CANONICAL COMPLETIONS OF
LATTICES AND ORTHOLATTICES

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Dedicated to Professor Gudrun Kalmbach

ABSTRACT. Every lattice, and ortholattice, can be represented as the closed elements of some Galois connection on a Boolean algebra. The canonical extension of this Boolean algebra yields a completion of the lattice, or ortholattice. We give a purely order theoretic characterization of this completion, and investigate its properties. While it preserves distributivity, it unfortunately preserves neither modularity nor orthomodularity.

1. Introduction

Let P and Q be posets, $\varphi \colon P \to Q$, and $\psi \colon Q \to P$. Following B i r k h o f f [1], the ordered pair (φ, ψ) is a *Galois connection* between P and Q if the maps are order inverting and for all $p \in P$, $q \in Q$ both $p \leq \psi\varphi p$ and $q \leq \varphi\psi q$. Here we shall be concerned only with the case that the posets P and Q are equal and Boolean.

Given a Galois connection (φ, ψ) on a Boolean algebra B, an element $x \in B$ is said to be *closed* if $x = \psi\varphi x$. It is well known that the closed elements form a bounded lattice under the partial ordering inherited from B. Further, if $\varphi = \psi$ and $x \cdot \varphi x = 0$ for all $x \in B$, the map φ is an *orthocomplementation* on this lattice of closed elements. The MacNeille completion [9] shows that every bounded lattice, and every ortholattice, can be embedded into the lattice of closed elements of a Galois connection on a Boolean algebra, and with a minor modification in the construction, this embedding can be chosen to be an isomorphism.

Having represented a bounded lattice, or ortholattice, L as the closed elements of some Galois connection (φ, ψ) on a Boolean algebra B, it is tempting to consider the canonical extension B^σ of this Boolean algebra in the sense of

Jónsson and Tarski [5]. Unfortunately, the operations φ, ψ are order inverting, and will not be well behaved under canonical extensions. Instead, we consider the maps φ^-, ψ^- defined by setting $\varphi^- x = (\varphi x)^-$ and $\psi^- x = (\psi x)^-$. These maps are order preserving, and are easily seen to be conjugates in the sense of [5]. Further, their canonical extensions are also conjugates [5], hence give rise to a Galois connection on B^σ. The Galois closed elements of B^σ then provide a completion of L, which we call the canonical completion.

It is our purpose here to give a purely order theoretic characterization of this completion, and to investigate its properties. This completion preserves distributivity, hence reduces to the usual canonical completion in the case of a Boolean algebra, but unfortunately preserves neither modularity nor orthomodularity.

2. Lattices of closed elements

In this section we show that every bounded lattice, and every ortholattice, is isomorphic to the closed elements of some Galois connection on a Boolean algebra. These results are probably not new, but we cannot find them in print. As they are easy consequences of M a c N e i l l e's [9] original work, we attribute the credit to him.

We begin by describing the familiar result that every bounded lattice can be embedded into the lattice of closed elements of some such Galois connection. Let M be a bounded lattice, and set B to be the power set of M. Define unary maps φ, ψ on B by setting φA to be the collection of all upper bounds in M of the subset $A \subseteq M$, and ψA to be the collection of all lower bounds in M of A. One easily checks that (φ, ψ) is a Galois connection on B, and the Galois closed elements are the normal ideals of M. As every principal ideal is normal, M can be embedded into the lattice of closed elements of B.

EXAMPLE 2.1. Letting ω denote the natural numbers with the usual ordering, and ω^d denote the dual of ω, consider the lattice M defined to be the ordinal sum of $\omega \times \omega$ and ω^d. In other words, M is the product of two copies of the natural numbers with an inverted copy of the natural numbers placed on top. With B and φ, ψ defined as above, the closed elements of B are isomorphic to the MacNeille completion of M. To find an isomorphic representation of M, it is natural to consider the subalgebra B_0 of $(B, +, \cdot, -, \varphi, \psi)$ generated by the principal ideals of M. Clearly, the restrictions of φ, ψ will form a Galois connection on B_0. Let A be the principal ideal $\{(0,0), (1,0)\}$. Then as φA is the set of upper bounds of A, we have $(\varphi A)^- = \{(0,n) : n \in \omega\}$. So $\varphi((\varphi A)^-) = \omega$. As ψ gives lower bounds, we have that $\psi \varphi((\varphi A)^-) = \omega \times \omega$. Thus, the closed elements of B_0 again comprise all normal ideals of M.

