have the FRNP.*

Proof. (i) $\Rightarrow$ (ii). Let $\mathcal{F}$ be a finite subalgebra of $\mathcal{U}$. As the atoms of $\mathcal{F}$ partition U we may define a function $f : U \longrightarrow [0, \infty)$ by setting $f(x) = \gamma(A)$ if A is an atom containing x. Clearly f is $\mathcal{F}$-measurable. We prove by induction on n that if $A_1, \ldots, A_n$ are distinct atoms of $\mathcal{F}$ with $\gamma(A_1) \leq \cdots \leq \gamma(A_n)$ then

(a) if $n \geq 2$, then $\gamma(A_1) \leq \gamma(\bigcup_{i=2}^{n} A_i)$.

(b) $\gamma(A_1)\nu(\bigcup_{i=1}^{n} A_i) = \mu(\bigcup_{i=1}^{n} A_i) - \mu(\bigcup_{i=2}^{n} A_i) + \gamma(A_1)\nu(\bigcup_{i=2}^{n} A_i)$.

(c) $\mu(A) = \int_A f d\nu$ where $A = \bigcup_{i=1}^{n} A_i$.

As every element of $\mathcal{F}$ can be expressed as the union of such an indexed family, this claim will establish our result.

$n = 1$. Part (a) is vacuous. As $\nu(\emptyset) = 0$ part (b) reduces to showing $\gamma(A_1)\nu(A_1) = \mu(A_1)$ which follows as $\mu << \nu$. And similarly part (c) follows as

$$\int_0^{\gamma(A_1)} \nu(\{x \in A_1 : f(x) \geq t\})dt = \gamma(A_1)\nu(A_1).$$

$n \geq 2$. Set $B = \bigcup_{i=2}^{n} A_1$. To establish part (a) note that if x is in B then $f(x) = \gamma(A_i)$ for some $i \geq 2$ and in particular $f(x) \geq \gamma(A_1)$. The indexed family of atoms $A_2, \ldots, A_n$ satisfies our inductive hypothesis and therefore $\mu(B) = \int_B f d\nu$. As

$$\int_B f d\nu \geq \int_0^{\gamma(A_1)} \nu(\{x \in B : f(x) \geq t\})dt = \gamma(A_1)\nu(B)$$

we have that $\mu(B) \geq \gamma(A_1)\nu(B)$. If $\nu(B) \neq 0$ division gives $\gamma(B) \geq \gamma(A_1)$. If $\nu(B) = 0$ then $\nu(A_2) = 0$ and hence $\gamma(A_2) = 0$. But $\gamma(A_1) \leq \gamma(A_2)$. In any event, part (a) is established. Note that A_1 and B are disjoint and we have just shown that $\gamma(A_1) \leq \gamma(B)$. So part (b) follows from our hypothesis (i). To establish part (c) note that $\int_A f d\nu$ is equal to

$$\int_0^{\gamma(A_1)} \nu(\{x \in \bigcup_{i=1}^n A_i : f(x) \geq t\})dt + \sum_{q=2}^n \int_{\gamma(A_{q-1})}^{\gamma(A_q)} \nu(\{x \in \bigcup_{i=1}^n A_i : f(x) \geq t\})dt.$$

Which in turn is equal to

$$\gamma(A_1)\nu\left(\bigcup_{i=1}^n A_i\right) + \sum_{q=2}^n [\gamma(A_q) - \gamma(A_{q-1})]\nu\left(\bigcup_{i=q}^n A_i\right).$$

Note that this expression is valid even if $\gamma(A_{q-1}) = \gamma(A_q)$ for some q. Rewriting, this expression is equal to

$$\sum_{q=1}^n \gamma(A_q)\nu\left(\bigcup_{i=q}^n A_i\right) - \sum_{q=2}^n \gamma(A_{q-1})\nu\left(\bigcup_{i=q}^n A_i\right).$$

