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Amalgamation of Ortholattices

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Abstract. We show that the variety of ortholattices has the strong amalgamation property and that the variety of orthomodular lattices has the strong Boolean amalgamation property, i.e. that two orthomodular lattices can be strongly amalgamated over a common Boolean subalgebra. We give examples to show that the variety orthomodular lattices does not have the amalgamation property and that the variety of modular ortholattices does not even have the Boolean amalgamation property. We further show that no non-Boolean variety of orthomodular lattices which is generated by orthomodular lattices of bounded height can have the Boolean amalgamation property.

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1. Introduction

Following the terminology of Grätzer [1, p. 252 ff], a V-formation in a class K of algebras is a quintuplet $(B, L_1, L_2, \varphi_1, \varphi_2)$ where B, L_1, L_2 are algebras in K and φ_i $(i = 1, 2)$ is an algebra-embedding of B into L_i. An amalgam of the V-formation in K is a triple (C, ψ_1, ψ_2) where $C \in K$, the ψ_i are algebra-embeddings of L_i into C satisfying $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$.

$$\begin{tikzcd} & L_1 \arrow[dr, "\psi_1"] & \\ B \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & C \\ & L_2 \arrow[ur, "\psi_2"'] & \end{tikzcd}$$

The amalgam is strong if, in addition, $\psi_1(L_1) \cap \psi_2(L_2) = \psi_1(\varphi_1(B))(= \psi_2(\varphi_2(B)))$ holds. The V-formation can be (strongly) amalgamated if there exists a (strong) amalgam of it. A class K has the (strong) amalgamation property iff every

V-formation in K can be (strongly) amalgamated. We will pay special attention to the case that B is a Boolean algebra. In this case we talk of Boolean amalgamation.

In this paper we study amalgamation in the class of ortholattices, orthomodular lattices and modular ortholattices. Throughout we abbreviate ortholattice as OL, orthomodular lattice as OML, and modular ortholattice as MOL. OLs stands not only for the plural of OL, but also for the class of all OLs, etc.

The question of amalgamation in these classes has so far received little attention. The simplest case was dealt with by MacLaren [8]. Here L_1, L_2 are OMLs and B is the two-element Boolean algebra. A strong amalgamation in OMLs is obtained by "identifying" the bounds in the disjoint union of L_1 and L_2. The construction has become known as the horizontal sum of L_1 and L_2. This is a very special case of Greechie's celebrated paste job [2]. His assumptions are that L_1 and L_2 are OMLs and that there exists an element $a \in B$ such that $\varphi_i(B)$ is the union of the principal ideal $[0, \varphi_i(a)]$ and the principal filter $[\varphi_i(a'), 1]$ in L_i. Strong amalgamation in OMLs is again obtained by "identifying" $\varphi_1(B)$ and $\varphi_2(B)$ in the disjoint union of L_1 and L_2. A considerably more complicated case was investigated by Schulte-Mönting [9]. Here it is again assumed that L_1 and L_2 are arbitrary OMLs but that for $i = 1, 2$ $\varphi_i(B)$ is a subalgebra of the centre of L_i. It is shown that in this case we also have strong amalgamation in OMLs.

In Section 2 of this paper we show that OLs have the strong amalgamation property. The proof is an easy adaptation of a well-known construction first used by Jónsson [4] to show that lattices have the strong amalgamation property. The bulk of the paper, in which we show that OMLs have the strong Boolean amalgamation property, is contained in Sections 6 and 7. The remaining results we have are negative. In Section 3 we show that OMLs do not have the amalgamation property. In our counter-example L_1 and L_2 are finite and B is MO3. (Recall that MOn is the MOL consisting of $2n$ incomparable elements and the bounds.) In Section 4 we show that MOLs do not have Boolean amalgamation and in Section 5 we show the same for every non-Boolean variety of OMLs which is generated by OMLs of bounded height. In both counter-examples B is an eight element Boolean algebra.

A note on notation. If f is a map then $f_{|X}$ is the restriction of f to X, id_X is the identity map of X, $|X|$ is the cardinal number of X. In the last two sections we use $A \leq B$ for "A is a subalgebra of B".

For background information concerning OMLs the reader is referred to [6]. Both authors gratefully acknowledge support by the Natural Sciences and Engineering Research Council of Canada, grant 0002985 (G.B.) and grant OGP0155640 (J.H.).

2. The Partial Amalgam, Amalgamation of OLs

The definition of amalgamation as given in the introduction is often cumbersome to work with. It can, in most cases, be replaced by the following simpler concept. Define a special V-formation to be a triple (B, L_1, L_2) where $B = L_1 \cap L_2$ is a subalgebra of both L_1 and L_2, confusing, as usual, the algebras with their underlying sets. A special V-formation gives rise to the V-formation $(B, L_1, L_2, \mathrm{id}_B, \mathrm{id}_B)$ and hence the concept of amalgamation as defined in the introduction can be applied to special V-formations. It turns out that under weak assumptions on a class K the existence of amalgams of V-formations and special V-formations are equivalent. We are sure the reader will find it easy to verify the following observation which makes this statement precise.

OBSERVATION. Let K be a class of algebras which is closed under isomorphisms and let $(B, L_1, L_2, \varphi_1, \varphi_2)$ be a V-formation in K. Then the following two statements are equivalent.

1. $(B, L_1, L_2, \varphi_1, \varphi_2)$ can be (strongly) amalgamated in K.
2. There exists a special V-formation (B, K_1, K_2) in K and two isomorphisms $f_i\colon K_i \to L_i$ satisfying $f_{i|B} = \varphi_i$ such that (B, K_1, K_2) can be (strongly) amalgamated in K.

The following construction of the partial amalgam of a special V-formation is well known and has been used before, see [4].

DEFINITION. Let L_1, L_2 be OLs and assume $B = L_1 \cap L_2$ is a subalgebra of both L_1 and L_2, i.e. that (B, L_1, L_2) is a special V-formation. Let $\leq_i$ be the partial ordering of L_i. Define a relation $\leq$ in $L_1 \cup L_2$ by setting $a \leq b$ if one of the following conditions is satisfied.