This example shows that the obvious approach to representing a bounded lattice as the closed elements of a Galois connection will not work without some modification. However, only a small amount of repair needs be done.

THEOREM 2.2. *Let M be a bounded lattice. Then there is a Boolean algebra B and a Galois connection (φ, ψ) on B such that M is isomorphic to the Galois closed elements of B.*

P r o o f. Consider the Boolean subalgebra B_I of the power set of M generated by the collection of all principal ideals of M, and the Boolean subalgebra B_F of the power set of M generated by all principal filters of M. For a subset $A \subseteq M$, define UA to be the collection of all upper bounds of A in M, and LA to be the collection of all lower bounds of A in M.

First, we show that if $A \in B_I$, then UA is a principal filter of M. Let T denote the collection of all elements of B_I which are finite intersections of principal ideals, or their set complements. We need at most one principal ideal in this representation as a finite intersection of principal ideals is principal, and as our lattice has a greatest element, we may assume there is at least one principal ideal in the representation. Using $a \downarrow$ for the principal ideal generated by a and $a \downarrow^-$ for its set complement, every element of T is of the form $(a_0 \downarrow) \cap (a_1 \downarrow)^- \cap \cdots \cap (a_n \downarrow)^-$ for some $0 \le n$. Note that if such an element is nonempty, it has a largest element, namely a_0, and therefore the collection of upper bounds form a principal filter of M. But every element of B_I is a finite union of elements of T, and as the upper bounds of a union of sets equals the intersection of upper bounds of the individual sets, our claim follows.

Now set $B = B_I \times B_F$ and define maps φ, ψ on B by setting

$$\varphi\big((X, Y)\big) = (M, UX),$$
$$\psi\big((X, Y)\big) = (LY, M).$$

Clearly both maps are order inverting, and as $X \subseteq LUX$ and $S \subseteq ULY$ for all subsets X, Y of M, it follows that the composites $\psi\varphi$ and $\varphi\psi$ are increasing maps. So (φ, ψ) is a Galois connection on B. The Galois closed elements of B are exactly the ones of the form (LUX, M), where $X \in B_I$. But we have shown that $X \in B_I$ implies UX is a principal filter of M, and hence that LUX is a principal ideal of M. Clearly every principal ideal arises in this fashion, so the map $a \rightsquigarrow (a\downarrow, M)$ is an isomorphism from M to the lattice of Galois closed elements of B. $\square$

We should note that the assumption of boundedness cannot be removed from the above result as the Galois closed elements of a Galois connection on a Boolean algebra necessarily form a bounded lattice. The least element is $\psi\varphi 0$ and the greatest element is $\psi\varphi 1 = 1$. The following result is due to B i r k h o f f [1, p. 123].

THEOREM 2.3. *If (φ, φ) is a Galois connection on a Boolean algebra B which satisfies $\varphi x \cdot \varphi \varphi x = \varphi \varphi 0$ for all $x \in B$, then φ is an orthocomplementation on the lattice of closed elements of B.*

P r o o f. Note first that for any x we have φx is closed. So φ is certainly a map from the closed elements to themselves. As φ is order inverting on all of B, it is also order inverting for closed elements. If x is closed, then $x = \varphi \varphi x$, so on the closed elements φ is period two. If x is closed, then $x = \varphi \varphi x$, so $\varphi x \cdot x = \varphi \varphi 0$. But meets in the lattice of closed elements agree with meets in B, so if x is closed, then the meet of x and φx in the lattice of closed elements is the smallest element in this lattice $\varphi \varphi 0$. By [7, p. 17], this is sufficient to show that φ is an orthocomplementation on the lattice of closed elements. $\square$

