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Regularity in Quantum Logic

John Harding[1]

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In 1996, Harding showed that the binary decompositions of any algebraic, relational, or topological structure X form an orthomodular poset Fact X. Here, we begin an investigation of the structural properties of such orthomodular posets of decompositions. We show that a finite set S of binary decompositions in Fact X is compatible if and only if all the binary decompositions in S can be built from a common n-ary decomposition of X. This characterization of compatibility is used to show that for any algebraic, relational, or topological structure X, the orthomodular poset Fact X is regular. Special cases of this result include the known facts that the orthomodular posets of splitting subspaces of an inner product space are regular, and that the orthomodular posets constructed from the idempotents of a ring are regular. This result also establishes the regularity of the orthomodular posets that Mushtari constructs from bounded modular lattices, the orthomodular posets one constructs from the subgroups of a group, and the orthomodular posets one constructs from a normed group with operators. Moreover, all these orthomodular posets are regular for the same reason. The characterization of compatibility is also used to show that for any structure X, the finite Boolean subalgebras of Fact X correspond to finitary direct product decompositions of the structure X. For algebraic and relational structures X, this result is extended to show that the Boolean subalgebras of Fact X correspond to representations of the structure X as the global sections of a sheaf of structures over a Boolean space. The above results can be given a physical interpretation as well. Assume that the true or false questions $\mathfrak{Q}$ of a quantum mechanical system correspond to binary direct product decompositions of the state space of the system, as is the case with the usual von Neumann interpretation of quantum mechanics. Suppose S is a subset of $\mathfrak{Q}$. Then a necessary and sufficient condition that all questions in S can be answered simultaneously is that any two questions in S can be answered simultaneously. Thus, regularity in quantum mechanics follows from the assumption that questions correspond to decompositions.

1. INTRODUCTION

One of the great departures of quantum theory from the classical is the existence of incompatible observables. Classically, one can conduct an

[1] Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003; e-mail: jharding@nmsu.edu.

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experiment which will answer questions about both the position and the momentum of an object. This is not the case with quantum mechanical systems. Quantum theory states that any measurement of position interferes in an essential way with momentum, and conversely.

Two questions of a physical system are said to be compatible if they can be answered simultaneously by a single experiment. More generally, a set S of questions is said to be compatible if there is a single experiment which will simultaneously answer all questions in X. Clearly if S is compatible, then any pair of questions in S is compatible. However, the question arises as to whether every pairwise compatible set is compatible, a condition referred to as regularity. In this paper we show that the regularity of a quantum mechanical system is a consequence of associating questions of the system with direct product decompositions of the state space.

It is a common practice to impose some structure on the collection $\mathfrak{Q}$ of true or false questions of a physical system. $\mathfrak{Q}$ is partially ordered by logical implication, and is given a complementation which corresponds to negating a question. This structure is usually assumed to at least be an orthomodular poset (Birkhoff and von Neumann, 1936; Mackey, 1963; Piron, 1976; Varadarajan, 1985). One can then use the structure on $\mathfrak{Q}$ to determine when a set S of questions is compatible—S is compatible if and only if S is contained in a Boolean subalgebra of $\mathfrak{Q}$ (Varadarajan, 1985, p. 55). Classically, every set of questions is compatible, which corresponds to the fact that the questions $\mathfrak{Q}$ of a classical system form a Boolean algebra.

As the compatibility of questions is reflected in the structure $\mathfrak{Q}$, the regularity of our system, or lack of it, is also reflected in the structure $\mathfrak{Q}$. This naturally leads to the definition of compatibility and regularity for arbitrary orthomodular posets. A subset S of an orthomodular poset P is compatible if S is contained in a Boolean subalgebra of P, and P is regular if every pairwise compatible subset of P is compatible.

In Harding (1996) it was shown that the collection of all binary decompositions of any algebraic, relational, or topological structure X form an orthomodular poset Fact X. So it is not unreasonable to assume that the true or false questions of a quantum mechanical system correspond to the binary direct product decompositions of the state space of the system. Indeed, this assumption is entirely in keeping with the standard approach to quantum mechanics, formulated by von Neumann (1932), where the state space is taken to be some Hilbert space $\mathcal{H}$ and the true or false questions correspond to closed subspaces of $\mathcal{H}$.