1. $a, b \in L_i$ and $a \leq_i b$.
2. $a \in L_i - L_j, b \in L_j - L_i$ $(i \neq j)$ and there exists $m \in B$ such that $a \leq_i m \leq_j b$.

It is easily seen that $\leq$ is a partial ordering of $L_1 \cup L_2$, and that if $a, b \in L_i$ then the join and meet of a and b in L_i is the same as in the partially ordered set $L_1 \cup L_2$ and that the union of the orthocomplementations in L_1 and L_2 is an orthocomplementation of $L_1 \cup L_2$. Thus $L_1 \cup L_2$ becomes an orthocomplemented poset which we call the partial amalgam of L_1 and L_2.

Simple examples show that the partial amalgam is not in general a lattice. It is, however, well known and easy to prove that the MacNeille completion of an orthocomplemented poset P carries a unique orthocomplementation extending the orthocomplementation of P, hence becomes an OL. Let $f\colon L_1 \cup L_2 \to C$ be the canonical embedding of this partial amalgam into its MacNeille completion and define $\psi_i = f_{|L_i}$. It is then obvious that (C, ψ_1, ψ_2) is a strong amalgam of the special V-formation (B, L_1, L_2). We thus obtain

THEOREM 1. *OLs have the strong amalgamation property.*

3. OMLs do not Have the Amalgamation Property

We consider the special V-formation (B, L_1, L_2) where L_1 and L_2 are given by the following Greechie diagrams.

L_1 L_2

The letters attached to the vertices denote atoms. Thus p is an atom of L_1 and a co-atom of L_2, etc. We assume that L_1, L_2 have the subalgebra generated by $\{p, q, s\}$ in common, but nothing else. Thus $B = L_1 \cap L_2$ is MO3.

Assume now that this special V-formation could be amalgamated in OMLs by (C, ψ_1, ψ_2). Identifying the elements of L_i with their images under ψ_i we would obtain

$$1 = m \vee q \leq m \vee q \vee e = m \vee b' = m \vee a \vee e.$$

Since $a \leq e'$ and $m \leq s \leq e'$ this would give

$$a \vee m = e'$$

and

$$r' = m \vee p' = m \vee a \vee c = e' \vee c = 1,$$

a contradiction. We thus have

THEOREM 2. *OMLs do not have the amalgamation property.*

4. MOLs do not Have Boolean Amalgamation

As we will see later a V-formation $(B, L_1, L_2, \varphi_1, \varphi_2)$ in OMLs can be strongly amalgamated in OMLs if B is a Boolean algebra. As opposed to this we will show in this section that a V-formation as above in MOLs cannot be amalgamated in MOLs even if B is an eight element Boolean algebra.

Let B be an eight-element Boolean algebra generated by the chain $0 < x < y < 1$. Let P be (the OL of subspaces of) a non-arguesian orthocomplemented projective plane. For the existence of these see [3, 10]. Define $L_1 = P \times 2$, where 2 is the two-element Boolean algebra and let a be an atom of P. Let L_2 be an arbitrary orthocomplemented projective plane; let m be a co-atom (line) of L_2 and let $e, f, g < m$ be atoms of L_2. Clearly there exist OL-embeddings $\varphi_i\colon B \to L_i$ satisfying

$$\varphi_1(x) = (0, 1), \quad \varphi_1(y) = (a, 1), \quad \varphi_2(x) = e, \quad \varphi_2(y) = m.$$

Assume now that the resulting V-formation could be amalgamated in MOLs by (C, ψ_1, ψ_2). Note that

$$\psi_1(0, 1) = \psi_1(\varphi_1(x)) = \psi_2(\varphi_2(x)) = \psi_2(e)$$

and

$$\psi_1(a, 1) = \psi_1(\varphi_1(y)) = \psi_2(\varphi_2(y)) = \psi_2(m).$$

Note furthermore that the sublattice $E = [(0, 1), (1, 1)]$ of L_1 is isomorphic with P and hence simple as a lattice. Also the sublattice $F = \{0, e, f, g, m\}$ of L_2 is simple. Now let $\varphi\colon C \to \prod_{i \in I} M_i$ be a subdirect representation of C, where φ is an OL-embedding of C into the product of the subdirectly irreducible MOLs M_i and if pr_i is the ith projection, the maps $\mathrm{pr}_i \circ \varphi$ are onto M_i. Since

$$\psi_1(0, 1) \neq \psi_1(a, 1)$$

there exists an index $i \in I$ such that

$$\mathrm{pr}_i(\varphi(\psi_1(0, 1))) \neq \mathrm{pr}_i(\varphi(\psi_1(a, 1))).$$

Thus the homomorphism $\mathrm{pr}_i \circ \varphi \circ \psi_1$ does not collapse the elements $(0, 1)$ and $(a, 1)$ of E. Since E is simple it follows that $\mathrm{pr}_i \circ \varphi \circ \psi_1$ restricted to E is a lattice embedding of E into M_i. Since

$$\mathrm{pr}_i(\varphi(\psi_2(e))) = \mathrm{pr}_i(\varphi(\psi_1(0, 1)))$$

and

$$\mathrm{pr}_i(\varphi(\psi_2(m))) = \mathrm{pr}_i(\varphi(\psi_1(a, 1)))$$

the homomorphism $\mathrm{pr}_i \circ \varphi \circ \psi_2$ does not collapse the elements e and m of F and it follows that the restriction of $\mathrm{pr}_i \circ \varphi \circ \psi_2$ to F is a lattice embedding of F into M_i. In particular,

$$\mathrm{pr}_i(\varphi(\psi_2(e))) \neq 0.$$

Since E is lattice-isomorphic with P it follows that $\mathrm{pr}_i(\varphi(\psi_1(E)))$ contains a four-element chain with smallest element

$$\mathrm{pr}_i(\varphi(\psi_1(0, 1))) = \mathrm{pr}_i(\varphi(\psi_2(e))) \neq 0.$$

It follows that M_i contains a five-element chain and hence, by [5], is arguesian. But P is isomorphic with a sublattice of M_i and is not arguesian, which is a contradiction. Thus we have

THEOREM 3. *MOLs do not have Boolean amalgamation.*

5. Boolean Amalgamation in OMLs of Bounded Height

We show in this section that the fact that OMLs have Boolean amalgamation is no longer true if one replaces the variety of all OMLs by a variety of OMLs generated by OMLs of bounded height. In order to be precise we make the following assumption.