COROLLARY 2.4. *If (φ, φ) is a Galois connection on a Boolean algebra B which satisfies $x \cdot \varphi x = 0$ for all $x \in B$, then φ is an orthocomplementation on the lattice of closed elements of B.*

THEOREM 2.5. *Let M be an ortholattice. Then there is a Boolean algebra B and a Galois connection (φ, φ) on B which satisfies $x \cdot \varphi x = 0$ for all $x \in B$, such that M is isomorphic to the ortholattice of closed elements of B.*

P r o o f. Let C be the subalgebra of the power set of M generated by the collection of all principal ideals of M. In the proof of Theorem 2.2 we showed that for any $A \in C$ the upper bounds of A form a principal filter of M, and hence $\{x' : x \in UA\}$ is a principal ideal of M. Therefore we can define a map φ on C by setting $\varphi A = \{x' : x \in UA\}$. Clearly φ is order inverting, and it is easily checked that $\varphi \varphi$ is increasing, so (φ, φ) is a Galois connection on C.

It is easy to show that $0 \in \varphi A$ for every $A \in C$, and therefore $A \cdot \varphi A \subseteq \{0\}$, with equality if and only if $0 \in A$. Consider the principal filter B consisting of all elements of C which contain 0, and note that B forms a Boolean algebra under the partial ordering inherited from C. As remarked above, φ is a map from C into B, and therefore the restriction of φ is a map from B to itself. As φ is order inverting on C, it is also order inverting on B. And as $A \leq \varphi \varphi A$ for all $A \in C$, this holds as well for all elements of B. Thus (φ, φ) is a Galois connection on the Boolean algebra B, and now $A \cdot \varphi A$ is equal to the zero of the Boolean algebra B for all $A \in B$. But the Galois closed elements of B are principal ideals of M, and it is clear that every principal ideal arises this way. It then follows easily that $a \rightsquigarrow a \downarrow$ is an ortholattice isomorphism from M to the ortholattice of closed elements of B. $\square$

3. Canonical extensions

In [5] J ó n s s o n and T a r s k i introduced the notion of a canonical extension

of a Boolean algebra with operators. We briefly describe that fragment of the theory needed for our purposes. The canonical extension of a Boolean algebra B is the embedding of B into the power set of its Stone space, which we denote as B^σ. We identify B with its image in B^σ, and freely speak of the open and closed elements of B^σ with the obvious meaning. Given a monotone unary operation f on B we define a unary operation f^σ on B^σ by setting

$$f^\sigma x = \bigcup_{x \geq y \in K} \; \bigcap_{y \leq a \in B} fa \, ,$$

where K denotes the collection of all closed elements of B^σ, and $\bigcup$, $\bigcap$ denote join and meet in the B^σ.

Given a unary operation f on a Boolean algebra B, we say that f is additive if it preserves binary joins, i.e., $f(x + y) = fx + fy$, completely additive if it preserves all existing joins, and an operator if it preserves binary joins and satisfies $f0 = 0$. For a unary map f, the unary operation f^- is defined by setting $f^- x = (fx)^-$, and the dual f^d of f is defined by setting $f^d x = \big(f(x^-)\big)^-$. Two unary maps f, g on B are called conjugates if $fx \cdot y = 0$ if and only if $x \cdot gy = 0$. The key point is that f, g are conjugates if and only if (f^-, g^-) is a Galois connection on B.

PROPOSITION 3.1. *Let f, g be monotone unary operations on a Boolean algebra B.*

(1) *If $a \in B$, then $f^\sigma a = fa$.*

(2) *If y is closed, then $f^\sigma y = \bigcap \{fa : y \leq a \in B\}$.*

(3) *If U is an up-directed subset of B, then $f^\sigma \big(\bigcup U\big) = \bigcup \{fu : u \in U\}$.*

(4) *If U is a down-directed subset of B, then $f^\sigma \big(\bigcap U\big) = \bigcap \{fu : u \in U\}$.*