Using the instance of part (b) we have just established as well as other instances which follow from the inductive hypothesis on the indexed family $A_q, \ldots, A_n$ gives the above expression equal to

$$\sum_{q=1}^n \left[\mu\left(\bigcup_{i=q}^n A_i\right) - \mu\left(\bigcup_{i=q+1}^n A_i\right) + \gamma(A_q)\nu\left(\bigcup_{i=q+1}^n A_i\right)\right] - \sum_{i=2}^n \gamma(A_{q-1})\nu\left(\bigcup_{i=q}^n A_i\right).$$

Which simplifies to

$$\sum_{q=1}^n \left[\mu\left(\bigcup_{i=q}^n A_i\right) - \mu\left(\bigcup_{i=q+1}^n A_i\right)\right],$$

which is equal simply to $\mu(\bigcup_{i=1}^n A_i)$.

(ii) $\Rightarrow$ (i). Surely if (μ, ν) have the FRNP then $\mu << \nu$. Suppose that A, B are disjoint sets with $\gamma(A) \leq \gamma(B)$. Let $\mathcal{F}$ be the subalgebra of $\mathcal{U}$ generated by A, B and let $f : U \longrightarrow [0, \infty)$ be an $\mathcal{F}$-measurable function with

$$\mu(F) = \int_F f d\nu \quad \text{for all} \quad F \in \mathcal{F}.$$

As A, B are atoms of $\mathcal{F}$, or empty, it follows that the restriction of f to A must be constant, as is the restriction of f to B. If $\nu(A) = 0$ then the particular value f takes on A is irrelevant to the above formula, as ν is null additive. But if $\nu(A) \neq 0$

then it follows easily that f must take the constant value $\gamma(A)$ on A. Therefore we may assume that $f(x) = \gamma(A)$ for all $x \in A$ and similarly $f(x) = \gamma(B)$ for all $x \in B$. As we assumed that $\gamma(A) \leq \gamma(B)$ we have that $\mu(A \cup B)$ is equal to

$$\int_0^{\gamma(A)} \nu(\{x \in A \cup B : f(x) \geq t\})dt + \int_{\gamma(A)}^{\gamma(B)} \nu(\{x \in A \cup B : f(x) \geq t\})dt.$$

Which in turn is equal to

$$\gamma(A)\nu(A \cup B) + (\gamma(B) - \gamma(A))\nu(B).$$

Note that this equality holds even if $\gamma(A) = \gamma(B)$. As $\mu << \nu$ we have $\gamma(B)\nu(B) = \mu(B)$ and therefore

$$\mu(A \cup B) - \mu(B) = \gamma(A)[\nu(A \cup B) - \nu(B)]. \qquad \square$$

6. Maxitive set functions

We now give an application of the results of the previous section. In particular, we show that maxitive set functions, an important class of capacities, are alternating capacities of infinite order.

Definition *Given an algebra $\mathcal{U}$, we say that a set function μ is maxitive on $\mathcal{U}$ if* $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ *for every $A, B \in \mathcal{U}$.*

As we shall see, maxitive set functions on $\mathcal{U}$ are related to a particular set function $\Gamma_{\mathcal{U}}$ which is defined by setting

$$\Gamma_{\mathcal{U}}(A) = \begin{cases} 1 & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases}$$

We shall often refer to $\Gamma_{\mathcal{U}}$ as Γ when the algebra $\mathcal{U}$ is clear from the context.

Lemma 6.1 *For any capacity μ on $\mathcal{U}$, the pair (μ, Γ) has the FRNP iff μ is maxitive.*

Proof. We first show that if μ is maxitive, then (μ, Γ) has FRNP. Surely $\mu << \Gamma$. Assume A, B are disjoint sets. We must show

$$\gamma(A) \leq \gamma(B) \Rightarrow \mu(A \cup B) - \mu(B) = \gamma(A)[\Gamma(A \cup B) - \Gamma(B)].$$

Assume $\gamma(A) \leq \gamma(B)$. This implies $\mu(A) \leq \mu(B)$ even in the case that one of A, B is the emptyset. Then as μ is maxitive $\mu(A \cup B) = \mu(B)$. Thus $\mu(A \cup B) - \mu(B) = 0$. If either A, B is the emptyset, then $\gamma(A) = 0$. If neither A, B is the emptyset then $\Gamma(A \cup B) - \Gamma(B) = 0$.