We assume that $n \geq 3$ is a natural number; that $\mathcal{V}$ is a variety of OMLs in which every chain in a subdirectly irreducible member of $\mathcal{V}$ has at most n elements and that there exists a subdirectly irreducible member L of $\mathcal{V}$ which contains an n-element chain. We show that such a variety does not have Boolean amalgamation even if B is the eight-element Boolean algebra.

Let B be an eight-element Boolean algebra generated by the chain $0 < x < y < 1$. Let L be a subdirectly irreducible member of $\mathcal{V}$ containing an n-element chain and let a be an atom of such a chain. Define $L_1 = L_2 = L \times 2$. Then there exist OL-embeddings $\varphi_i\colon B \to L_i$ satisfying

$$\varphi_1(x) = (0, 1), \quad \varphi_1(y) = (a, 1), \quad \varphi_2(x) = (a', 0), \quad \varphi_2(y) = (1, 0).$$

We show that the resulting V-formation cannot be amalgamated in $\mathcal{V}$.

Assume now that (C, ψ_1, ψ_2) was an amalgam of the above V-formation in $\mathcal{V}$. Define $E = [\varphi_1(x), 1] = [(0, 1), (1, 1)]$ and $F = [0, \varphi_2(y)] = [(0, 0), (1, 0)]$. Clearly E and F are lattice isomorphic with L and L is chain-finite and subdirectly irreducible as an OML, hence simple as a lattice. Thus $\psi_1(E)$ and $\psi_2(F)$ are simple as lattices. Note that

$$\begin{aligned}
&\psi_2(\varphi_2(x)) = \psi_2(a', 0) \in \psi_2(F),\\
&\psi_2(\varphi_2(x)) = \psi_1(\varphi_1(x)) = \psi_1(0, 1) \in \psi_1(E),\\
&\psi_1(\varphi_1(y)) = \psi_1(a, 1) \in \psi_1(E),\\
&\psi_1(\varphi_1(y)) = \psi_2(\varphi_2(y)) = \psi_2(1, 0) \in \psi_2(F).
\end{aligned}$$

Thus

$$\psi_2(\varphi_2(x)), \psi_1(\varphi_1(y)) \in \psi_1(E), \psi_2(F).$$

Now let $\varphi\colon C \to \prod_{i \in I} M_i$ be a subdirect product representation of C by subdirectly irreducible OMLs M_i and $w_i = \mathrm{pr}_i \circ \varphi$. Since

$$\psi_2(\varphi_2(x)) < \psi_2(\varphi_2(y)) = \psi_1(\varphi_1(y)),$$

there exists an index $i \in I$ such that

$$w_i(\psi_2(\varphi_2(x)) < w_i(\psi_1(\varphi_1(y))).$$

Since $\psi_2(\varphi_2(x))$ and $\psi_1(\varphi_1(y))$ both belong to $\psi_1(E)$ and $\psi_2(E)$ it follows that the restriction of w_i to $\psi_1(E)$ and to $\psi_2(F)$ are lattice embeddings. But $w_i(\psi_1(E))$ contains an n-element chain with smallest element $w_i(\psi_1(\varphi_1(x)))$. Since $\varphi_2(0) < \varphi_2(x)$ in F we obtain

$$w_i(\psi_2(\varphi_2(0))) < w_i(\psi_2(\varphi_2(x))) = w_i(\psi_1(\varphi_1(x))) \neq 0.$$

Thus M_i contains an $(n+1)$-element chain contradicting Jónsson's celebrated lemma [1]. Thus our V-formation cannot be amalgamated in $\mathcal{V}$. We thus have

THEOREM 4. *If $\mathcal{V}$ is a non-Boolean variety of OMLs generated by OMLs of bounded height then $\mathcal{V}$ does not have Boolean amalgamation.*

6. Boolean Amalgamation, Preliminaries

LEMMA 6.1. *Let L_1, L_2 be OMLs, $B = L_1 \cap L_2 \leq L_1, L_2$, B Boolean, $a \in L_1 \cup L_2$. Then for $i = 1, 2$ there exist OMLs M_i, Boolean algebras $B_i \leq M_i$, OL-embeddings $\alpha_i\colon L_i \to M_i$ and an isomorphism $\gamma\colon B_1 \to B_2$ such that*

1. $\alpha_i(B) = \alpha_i(L_i) \cap B_i \leq B_i$,
2. $\alpha_{2|B} = \gamma \circ \alpha_{1|B}$,
3. *if L_i is infinite then $|L_i| = |M_i|$,*
4. *if $e \in L_i$ and $m = \max\{b \in B \mid b \leq e\}$ then $\alpha_i(m) = \max\{b \in B_i \mid b \leq \alpha_i(e)\}$,*
5. *if $a \in L_i$ then $\{b \in B_i \mid b \leq \alpha_i(a)\}$ has a maximum.*

Proof. Define

$$X = \{x \in B \mid x \leq a\}.$$

For $c \in L_i$ define $\tilde{c} \in L_i^X$ by

$$\tilde{c}(x) = c \text{ for all } x \in X.$$

Define

$$a^* = \mathrm{id}_X,$$
$B[a]$ is the subalgebra of B^X generated by $\{a^*\} \cup \{\tilde{c} \mid c \in B\}$,
$L_i[a]$ is the subalgebra of L_i^X generated by $\{a^*\} \cup \{\tilde{c} \mid c \in L_i\}$.