(5) *If f is an operator, then f^σ is completely additive.*

(6) *If f, g are conjugates, then both are operators.*

(7) *If f, g are conjugates, then so are f^σ, g^σ.*

(8) *If f is an operator, then $\big(f^d\big)^\sigma = \big(f^\sigma\big)^d$.*

P r o o f. With the exception of the third and fourth, the first seven statements appear in [5]. The third and fourth are well known and easy to verify. The final statement appears in [4, Lemma 5.6]. □

We say that (B, f, g) is a conjugated algebra if B is a Boolean algebra, and f, g are unary maps on B which are conjugates. For any such conjugated algebra, (f^-, g^-) is a Galois connection on B, and therefore we may speak of the Galois closed elements of a conjugated algebra. Note that by part (7) of the above proposition, the canonical extension $(B^\sigma, f^\sigma, g^\sigma)$ is also a conjugated algebra, and so we may consider its lattice of closed elements as well.

THEOREM 3.2. *Let L be the lattice of closed elements of a conjugated algebra B and L^σ be the lattice of closed elements of the canonical extension B^σ.*

(C1) *L^σ is a completion of L.*

(C2) *Each element of L^σ is a meet of joins of elements of L.*

(C3) *Each element of L^σ is a join of meets of elements of L.*

(C4) *If $S, T \subseteq L$ and $\prod S \leq \sum T$, then $\prod S' \leq \sum T'$ for some finite $S' \subseteq S, T' \subseteq T$.*

(C5) *$\prod_I \sum_{J_i} a_{ij} = \sum_\alpha \prod_I a_{i\alpha(i)}$ if each $\{a_{ij} : j \in J_i\}$ is an up-directed subset of L.*

(C6) *$\sum_I \prod_{J_i} a_{ij} = \prod_\alpha \sum_I a_{i\alpha(i)}$ if each $\{a_{ij} : j \in J_i\}$ is a down-directed subset of L.*

P r o o f. We use $\sum$, $\prod$ for joins and meets in L^σ and $\bigcup$, $\bigcap$ for joins and meets in B^σ.

(1) The underlying Boolean algebra of B^σ is complete, so the Galois closed elements form a complete lattice. The meet of x, y in L is given by their meet in the Boolean algebra underlying B, and the join of x, y in L is given by $g^d f(x \cup y)$. But B is a subalgebra of B^σ, and hence L is a sublattice of L^σ.

CLAIM. *If T is an up-directed subset of L, then $\sum T = \bigcup T$.*

By definition, $\sum T = (g^\sigma)^d f^\sigma(\bigcup T)$. From the above proposition, f is an operator, hence f^σ is completely additive, and $(g^\sigma)^d = (g^d)^\sigma$. So $\sum T = (g^d)^\sigma(\bigcup\{ft : t \in T\})$. As T is up-directed, so also is $\{ft : t \in T\}$. By part (3) of the above proposition $\sum T = \bigcup\{g^d ft : t \in T\}$. But $T \subseteq L$, so $g^d ft = t$, and our claim follows.

(2) Each element of L^σ is Galois closed, hence of the form $(g^\sigma x)^-$ for some $x \in B^\sigma$. But
$$(g^\sigma x)^- = \bigcap_{x \geq y \in K} \bigcup_{y \leq a \in B} (ga)^-.$$
Each $(ga)^-$ is Galois closed in B, hence in L. But $\{(ga)^- : y \leq a \in B\}$ is up-directed, and the result follows from the claim.

(3) Each element of L^σ equals $(g^\sigma)^d x$ for some $x \in B^\sigma$. But $(g^\sigma)^d = (g^d)^\sigma$ and
$$(g^d)^\sigma x = \bigcup_{x \geq y \in K} \bigcap_{y \leq a \in B} g^d a.$$
Each $g^d a$ is Galois closed in B and hence in L, and meets in B^σ agree with meets in L^σ. So $(g^d)^\sigma x$ is a join (in B^σ) of meets of elements of L. As $(g^d)^\sigma x$ is in L^σ, it must also be the join of this family in L^σ.

(4) We may assume that T is up-directed. Then $\prod S = \bigcap S$ and by our claim $\sum T = \bigcup T$. Our result then follows directly from the corresponding result for the canonical extension B^σ of the Boolean algebra B.