Next we show that if (μ, Γ) have the FRNP then μ is maxitive. Assume that $\mu(A) \leq \mu(B)$. If B is empty, then $\mu(A \cup B) = \mu(A) = 0$ and this is equal to the maximum of $\{\mu(A), \mu(B)\}$. If B is not empty, then by the above characterization of the FRNP we have

$$\mu(A \cup B) - \mu(B) = \gamma(A)[\Gamma(A \cup B) - \Gamma(B)].$$

And as $B \neq \emptyset$ we have $\Gamma(A \cup B) - \Gamma(B) = 0$. Thus $\mu(A \cup B) = \mu(B)$ and as $\mu(B) = \max\{\mu(A), \mu(B)\}$ our result is established. □

Using Theorem 5.1 we obtain an alternative proof of the following result given in Nguyen et al. (1997).

Theorem 6.2 *If μ is maxitive on $\mathcal{U}$, then μ is alternating of infinite order.*

Proof. As μ is maxitive, the pair (μ, Γ) has the FRNP and hence the LRNP. Once it is established that Γ is alternating of infinite order on $\mathcal{U}$, our result will follow from Theorem 5.1. It is not difficult to give an elementary proof of this based on the fact that for any $n \geq 1$ there are exactly as many subsets of $\{1, \dots, n\}$ of even cardinality as odd. But we shall instead direct the reader to the more general result contained in Proposition 3 of Marinacci (1996). □

7. Some counter-examples

In this section we show that RNP $\not\Rightarrow$ FRNP and FRNP $\not\Rightarrow$ RNP. In view of the fact that RNP $\Rightarrow$ LRNP and FRNP $\Rightarrow$ LRNP this also provides that LRNP $\not\Rightarrow$ RNP and LRNP $\not\Rightarrow$ FRNP.

Theorem 7.1 *Let $\mathcal{L}$ be the Lebesgue measurable subsets of the real interval $(0,1)$ and ν be Lebesgue measure on $\mathcal{L}$. There exists a map $\mu : \mathcal{L} \longrightarrow [0,1]$ such that*

(i) *μ is monotone and $\mu(\emptyset) = 0$.*

(ii) *μ is finitely additive.*

(iii) $\mu << \nu$

(iv) *μ is not countably additive.*

(v) *μ takes only the values 0 and 1.*

(vi) *(μ, ν) has the FRNP.*

(vii) *(μ, ν) does not have the RNP.*

Proof. Let $\mathcal{I} = \{A \in \mathcal{L} : \nu(A) = 0\}$, i.e. $\mathcal{I}$ is all sets of Lebesgue measure zero. For each natural number $n \geq 1$ define a subset A_n by setting

$$A_n = [\frac{1}{n+1}, \frac{1}{n}) \quad \text{for} \quad n \geq 1.$$

Obviously the A_n's are a countable family of pairwise disjoint Lebesgue measurable sets whose union is all of $(0,1)$. Consider the set

$$\mathcal{J} = \mathcal{I} \cup \{A_n : n \geq 1\}.$$

No finite union of members of $\mathcal{J}$ equals all of $(0,1)$ and $\mathcal{J}$ is contained in $\mathcal{L}$. Thus $\mathcal{J}$ generates a proper ideal of $\mathcal{L}$ and hence is contained in a maximal proper ideal $\mathcal{P}$ of $\mathcal{L}$. Note that as $\mathcal{P}$ is a maximal proper ideal of $\mathcal{L}$ we have that for any $A \in \mathcal{L}$ exactly one of A and A^c is in $\mathcal{P}$. The notation A^c denotes the complement of the set A. So we can define a map $\mu : \mathcal{L} \longrightarrow [0,1]$ by setting

$$\mu(A) = \begin{cases} 0 & \text{if } A \in \mathcal{P} \\ 1 & \text{if } A \notin \mathcal{P} \end{cases}$$