Here we will determine the meaning of compatibility in an orthomodular poset Fact X. Let S be a finite subset of Fact X. We show that S is compatible if and only if the binary decompositions of X which comprise the elements of S can all be built from a common n-ary decomposition of X. With this description of compatibility, we can characterize the finite Boolean subalgebras of Fact X for any algebraic, relational, or topological structure X. We show there is a bijection between the collection of all finite Boolean subalgebras of Fact X and the collection of equivalence classes of finitary direct product decompositions of X. Here, two such decompositions are equivalent if one can be obtained from the other by permuting the factors, then relabeling the elements of the factors. For algebraic and relational structures we go farther. We show there is a bijection between the collection of all Boolean subalgebras of Fact X, finite or otherwise, and the equivalence classes of representations of X as the global sections of a sheaf over a Boolean space.

This description of compatibility can also be used to study the regularity of Fact X. We show that for any algebraic, relational, or topological structure X, the orthomodular poset Fact X is regular. In fact, we show a good deal more than this. We show that for any relation algebra R, the orthomodular poset $R^{(2)}$ (Harding, 1996) is regular. As many familiar orthomodular posets arise from decompositions, we have several corollaries to this result. In particular, the orthomodular posets which arise from the splitting subspaces of an inner product space are regular, the orthomodular posets constructed from the idempotents of a ring are regular, the orthomodular posets that Mushtari constructs from bounded modular lattices are regular, the orthomodular posets one constructs from the subgroups of a group are regular, and the orthomodular posets one constructs from the projections of a normed group with operators are regular. In many of these cases it is not difficult to establish regularity directly, and this has been done for orthomodular posets of splitting subspaces and orthomodular posets of idempotents of a ring. The point here is that these orthomodular posets are all regular for the same reason—they arise from decompositions.

This paper is organized in the following fashion. The second section contains the necessary background information to make the paper reasonably self-contained. Section 2.1 is devoted to orthomodular posets and regularity. The important notion of regularity for orthomodular posets was introduced by Brabec (1979), Brabec and Pták (1982), Neubrunn and Pulmannová (1983), and Pulmannová (1981). All of the results on regularity in this section can be found in the book of Pták and Pulmannová (1991). Section 2.2 contains a few brief facts about relation algebras. As this paper is about the algebra of decompositions, one should not be surprised that the algebra of equivalence relations plays a fundamental role, and relation algebras are direct generalizations of the algebra of all binary relations on a set X. Our results are established in the generality of relation algebras so as to include some important special cases. Section 2.3 is a brief review of results established in Harding (1996). Here we outline how one constructs an orthomodular poset $R^{(2)}$ from a relation algebra R.

In Section 3 we begin our investigation of compatibility in Fact X, for X a set. We develop a characterization of Boolean subalgebras of Fact X in terms of certain families of equivalence relations on X which we call Boolean subsystems. A Boolean subsystem is a family of pairwise permuting equivalence relations which form a Boolean sublattice of the lattice of all equivalence relations on X. This result then has two immediate consequences. It allows us to characterize compatible subsets of Fact X as ones which correspond to subsets of Boolean subsystems. And as finite Boolean subsystems correspond to equivalence classes of finitary direct decompositions of X, the finite Boolean subalgebras of Fact X also correspond to equivalence classes of finitary direct product decompositions of X. While we have described these results as they pertain to decompositions of a set, they are proved in the generality of arbitrary relation algebras.

In Section 4 we show that for any set X, the orthomodular poset Fact X is regular. Again, this result is established in the much more general setting of relation algebras.

In Section 5 we extend the results of Sections 3 and 4 to apply to the decompositions Fact X of any algebraic, relational, or topological structure X, as well as to many of the other orthomodular posets which arise from decompositions. Finally, in Section 6 we extend the above-mentioned correspondence between equivalence classes of finitary direct decompositions of a set X and finite Boolean subalgebras of Fact X. We show that there is a correspondence between the collection of all Boolean subalgebras of Fact X and equivalence classes of representations of X as the global sections of a sheaf over a Boolean space. This result is then extended to algebraic and relational structures, but not to topological structures.