(5) By the above claim $\prod_I \sum_{J_i} a_{ij} = \bigcap_I \bigcup_{J_i} a_{ij}$. Using the fact that B^σ is completely distributive, we may write this as $\bigcup_\alpha \bigcap_I a_{i\alpha(i)}$. This is a join (in B^σ) of elements of L^σ. But it lies in L^σ, and therefore is the join in L^σ as well.

(6) By the manner in which joins in L^σ are formed,

$$\sum_I \prod_{J_i} a_{ij} = (g^\sigma)^d f^\sigma \left[\bigcup_I \bigcap_{J_i} a_{ij} \right].$$

Parts (4) and (5) of the above proposition yield

$$\sum_I \prod_{J_i} a_{ij} = (g^\sigma)^d \left[\bigcup_I \bigcap_{J_i} f a_{ij} \right].$$

Using the fact that B^σ is completely distributive and that $(g^\sigma)^d$ is completely multiplicative (as g^σ is completely additive), we have

$$\sum_I \prod_{J_i} a_{ij} = \bigcap_\alpha (g^\sigma)^d \left[\bigcup_I f a_{i\alpha(i)} \right].$$

Using again that f^σ is completely additive gives

$$\sum_I \prod_{J_i} a_{ij} = \prod_\alpha (g^\sigma)^d f^\sigma \left[\bigcup_I a_{i\alpha(i)} \right],$$

which yields our result. $\qquad\square$

THEOREM 3.3. *Assume L is a sublattice of L^σ, M is a sublattice of M^σ, and both couples satisfy (C1) through (C6) of the previous theorem. If L is isomorphic to M, then there is a unique isomorphism between L^σ and M^σ extending the one between L and M.*

P r o o f . Let $h\colon L \to M$ and $m\colon M \to L$ be mutually inverse isomorphisms. Set

$$h^\sigma x = \sum_{x \ge y \in K} \prod_{y \le a \in L} ha,$$

where K is the collection of all elements of L^σ which are meets of elements of L. We call K the closed elements of L^σ. Similarly, we define a map m^σ using the closed elements K' of M^σ. Clearly h^σ, m^σ are monotone and extend h, m.

Next, we show that h^σ, m^σ restrict to mutually inverse isomorphisms between K and K'. This will follow from symmetry and the definition of closed elements

if we show for $y \in K$ and $a \in L$ that $y \leq a$ iff $h^\sigma y \leq ha$. One direction is clear as h^σ is monotone. Conversely, if $h^\sigma y \leq ha$, then $\prod \{hb \colon y \leq b \in L\} \leq ha$. By the compactness property (C4) we have $hb \leq ha$ for some $y \leq b \in L$, and hence $y \leq b \leq a$.

Finally, we show that h^σ, m^σ are mutually inverse isomorphisms between L^σ, M^σ. By property (C3), elements of L^σ and M^σ are all joins of closed elements. Using symmetry, it is enough to show that for $x \in L$ and $y \in K$ that $y \leq x$ iff $h^\sigma y \leq h^\sigma x$. One direction follows from monotonicity. For the other, assume $h^\sigma y \leq h^\sigma x$. Enumerating $\{y \in K \colon y \leq x\}$ as $(y_i)_I$ and enumerating each $\{a \in L \colon y_i \leq a\}$ as $(a_{ij})_{J_i}$ the definition of $h^\sigma x$ gives

$$h^\sigma y \leq \sum_I \prod_{J_i} ha_{ij} \,.$$

Then by property (C6)

$$h^\sigma y \leq \prod_\alpha \sum_I ha_{i\alpha(i)} \,.$$

So for each α, the compactness property (C4) provides a finite subset $I_\alpha \subseteq I$ with

$$h^\sigma y \leq \sum_{I_\alpha} ha_{i\alpha(i)} \,.$$

But m^σ is monotone, restricts to a lattice homomorphism on M, and is an inverse of h^σ on K'. So for each α

$$y \leq \sum_{I_\alpha} a_{i\alpha(i)} \,.$$

Therefore $y \leq \prod_\alpha \sum_I a_{i\alpha(i)}$. Applying (C6) $y \leq \sum_I \prod_{J_i} a_{ij}$, which by property (C3) is equal to x. $\qquad\square$