Define relations θ_i in $L_i[a]$ by

$$f \theta_i g \Leftrightarrow \text{there exists } k \in X \text{ such that } [k, \to] \subseteq \{x \in X \mid f(x) = g(x)\}.$$

Here $[k, \to] = \{x \in X \mid k \leq x\}$. Noting that θ_i is a congruence in $L_i[a]$ define

$$M_i = L_i[a]/\theta_i,$$
$$B_i = \{f/\theta_i \mid f \in B[a]\}.$$

Clearly $B[a]$ is a subalgebra of $L_i[a]$ $(i = 1, 2)$ and hence $B_i \leq M_i$. But $B[a]$ is a subalgebra of the Boolean algebra B^X and hence B_i is Boolean. It is easy to see that the map

$$\alpha_i\colon L_i \to M_i \quad \text{defined by } \alpha_i(c) = \tilde{c}/\theta_i$$

is an OL-embedding. Clearly $\alpha_i(B) \subseteq \alpha_i(L_i) \cap B_i$. Assume $c \in L_i$ and $\alpha_i(c) \in B_i$. Then $\tilde{c}/\theta_i = f/\theta_i$ for some $f \in B[a]$ and hence there exists $k \in X$ such that $[k, \rightarrow] \subseteq \{x \in X \mid \tilde{c}(x) = f(x)\}$. In particular, $\tilde{c}(k) = f(k)$ and thus $c = f(k) \in B$. Then as B_i and $\alpha_i(L_i)$ are subalgebras of M_i so also is their intersection. We have thus proved

(1) $\alpha_i(B) = \alpha_i(L_i) \cap B_i \leq B_i$.

Note that the restrictions of θ_1 and θ_2 to $B[a]$ agree, thus there is an isomorphism

$$\gamma\colon B_1 \rightarrow B_2 \text{ such that } \gamma(f/\theta_1) = f/\theta_2 \text{ for all } f \in B[a].$$

It is easy to see that

(2) $\alpha_{2|B} = \gamma \circ \alpha_{1|B}$.

As M_i is generated by $\alpha_i(L_i) \cup \{a^*/\theta_i\}$ it follows that

(3) if L_i is infinite then $|L_i| = |M_i|$.

(4) If $e \in L_i$ and $m = \max\{b \in B \mid b \leq e\}$ then $\alpha_i(m) = \max\{b \in B_i \mid b \leq \alpha_i(e)\}$.

Note that $m \in B$ and hence $\alpha_i(m) \in B_i$. But

$$\tilde{m}(x) = m \leq e = \tilde{e}(x) \quad \text{for all } x \in X.$$

Hence $\alpha_i(m) \leq \alpha_i(e)$. Assume $b \in B_i$. Then there exists $f \in B[a]$ with $b = f/\theta_i$. As

$$f(x) \leq e \Leftrightarrow f(x) \leq m,$$

thus

$$f/\theta_i \leq \tilde{e}/\theta_i \Leftrightarrow f/\theta_i \leq \tilde{m}/\theta_i.$$

Hence $b \leq \alpha_i(e) \Leftrightarrow b \leq \alpha_i(m)$, proving (4).

(5) $a^*/\theta_i = \max\{b \in B_i \mid b \leq \alpha_i(a)\}$.

Note that $a^* \in B[a]$ and hence $a^*/\theta_i \in B_i$. Also

$$a^*(x) = x \leq a = \tilde{a}(x) \quad \text{for all } x \in X$$

and hence

$$a^*/\theta_i \leq \tilde{a}/\theta_i = \alpha_i(a).$$

Assume $b \in B_i$ and $b \leq \alpha_i(a)$. Then there exists $f \in B[a]$ such that $b = f/\theta_i$. Since $B[a]$ is a Boolean algebra generated by the subalgebra $\{b \mid b \in B\}$ and the singleton $\{a^*\}$, there exist $c, d \in B$ such that

$$f = (a^* \wedge \tilde{c}) \vee (a^{*'} \wedge \tilde{d}).$$

Since $b \leq \alpha_i(a)$ there exists $k \in X$ such that

$$[k, \to] \subseteq \{x \in X \mid f(x) \leq \tilde{a}(x)\}.$$

But for $x \in X$,

$$f(x) \leq \tilde{a}(x) \Leftrightarrow (x \wedge c) \vee (x' \wedge d) \leq a \Leftrightarrow x' \wedge d \leq a,$$

hence $k' \wedge d \leq a$ and $d \leq a$. Thus if $d \leq x \in X$ then

$$f(x) = (x \wedge c) \vee (x' \wedge d) \leq x = a^*(x),$$

hence

$$[d, \to] \subseteq \{x \in X \mid f(x) \leq a^*(x)\}$$

and $b = f/\theta_i \leq a^*/\theta_i$, proving (5) and the lemma. □

LEMMA 6.2. *Let L_1, L_2 be OMLs, $B = L_1 \cap L_2 \leq L_1, L_2$, B Boolean, $a \in L_1 \cup L_2$. Then there exist OMLs $L_1(a), L_2(a)$ and a Boolean algebra $B(a)$ such that*

1. $B \leq B(a) = L_1(a) \cap L_2(a) \leq L_1(a), L_2(a)$,
2. $L_i \leq L_i(a)$,
3. *if L_i is infinite then* $|L_i| = |L_i(a)|$,
4. *if $e \in L_1 \cup L_2$ and $m = \max\{b \in B \mid b \leq e\}$ then $m = \max\{b \in B(a) \mid b \leq e\}$,*
5. $\{b \in B(a) \mid b \leq a\}$ *has a maximum,*
6. $B(a) \cap (L_1 \cup L_2) = B$.

Proof. Let $M_i, B_i, \alpha_i, \gamma$ be as in the previous lemma. Choose sets A, D_1, D_2 which are pairwise disjoint and disjoint with $L_1 \cup L_2$ such that their cardinal numbers allow the existence of bijections $\varphi_1, \delta_1, \delta_2$ where

$$\varphi_1\colon\ A \to B_1 - \alpha_1(B),$$
$$\delta_i\colon\ D_i \to M_i - (\alpha_i(L_i) \cup B_i).$$

Define $\varphi_2 = \gamma \circ \varphi_1$. As $\gamma\colon\ B_1 \to B_2$ is an isomorphism with $\gamma \circ \alpha_{1|B} = \alpha_{2|B}$ we thus have that φ_2 is a bijection

$$\varphi_2\colon\ A \to B_2 - \alpha_2(B).$$

For $i = 1, 2$ define $L_i(a) = L_i \cup A \cup D_i$ and define maps $f_i\colon\ L_i(a) \to M_i$ by

$$f_{i|L_i} = \alpha_i,$$
$$f_{i|A} = \varphi_i,$$
$$f_{i|D_i} = \delta_i.$$

Note that

$$f_i(L_i) = \alpha_i(L_i),$$
$$f_i(A) = B_i - \alpha_i(B),$$
$$f_i(D_i) = M_i - (\alpha_i(L_i) \cup B_i).$$

By assumption, $\alpha_i(B) = \alpha_i(L_i) \cap B_i$ and thus $f_i(A) = B_i - \alpha_i(L_i)$. It is easy to see that the sets $f_1(L_1)$, $f_1(A)$, $f_1(D_1)$ are pairwise disjoint with union M_1 and the sets $f_2(L_2)$, $f_2(A)$, $f_2(D_2)$ are pairwise disjoint with union M_2. As the restrictions of f_i to L_i, A, D_i are bijections it follows that

(1) $f_i\colon L_i(a) \to M_i$ is a bijection.