It is well known that μ is a homomorphism from $\mathcal{L}$ to the two element Boolean algebra $\{0,1\}$. Then considered as a set function, it follows that μ is monotone, $\mu(\emptyset) = 0$, and μ is finitely additive. But there is a countable family of pairwise disjoint sets A_n in $\mathcal{P}$ whose union is not in $\mathcal{P}$. So μ is not countably additive. Clearly μ takes only the values 0 and 1, and as $\mathcal{P}$ was constructed to contain all sets of Lebesgue measure zero, $\mu << \nu$. Thus the first five properties have been established. The sixth property follows from the finite additivity of μ and ν. We have only to show that (μ, ν) does not have the RNP. Suppose that $f : (0,1) \longrightarrow [0,\infty)$ is an $\mathcal{L}$ measurable function. Define a map $\pi : \mathcal{L} \longrightarrow [0,\infty)$ by setting

$$\pi(A) = \int_A f d\nu.$$

Translating the definition of this integral we have

$$\pi(A) = \int_0^\infty \nu(\{a \in A : f(a) \geq t\}) dt.$$

As ν is usual Lebesgue measure, $\int_0^\infty \nu(\{a \in A : f(a) \geq t\}) dt$ is nothing other the usual Lebesgue integral. But the Lebesgue integral is countably additive, more precisely, the function π defined above is countably additive. Thus π could not possibly equal μ as μ is not countably additive. So (μ, ν) does not have the RNP. $\square$

Having shown that FRNP $\not\Rightarrow$ RNP, even for finitely additive functions, we next provide an example that shows RNP $\not\Rightarrow$ FRNP, even for finite algebras.

Theorem 7.2 *There is an eight element algebra* $\mathcal{U}$ *and maps* $\mu, \nu : \mathcal{U} \longrightarrow [0,\infty)$ *so that* (μ,ν) *has the RNP but* (μ,ν) *does not have the FRNP.*

Proof. Let A, B, C be any disjoint non-empty sets. Let U be their union and $\mathcal{U}$ be the algebra of subsets of U generated by A, B, C. Then

$$\mathcal{U} = \{\emptyset, A, B, C, A \cup B, A \cup C, B \cup C, U\}.$$

Define $\nu : \mathcal{U} \longrightarrow [0, \infty)$ by specifying $\nu(\emptyset) = 0$, $\nu(A) = 1$, $\nu(B) = 1$, $\nu(C) = 2$, $\nu(A \cup B) = 1$, $\nu(A \cup C) = 2$, $\nu(B \cup C) = 2$ and $\nu(U) = 3$. As A, B, C partition U we may define a function $f : U \longrightarrow [0, \infty)$ by setting

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ 2 & \text{if } x \in B \\ 3 & \text{if } x \in C \end{cases}$$

Then f is $\mathcal{U}$-measurable. Define $\mu : \mathcal{U} \longrightarrow [0, \infty)$ by setting $\mu(A) = \int_A f d\nu$. By our construction, we have (μ, ν) has the RNP. We need a few calculations.

$$\mu(C) = \int_C f d\nu = \int_0^3 \nu(C) dt = 3 \times 2 = 6.$$

$$\mu(A \cup B) = \int_0^1 \nu(A \cup B) dt + \int_1^2 \nu(B) dt = 1 + 1 = 2.$$

$$\mu(U) = \int_0^1 \nu(U) dt + \int_1^2 \nu(B \cup C) dt + \int_2^3 \nu(C) dt = 3 + 2 + 2 = 7.$$

Recall that $\nu(C) = 2$, $\nu(A \cup B) = 1$ and $\nu(U) = 3$. Therefore $\gamma(C) = 3$ and $\gamma(A \cup B) = 2$. So $\gamma(A \cup B) \leq \gamma(C)$. But replacing the above values yields

$$\mu(U) - \mu(C) \neq \gamma(A \cup B)[\nu(U) - \nu(C)].$$

As ν is null additive, Theorem 5.2 proves that (μ, ν) does not have the FRNP. □

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