Thus every bounded lattice L has a completion which satisfies (C1) through (C6), and this completion is determined up to isomorphism by these properties. We call such a completion a canonical completion of L, and use L^σ to denote one such canonical completion. It follows from theorem 2.5 that every ortholattice can be represented as the Galois closed elements of some conjugated algebra (B, f, f) which satisfies $x \leq fx$. As the canonical completion $(B^\sigma, f^\sigma, f^\sigma)$ is also a conjugated algebra which satisfies $x \leq f^\sigma x$, its closed elements form an ortholattice as well. So each ortholattice has an ortholattice completion satisfying (C1) through (C6), and this completion is determined up to isomorphism by these properties. So we may speak of the canonical completion L^σ of an ortholattice L as well.

PROPOSITION 3.4.

(1) L^σ *is atomic, but need not be atomistic.*

(2) L^σ *need not be meet continuous or algebraic.*

(3) *If L is distributive, then L^σ is completely distributive and doubly algebraic.*

(4) L^σ *need not be modular, even if L is a modular lattice, or modular ortholattice.*

(5) L^σ *need not be orthomodular, even if L is orthomodular.*

P r o o f . (1) To show atomicity, it is enough to show each non-zero closed element of L^σ has an atom beneath it. Suppose $y \in L^\sigma$ is closed. Then $\{a \in L : y \leq a\}$ is a proper filter of L. Extend this to a maximal proper filter F of L. Then $\prod F$ is non-zero by the compactness property, lies beneath y, and is an atom by the maximality of F. That elements of L need not be joins of atoms is easily provided by the fact that the canonical extension of a finite lattice is itself.

(2) Construct a lattice L as follows. The underlying set of L consists of the elements $0, 1, a$ and the set of all ordered pairs of integers (m, n) such that $m + n \leq 0$. We define a partial ordering on L such that $0, 1$ are the bounds of L, a is incomparable to all but the bounds, and $(m, n) \leq (m', n')$ iff $m \leq m'$ and $n \leq n'$. It is easy to verify that L is a lattice. Set I to be the collection of all integers, and for each $i \in I$, set $J_i = \{j : j \leq -i\}$. Choose a map α such that $\alpha(i) \in J_i$ for each integer i. Setting $n = \alpha(0)$ we have that the join of $(0, n)$ and $(1 - n, \alpha(1 - n))$ in L is equal to 1. It follows from condition (C6) that $\sum_I \prod_{J_i} (i, j) = 1$. But $a \cdot \prod_{J_i} = 0$ for each $i \in I$. So L^σ is not meet continuous, and hence not algebraic.

(3) It is enough to provide a completion of a distributive lattice which is completely distributive, doubly algebraic, and satisfies (C1) through (C6). Such is provided by the collection of all order ideals of the Priestly space. See [2] for a complete account.

(4) Let L be the modular ortholattice of all finite or cofinite dimensional subspaces of a Hilbert space. From K a p l a n s k y 's result that a complete modular ortholattice is a continuous geometry, L cannot be embedded into a complete modular ortholattice [7, p. 182]. Thus L^σ is not modular, and as the lattice reduct of L^σ must be the canonical completion of the lattice reduct of L, this establishes the claim for modular lattices as well.