Define now operations $\vee_i, \wedge_i, {}^{\prime i}, 0_i, 1_i$ in $L_i(a)$ by

$$b \vee_i c = f_i^{-1}(f_i(b) \vee f_i(c)),$$
$$b \wedge_i c = f_i^{-1}(f_i(b) \wedge f_i(c)),$$
$$b^{\prime i} = f_i^{-1}(f_i(b)'),$$
$$0_i = f_i^{-1}(0),$$
$$1_i = f_i^{-1}(1),$$

where the operations on the right-hand side of the equations are taken in M_i. It is clear that with these definitions $L_i(a)$ becomes an OML and f_i becomes an OL-isomorphism between $L_i(a)$ and M_i.

(2) In L_i the original operations $\vee, \wedge, {}', 0, 1$ agree with $\vee_i, \wedge_i, {}^{\prime i}, 0_i, 1_i$, so $L_i \leq L_i(a)$.

For $b, c \in L_i$,

$$\begin{aligned} b \vee_i c &= f_i^{-1}(f_i(b) \vee f_i(c)) && (\vee \text{ in } M_i) \\ &= f_i^{-1}(\alpha_i(b) \vee \alpha_i(c)) && (\text{same}) \\ &= f_i^{-1}(\alpha_i(b \vee c)) && (\vee \text{ in } L_i) \\ &= f_i^{-1}(f_i(b \vee c)) \\ &= b \vee c. \end{aligned}$$

$$\begin{aligned} b^{\prime i} &= f_i^{-1}(f_i(b)') && ({}' \text{ in } M_i) \\ &= f_i^{-1}(\alpha_i(b)') && (\text{same}) \\ &= f_i^{-1}(\alpha_i(b')) && ({}' \text{ in } L_i) \\ &= f_i^{-1}(f_i(b')) \\ &= b'. \end{aligned}$$

That the remaining operations are the same is a consequence of this.

(3) $f_{i|B\cup A}\colon\ B \cup A \to B_i$ is a bijection.

This follows as f_i is one-one, $f_i(A) = B_i - \alpha_i(B)$ and $f_i(B) = \alpha_i(B)$. As $\gamma \circ \alpha_{1|B} = \alpha_{2|B}$ and $\gamma \circ \varphi_1 = \varphi_2$ it is easy to see that

(4) $f_2(b) = \gamma(f_1(b))$ for all $b \in B \cup A$,

(5) $f_2^{-1}(b) = f_1^{-1}(\gamma^{-1}(b))$ for all $b \in B_2$.

It follows from (4) and (5) that in $B \cup A = L_1(a) \cap L_2(a)$ the OL-operations of $L_1(a)$ and $L_2(a)$ coincide and yield elements of $B \cup A$. If we define $B(a)$ to be the subalgebra $B \cup A$ of L_i we thus have

(6) $f_{i|B\cup A}\colon\ B(a) \to B_i$ is an isomorphism, hence

(7) $B \leq B(a) = L_1(a) \cap L_2(a) \leq L_1(a), L_2(a)$ and $B(a)$ is Boolean.

Since f_i is an isomorphism between $L_i(a)$ and M_i the cardinal number of $L_i(a)$ equals that of M_i. By our choice of M_i, if L_i is infinite then $|L_i| = |M_i|$. Thus

(8) If L_i is infinite then $|L_i| = |L_i(a)|$.

(9) If $e \in L_1 \cup L_2$ and $m = \max\{b \in B \mid b \leq e\}$ then $m = \max\{b \in B(a) \mid b \leq e\}$.

Assume $e \in L_i$, $b \in B(a)$ and $b \leq e$. We have to show that $b \leq m$. Clearly $f_i(b) \leq f_i(e) = \alpha_i(e)$. By our choice of α_i we have $\alpha_i(m) = \max\{b \in B_i \mid b \leq \alpha_i(e)\}$ and hence $f_i(b) \leq \alpha_i(m) = f_i(m)$. As $f_{i|B\cup A}$ is an isomorphism $b \leq m$, proving (9).

(10) $\{b \in B(a) \mid b \leq a\}$ has a maximum.

Assume $a \in L_i$. By our choice of B_i the set $\{b \in B_i \mid b \leq \alpha_i(a)\}$ has a maximum m. Then $f_i^{-1}(m) \in B(a)$ and as α_i is an embedding $f_i^{-1}(m) = \alpha_i^{-1}(m) \leq a$. Assume that $b \in B(a)$ and $b \leq a$. Then $f_i(b) \in B_i$ and $f_i(b) \leq f_i(a) = \alpha_i(a)$ and hence $f_i(b) \leq m$. As $f_{i|B\cup A}$ is an isomorphism $b \leq f_i^{-1}(m)$, proving (10) and hence Lemma 6.2. $\square$

The OMLs $L_1(a), L_2(a)$ in the previous lemma are, of course, not completely determined by L_1, L_2 and a. Using AC we may, however, assume that specific OMLs $L_1(a), L_2(a)$ are chosen for every L_1, L_2, a as in the assumption of Lemma 6.2. We will assume this in the proof of the next lemma.