2. PRELIMINARIES

If a, b are elements of a partially ordered set, we shall write $a + b$ to denote the supremum of $\{a, b\}$, if such exists, and $a \cdot b$ to denote the infimum of $\{a, b\}$, if such exists. While we will never have cause to consider suprema and infima of infinite sets, we will find it convenient to write ΣF and ΠF for the supremum and infimum of a finite set, provided these exist. Finally, the set complement of a set A will be denoted by $\neg A$. The universal set in which this complementation is to be taken will always be clear from the context.

2.1. Orthomodular Posets

Definition 2.1.1. An orthomodular poset is a structure $(P, \leq, 0, 1, \perp)$ such that:

(i) $(P, \leq, 0, 1)$ is a bounded partially ordered set.
(ii) $\perp; P \to P$ is an order-inverting complementation of period two.
(iii) If $a, b \in P$ and $a \leq b^{\perp}$, then $a + b$ exists in P.
(iv) if $a, b \in P$ and $a \leq b^{\perp}$, then $b^{\perp} = a + (a + b)^{\perp}$.

Several important consequences of this definition are worthy of note. These are well-known results, and complete proofs are given in Beran (1984) and Pták and Pulmannová (1991).

Lemma 2.1.2. Let P be an orthomodular poset.

(i) If $a, b \in P$ and $a + b$ exists, then $a^{\perp} \cdot b^{\perp}$ exists and is equal to $(a + b)^{\perp}$.
(ii) If $a, b \in P$ and $a \cdot b$ exists, then $a^{\perp} + b^{\perp}$ exists and is equal to $(a \cdot b)^{\perp}$.
(iii) If $\{a_1, \ldots, a_n\} \subseteq P$ and $a_i \leq a_j^{\perp}$ for $i \neq j$, then $\Sigma \{a_1, \ldots, a_n\}$ exists.
(iv) If $a, b \in P$ are such that $a \leq b^{\perp}$ and $a + b = 1$, then $a = b^{\perp}$.

We shall often refer to the relationship $a \leq b^{\perp}$ by saying that a is orthogonal to b. The final condition in Definition 2.1.1, and its equivalent formulation given as the final statement in Lemma 2.1.2, are known as the orthomodular law.

Definition 2.1.3. If P is an orthomodular poset and $S \subseteq P$, we say that S is a subalgebra of P if S is closed under orthocomplementation and finite orthogonal joins. Any subalgebra of P is naturally an orthomodular poset in its own right. If a subalgebra of P happens to be a Boolean algebra, we say it is a Boolean subalgebra of P.

Lemma 2.1.4. If B is a Boolean subalgebra of P, then any two elements of B have a join in P, and joins taken in B agree with joins taken in P. As B is closed under orthocomplementation, similar remarks are valid for meets.

Proof. Let a, b be elements in B. As B is Boolean, there are elements p, q, r in B such that (i) p, q, r are pairwise orthogonal in B, (ii) the join of $\{p, r\}$ in B equals a, and (iii) the join of $\{q, r\}$ in B equals b. Then as B is a subalgebra of P, we have that (i) p, q, r are pairwise orthogonal in P, (ii) the join of $\{p, r\}$ in P equals a, and (iii) the join of $\{q, r\}$ in P equals b. It follows that the join of $\{p, q, r\}$ exists in P and is the join of $\{a, b\}$ in P. But this element is in B, and hence is the join of $\{a, b\}$ in B as well. ■

Definition 2.1.5. A subset $A \subseteq P$ is said to be compatible[2] if there is a Boolean subalgebra of P which contains A. A subset $A \subseteq P$ is said to be

[2] Many authors, including Pták and Pulmannová (1991), give a technical definition of compatibility which they prove to be equivalent to the definition given above. Some authors also refer to this property as full compatibility or f-compatibility, leaving compatible to mean pairwise compatible.

pairwise compatible if any two elements of A are compatible. Clearly any compatible set is pairwise compatible. An orthomodular poset is called regular if every pairwise compatible set is compatible.