(5) Let L be an orthomodular lattice containing two increasing sequences $(x_n)_\omega$ and $(y_n)_\omega$ such that for all natural numbers n we have (i) $x_n \leq y_n$, (ii) $x'_{n+1} \cdot y_n = 0$, and (iii) $y_0 \not\leq x_n$. Such an L is provided by applying Kalmbach's

construction [3, 6] to the lattice $(\omega + 1) \times 2$. Define elements x, y of L^{σ} by $x = \sum_{\omega} x_n$ and $y = \sum_{\omega} y_n$. Obviously $x \leq y$, with the inequality strict by (iii) and the compactness property (C4). Then $x' \cdot y$ is equal to $\prod_n x'_n \cdot \sum_n y_n$, which is a meet of joins of up-directed subsets of L (all but one such subset being a singleton). Applying (C5) yields $x' \cdot y$ equal to $\sum_n \prod_m x'_m \cdot y_n$, which by (ii) is equal to 0. So L^{σ} is not orthomodular. $\qquad\qquad\square$

4. Miscellaneous

The hope of finding an orthomodular completion for orthomodular lattices was the author's original motivation for this study. The results of the previous section show that the canonical completion is not a candidate. However, the following digression shows that any hope of finding an orthomodular completion must necessarily abandon regularity. (Recall that an embedding is regular if it preserves all existing joins and meets.) The origins of this result lie in a paper by P a l k o [10].

PROPOSITION 4.1. *Any regular embedding of an orthomodular lattice into a complete orthomodular lattice factors, as a pair of regular embeddings, through the MacNeille completion.*

P r o o f . Suppose φ is a regular embedding of L into a complete orthomodular lattice C. For each normal ideal N of L define $\beta N = \sum \varphi[N]$. Clearly $\varphi = \beta \circ i$, where i is the regular embedding of L into its MacNeille completion. We need only show β is a regular embedding.

For a normal ideal N, the orthocomplement $N' = \{u' : u \in UN\}$. It follows that $\beta(N') \leq (\beta N)'$. Equality will follow from the orthomodularity of C if we can show $\beta(N') + \beta N = 1$. But this term is equal to $\varphi[N' \cup N]$, and as the join of $N' \cup N$ in L equals 1, and φ is regular, equality in the above follows.

For a family of normal ideals N_i ($i \in I$) the obvious monotonicity of β shows $\beta\left(\bigcap_I N_i\right) \leq \prod_I \beta N_i$. Again, equality will follow from orthomodularity if we can show $\beta\left(\bigcap_I N_i\right) + \left(\prod_I \beta N_i\right)' = 1$. As β is compatible with orthocomplementation, this term is equal to $\beta\left(\bigcap_I N_i\right) + \sum_I \beta(N'_i)$, which in turn is equal to $\sum \varphi\left[\bigcap_I N_i \cup \bigcup_I N'_i\right]$. From the regularity of φ this join equals 1, so β preserves arbitrary meets, and hence is regular.

Finally, to see that β is an embedding, suppose that M, N are normal ideals. If $\beta M = \beta N$, then for every $u \in UM$ and $n \in N$, we have $iu \geq M$ and $N \geq in$. As β is monotone, $\beta iu \geq \beta M = \beta N \geq \beta in$. Then as $\beta \circ i$ equals the embedding

φ, we have $u \geq n$. As this holds for each $n \in N$, it follows that $u \in UN$. By symmetry, $UM = UN$, and as these are normal ideals, $M = N$. $\qquad\square$

COROLLARY 4.2. *A variety of orthomodular lattices admits a regular completion iff it is closed under MacNeille completions.*

As a final comment, we note the following.

THEOREM 4.3. *There is a modal algebra which has no completion in the variety it generates.*

P r o o f. Let L be a modular ortholattice which cannot be embedded into a complete modular ortholattice (see the proof of Proposition 3.4 part 4). We have seen that there is a conjugated algebra (B, f, f), which satisfies $x \leq fx$, such that L is isomorphic to the ortholattice of Galois closed elements of (B, f, f). Clearly such a conjugated algebra may be considered as a reflexive modal algebra. As the operations in the ortholattice of Galois closed elements of (B, f, f) are defined in terms of the operations of (B, f, f), we may express the modularity of the ortholattice of Galois closed elements as an identity in the language of (B, f, f). Any other algebra in the variety generated by (B, f, f) will have its Galois closed elements form a modular ortholattice. As the Galois closed elements of a complete conjugated algebra form a complete lattice, our result follows. $\qquad\square$

An example of K r a m e r and M a d d u x [8] is of a similar nature, but they only consider completions in which the operations are completely additive.

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