LEMMA 6.3. *Let L_1, L_2 be OMLs, $B = L_1 \cap L_2 \leq L_1, L_2$, B Boolean. Then there exist OMLs L_1', L_2' and a Boolean algebra B' such that*

1. $B \leq B' = L_1' \cap L_2' \leq L_1', L_2'$,
2. $L_i \leq L_i'$ $(i = 1, 2)$,
3. $|L_i'| \leq \max\{|L_1|, |L_2|\}$,
4. *if $e \in L_1 \cup L_2$ and $m = \max\{b \in B \mid b \leq e\}$ then $m = \max\{b \in B' \mid b \leq e\}$,*
5. *for every $a \in L_1 \cup L_2$ the set $\{b \in B' \mid b \leq a\}$ has a maximum,*
6. $B' \cap (L_1 \cup L_2) = B$.

Proof. If L_1 or L_2 is finite then B is finite and we may choose $L_i' = L_i$ and $B' = B$. We may thus assume that L_1 and L_2 are infinite. Define $\lambda = \max\{|L_1|, |L_2|\}$. Clearly $\lambda = |L_1 \cup L_2|$. Enumerate the elements by λ (considered as an initial ordinal). Thus $L_1 \cup L_2 = \{a_\mu \mid \mu < \lambda\}$. For every $\mu \leq \lambda$ define recursively OMLs $L_{1\mu}, L_{2\mu}$ and Boolean algebras B_μ by

$$B_0 = B, \quad L_{10} = L_1, \quad L_{20} = L_2,$$
$$B_{\nu+1} = B_\nu(a_\nu), \quad L_{1\nu+1} = L_{1\nu}(a_\nu), \quad L_{2\nu+1} = L_{2\nu}(a_\nu),$$
$$B_\mu = \bigcup_{\nu<\mu} B_\nu, \quad L_{1\mu} = \bigcup_{\nu<\mu} L_{1\nu}, \quad L_{2\mu} = \bigcup_{\nu<\mu} L_{2\nu} \quad \text{if } \mu \text{ is a limit ordinal.}$$

Define

$$B' = B_\lambda, \quad L_1' = L_{1\lambda} \quad \text{and} \quad L_2' = l_{2\lambda}.$$

It is an easy exercise to show that B', L_1', L_2' have the desired properties. □

As before we may again assume that given L_1, L_2, B as in the previous lemma specific B', L_1', L_2' are chosen satisfying the previous lemma.

LEMMA 6.4. *Let L_1, L_2 be OMLs, $B = L_1 \cap L_2 \leq L_1, L_2$, B Boolean. Then there exist OMLs $\overline{L}_1, \overline{L}_2$ and a Boolean algebra $\overline{B}$ such that*
1. $B \leq \overline{B} = \overline{L}_1 \cap \overline{L}_2 \leq \overline{L}_1, \overline{L}_2$,
2. $L_i \leq \overline{L}_i$ $(i = 1, 2)$,
3. *for every $a \in \overline{L}_1 \cup \overline{L}_2$ the set $\{b \in B' \mid b \leq a\}$ has a maximum,*
4. $\overline{B} \cap (L_1 \cup L_2) = B$.

Proof. For $n < \omega$ define recursively OMLs L_{1n}, L_{2n} and Boolean algebras B_n by

$$B_0 = B, \quad L_{10} = L_1, \quad L_{20} = L_2,$$
$$B_{n+1} = B_n', \quad L_{1n+1} = L_{1n}', \quad L_{2n+1} = L_{2n}'.$$

Put

$$\overline{B} = \bigcup_{n<\omega} B_n, \quad \overline{L}_1 = \bigcup_{n<\omega} L_{1n} \quad \text{and} \quad \overline{L}_2 = \bigcup_{n<\omega} L_{2n}.$$

It is again an easy exercise to see that $\overline{B}$ and the $\overline{L}_i$ have the desired properties. □

7. Boolean Amalgamation of OMLs

THEOREM 5. *Let $(B, L_1, L_2, \varphi_1, \varphi_2)$ be a V-formation in OMLs, where B is a Boolean algebra. Then this V-formation can be strongly amalgamated on OMLs.*

Proof. It follows from the observation in Section 2 that we may actually assume that $B = L_1 \cap L_2$ is a subalgebra of both L_1 and L_2 and that $\varphi_{1|B} = \varphi_{2|B} = \mathrm{id}_B$, i.e. that (B, L_1, L_2) is a special V-formation. Choose $\overline{L}_1, \overline{L}_2, \overline{B}$ as in Lemma 6.4 and assume that the new special V-formation $(\overline{B}, \overline{L}_1, \overline{L}_2)$ can be strongly amalgamated in OMLs by (C, ψ_1, ψ_2). Then it is easy to see that $(C, \psi_{1|L_1}, \psi_{2|L_2})$ strongly amalgamates the original V-formation. We may thus work under the assumptions

(1) $B = L_1 \cap L_2 \leq L_1, L_2$

(2) for every $a \in L_1 \cup L_2$, $\underline{a} = \max\{b \in B \mid b \leq a\}$ exists.

Let $P = L_1 \cup L_2$ be the partial amalgam as defined in Section 2. It is then easy to see that for every $a \in P$,

$$\overline{a} = \min\{b \in B \mid a \leq b\} \text{ exists} \quad \text{and} \quad \overline{a'} = (\underline{a})', \underline{a'} = (\overline{a})'.$$

Furthermore, it is an easy consequence of the definition of the partial amalgam that

(3) if $a \in L_i, b \in L_j$ $(i \neq j)$ then in P: $a \leq b \Leftrightarrow \overline{a} \leq b \Leftrightarrow a \leq \underline{b} \Leftrightarrow \overline{a} \leq \underline{b}$.

For $X \subseteq P$ let uX (or $u(X)$) be the set of all upper bounds of X in P and let lX (or $l(X)$) be the set of all lower bounds of X in P. If $a, b \in P$ we define

$$[a, b]_i = \{x \in L_i \mid a \leq x \leq b\}.$$

(4) If $a \in L_1, b \in L_2$ then

$$\begin{aligned}
&L_1 \cap u\{a, b\} = [a \vee \overline{b}, 1]_1, \qquad L_2 \cap u\{a, b\} = [\overline{a} \vee b, 1]_2, \quad \text{in particular}\\
&u\{a, b\} = [a \vee \overline{b}, 1]_1 \cup [\overline{a} \vee b, 1]_2,\\
&L_1 \cap l\{a, b\} = [0, a \wedge \underline{b}]_1, \qquad L_2 \cap l\{a, b\} = [0, \underline{a} \wedge b]_2, \quad \text{in particular}\\
&l\{a, b\} = [0, a \wedge \underline{b}]_1 \cup [0, \underline{a} \wedge b]_2.
\end{aligned}$$

Note that $a, \overline{b} \in L_1$ and hence $a \vee \overline{b}$ exists in L_1 and hence in P and the same for the remaining joins and meets. Clearly

$$L_1 \cap u\{a, b\} \supseteq [a \vee \overline{b}, 1]_1.$$

If $a, b \leq x \in L_1$ then, by (3), $a \vee \overline{b} \leq x$, proving the first equation. The second equation follows by symmetry and the third is a consequence of the first two. The rest follows by duality.

(5) For $a \in L_1, b \in L_2$,

$$\begin{aligned}
&L_1 \cap l(u\{a, b\}) = [0, (a \vee \overline{b}) \wedge (\underline{\overline{a} \vee b})]_1 \quad \text{and}\\
&L_2 \cap l(u\{a, b\}) = [0, (\underline{a \vee \overline{b}}) \wedge (\overline{a} \vee b)]_2, \quad \text{in particular}\\
&l(u\{a, b\}) = [0, (a \vee \overline{b}) \wedge (\underline{\overline{a} \vee b})]_1 \cup [0, (\underline{a \vee \overline{b}}) \wedge (\overline{a} \vee b)]_2.
\end{aligned}$$

We show the first equality. By (4) we have

$$l(u\{a, b\}) = l([a \vee \overline{b}, 1]_1 \cup [\overline{a} \vee b, 1]_2) = l\{a \vee \overline{b}, \overline{a} \vee b\}$$

hence by (4)

$$L_1 \cap l(u\{a, b\}) = [0, (a \vee \overline{b}) \wedge (\underline{\overline{a} \vee b})]_1.$$

Recall that a normal ideal in P is a set $A \subseteq P$ satisfying $A = l(u(A))$ or equivalently, for which there exists a set $X \subseteq P$ such that $A = l(X)$. We define a normal ideal A to be finitely generated iff there exists a finite set $F \subseteq P$ such that $A = l(u(F))$.

(6) For $A \subseteq P$ the following are equivalent.

1. A is a finitely generated normal ideal.
2. There exist $a \in L_1, b \in L_2$ such that
   (a) $L_1 \cap A = [0, a]_1$, $L_2 \cap A = [0, b]_2$,
   (b) $a = (a \vee \overline{b}) \wedge (\underline{\overline{a} \vee b})$ and $b = (\underline{a \vee \overline{b}}) \wedge (\overline{a} \vee b)$.
3. There exist $c \in L_1, d \in L_2$ such that $L_1 \cap A = [0, c \wedge \underline{d}]_1$, $L_2 \cap A = [0, \underline{c} \wedge d]_2$.

$1 \Rightarrow 2$. By assumption there exists a finite set F such that $A = l(u(F))$. Define

$$x = \bigvee(F \cap L_1), \; y = \bigvee(F \cap L_2).$$

Clearly $u(F) = u\{x, y\}$. Thus $A = l(u\{x, y\})$ and, by (5),

$$L_1 \cap A = [0, (x \vee \overline{y}) \wedge (\underline{\overline{x} \vee y})]_1, \qquad L_2 \cap A = [0, (\underline{x \vee \overline{y}}) \wedge (\overline{x} \vee y)]_2.$$

With

$$a = (x \vee \overline{y}) \wedge (\underline{\overline{x} \vee y}), \quad b = (\underline{x \vee \overline{y}}) \wedge (\overline{x} \vee y)$$

we obtain 2(a). But $a \geq x \wedge \underline{(\overline{x})} = x \wedge \overline{x} = x$ and $b \geq y \wedge \overline{y} = y$. Thus

$$A = l(u\{x, y\}) \subseteq l(u\{a, b\}) \subseteq l(u(A)) = A,$$

hence $A = l(u\{a, b\})$ and, by (5),

$$L_1 \cap A = [0, (a \vee \overline{b}) \wedge (\underline{\overline{a} \vee b})]_1, \qquad L_2 \cap A = [0, (\underline{a \vee \overline{b}}) \wedge (\overline{a} \vee b)]_2.$$

These equations together with 2(a) clearly imply 2(b).

$2 \Rightarrow 3$. Put $c = a \vee \overline{b}$, $d = \overline{a} \vee b$.

$3 \Rightarrow 1$. By (4), $A = l\{c, d\}$ and hence A is a normal ideal. Clearly $u(A) = u\{c \wedge \underline{d}, \underline{c} \wedge d\}$ and hence

$$A = l(u(A)) = l(u\{c \wedge \underline{d}, \underline{c} \wedge d\}).$$

Thus A is finitely generated.

Let C be the set of all finitely generated normal ideals; let N be the set of all normal ideals, the standard form of the MacNeille completion of P, and let f be the canonical embedding of P into N, i.e. $f(a) = [0, a]$.

(7) C is a sub-OL of N generated by $f(P)$.

Assume $A, B \in C$. Then there exist finite sets $F, G \subseteq P$ such that $A = l(u(F))$, $B = l(u(G))$. Clearly

$$F \cup G \subseteq A \cup B \subseteq l(u(F \cup G)).$$

Thus

$$l(u(A \cup B)) \subseteq l(u(l(u(F \cup G)))) = l(u(F \cup G))$$

and

$$l(u(F \cup G)) \subseteq l(u(A \cup B)),$$

and hence

$$l(u(F \cup G)) = l(u(A \cup B)).$$

Since the last set is the join of A and B in N it follows that this join belongs to C. The orthocomplement $A^\perp$ of $A \in N$ is defined by $A^\perp = l(A')$ where $A' = \{a' \mid a \in A\}$. As $A \in C$, by (6), A is of the form

$$A = [0, a]_1 \cup [0, b]_2 \quad \text{with } a \in L_1, b \in L_2.$$

It follows that

$$A' = [a', 1]_1 \cup [b', 1]_2 \quad \text{and} \quad A^\perp = l\{a', b'\}.$$

Thus, by (4),

$$L_1 \cap A^\perp = [0, a' \wedge \underline{(b')}]_1 \quad \text{and} \quad L_2 \cap A^\perp = [0, \underline{(a')} \wedge b']_2$$

and $A^\perp \in C$ by (6). Thus C is a subalgebra of N. If $A \in C$ then, by (6), there exists $c \in L_1$, $d \in L_2$ such that

$$L_1 \cap A = [0, c \wedge \underline{d}]_1 \quad \text{and} \quad L_2 \cap A = [0, \underline{c} \wedge d]_2.$$

By (4) this implies that

$$A = l\{c, d\} = [0, c] \cap [0, d] = f(c) \cap f(d).$$

Thus A is the meet of two elements of $f(P)$ and hence C is generated by $f(P)$, proving (7).

(8) If $u, v \in P$ and if $u \vee v$ exists in P then $\overline{u \vee v} = \overline{u} \vee \overline{v}$, and dually.

Clearly $u \vee v \leq \overline{u} \vee \overline{v} \in B$ and hence $\overline{u \vee v} \leq \overline{u} \vee \overline{v}$. Also $u, v \leq \overline{u \vee v} \in B$ hence $\overline{u}, \overline{v} \leq \overline{u \vee v}$ and $\overline{u} \vee \overline{v} \leq \overline{u \vee v}$, proving (8).

(9) C is an OML.

Assume $A, B \in C$, $A \subseteq B$ and $A^\perp \cap B = \{0\}$. We have to show that $A = B$. By (6) there exist elements $a, c \in L_1$, $b, d \in L_2$ such that

$$L_1 \cap A = [0, a]_1 \quad \text{and} \quad L_2 \cap A = [0, b]_2,$$
$$L_1 \cap B = [0, c \wedge \underline{d}]_1 \quad \text{and} \quad L_2 \cap B = [0, \underline{c} \wedge d]_2, \quad \text{with}$$
$$a = (a \vee \overline{b}) \wedge (\underline{\overline{a} \vee b}) \quad \text{and} \quad b = (\underline{a \vee \overline{b}}) \wedge (\overline{a} \vee b).$$

Clearly $A^\perp = l\{a', b'\}$ and, by (4),

$$L_1 \cap A^\perp = [0, a' \wedge (\overline{b})']_1 \quad \text{and} \quad L_2 \cap A^\perp = [0, (\overline{a})' \wedge b']_2.$$

Since $A \subseteq B$ we have $a \le c \wedge \underline{d}$ and $b \le \underline{c} \wedge d$, and hence

$$a \vee \overline{b} \le c \wedge (\overline{a} \vee \overline{b}) \quad \text{and} \quad \overline{a} \vee b \le d \wedge (\overline{a} \vee \overline{b}).$$

Since $A^\perp \cap B = \{0\}$ we have

$$a' \wedge (\overline{b})' \wedge c \wedge \underline{d} = 0 \quad \text{and} \quad (\overline{a})' \wedge b' \wedge \underline{c} \wedge d = 0.$$

But $\overline{a}$ commutes with $\overline{b}$, hence

$$a' \wedge (\overline{b})' \wedge c \wedge (\overline{a} \vee \overline{b}) = a' \wedge (\overline{b})' \wedge c \wedge \overline{a} \le a' \wedge (\overline{b})' \wedge c \wedge \underline{d} = 0,$$

$$(\overline{a})' \wedge b' \wedge d \wedge (\overline{a} \vee \overline{b}) = (\overline{a})' \wedge b' \wedge d \wedge \overline{b} \le (\overline{a})' \wedge b' \wedge \underline{c} \wedge d = 0.$$

Thus

$$a \vee \overline{b} = c \wedge (\overline{a} \vee \overline{b}) \quad \text{and} \quad \overline{a} \vee b = d \wedge (\overline{a} \vee \overline{b}).$$

Furthermore

$$\begin{aligned}
(a \vee \overline{b}) \wedge (\underline{\overline{a} \vee b}) &= (a \vee \overline{b}) \wedge \underline{d \wedge (\overline{a} \vee \overline{b})} \\
&= (a \vee \overline{b}) \wedge \underline{d} \wedge (\overline{a} \vee \overline{b}) \quad \text{(by (8))} \\
&= (a \vee \overline{b}) \wedge \underline{d} \\
&= a \vee (\overline{b} \wedge \underline{d}) \quad (\text{as } a, \overline{b} \text{ commute with } \underline{d}) \\
&\le c \wedge \underline{d}
\end{aligned}$$

and

$$\begin{aligned}
a' \wedge ((\overline{b})' \vee (\underline{d})') \wedge c \wedge \underline{d} &= a' \wedge (\overline{b})' \wedge c \wedge \underline{d} \quad (\text{as } \overline{b} \text{ commutes with } \underline{d}) \\
&= 0 \quad (\text{as } a \vee \overline{b} \le c).
\end{aligned}$$

Thus

$$a = (a \vee \overline{b}) \wedge (\underline{\overline{a} \vee b}) = c \wedge \underline{d}.$$

By symmetry we obtain

$$b = (\underline{a \vee \overline{b}}) \wedge (\overline{a} \vee b) = \underline{c} \wedge d.$$

The last two equations prove $A = B$ and hence (9).

Define now $\psi_i = f_{|L_i}$. Then it is clear that (C, ψ_1, ψ_2) strongly amalgamates our V-formation, proving Theorem 5. □

8. Concluding Remarks

There is an abundance of open problems connected with the questions dealt with in this paper. Here are some samples.

We have considered only very few varieties of OLs, namely OLs, OMLs, OMLs generated by members of bounded height and MOLs. What about other varieties of OLs? are there any which have (strong, Boolean) amalgamation?

In Sections 6 and 7 we proved that OMLs have strong Boolean amalgamation. In our example showing that OMLs do not have the amalgamation property, B was MO3. Is there a counter-example in which B is MO2? Since every variety of OMLs which does not consist of Boolean algebras only contains MO2, the question seems natural. We do not know the answer.

Connected with the question of amalgamation in a variety is the question whether epimorphisms are surjective, see [7]. It is well known and easy to prove that in every variety with the strong amalgamation property epimorphisms are surjective. Thus epimorphisms in OLs are surjective. We do not know whether epimorphisms in OMLs are surjective.

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