The important notion of regularity was introduced by Brabec (1979), Brabek and Pták (1982), Neubrunn and Pulmannová (1983), and Pulmannová (1981). The treatment of regularity given here closely follows the book of Pták and Pulmannová (1991), where proofs of Propositions 2.1.6 and 2.1.7 as well as Example 2.1.9 can be found. Some slight reformulation of the results has been made to make them easier to apply in the sequel. Before proceeding to the following proposition, it is necessary to introduce some notation. If $\{a_0, \ldots, a_{n-1}\}$ is a subset of an orthomodular poset P, we let a_i^0 be a_i and a_i^1 be $a_i^{\perp}$. As is customary in set theory, we consider n to be the set $\{0, 1, \ldots, n-1\}$ and 2 to be the set $\{0, 1\}$. So for each $\alpha \in 2^n$, the set $\{a_i^{\alpha(i)}: i = 1, 2, \ldots, n-1\}$ contains each element a_i or its orthocomplement.

Proposition 2.1.6. Let P be an orthomodular poset and $A = \{a_0, a_1, \ldots, a_{n-1}\}$ be a finite subset of P. Then A is compatible if and only if for each $\alpha \in 2^n$ the family $\{a_i^{\alpha(i)}: i = 1, 2, \ldots, n-1\}$ has a meet in P and

$$\sum_{\alpha \in 2^n} \prod_{i=0}^{n-1} a_i^{\alpha(i)} = 1$$

Note that if the meets involved exist, then the join is guaranteed to exist, as it is an orthogonal join.

Proposition 2.1.7. For an orthomodular poset P, the following are equivalent:

(i) P is regular.
(ii) If $\{a, b, c\}$ is a pairwise compatible set, then $\{a, b \cdot c\}$ is compatible.

Note that the existence of $b \cdot c$ in condition (ii) is ensured, as b, c are compatible.

Corollary 2.1.8. An orthomodular poset P is regular iff every pairwise compatible subset of P with three elements is compatible.

Proof. The condition is surely necessary. To see that it is sufficient, we will verify the second condition of Proposition 2.1.7. Indeed, if $\{a, b, c\}$ is pairwise compatible, then by assumption it is compatible. So $\{a, b, c\}$ is contained in some Boolean subalgebra B of P. Then $\{a, b \cdot c\}$ is also contained in B and hence is compatible. ∎

Example 2.1.9. The following is an example of an orthomodular poset which is not regular. Let P be the set of all subsets of $\{1, 2, 3, 4, 5, 6, 7,$

8} which have even cardinality. Partially order P by set inclusion, and let orthocomplementation be set complementation. Then P is an orthomodular poset. In fact, it is a subalgebra of the orthomodular poset of all subsets of {1, 2, 3, 4, 5, 6, 7, 8} in the sense of Definition 2.1.3. One can verify that {{1, 2, 3, 4}, {1, 2, 5, 6}, {1, 3, 5, 7}} is a pairwise compatible set which is not compatible.

The above example also shows that a subalgebra of a regular orthomodular poset need not be regular. It will be of importance to us to have conditions sufficient to guarantee that a subalgebra of a regular orthomodular poset is regular. This task will comprise the remainder of the section.

Lemma 2.1.10. Let P be an orthomodular poset and S be a subalgebra of P. If x, y are elements of S which are compatible in S, then x, y are compatible in P. Further, the meet of x, y in S exists and is equal to the meet of x, y in P.

Proof. If x, y are compatible in S, they are in some Boolean subalgebra B of S. But B is also a Boolean subalgebra of P. Thus x, y are compatible in P. To see our further remark, note that the meet of x, y in B exists, and as B is a Boolean subalgebra of both S and P, it follows by Lemma 2.1.4 that the meet of x, y in S equals the meet of x, y in B equals the meet of x, y in P. ■

Definition 2.1.11. Let P be an orthomodular poset and S be a subalgebra of P. We say that S is a compatible subalgebra of P if whenever elements x, y of S are compatible in P, they are compatible in S.

Example 2.1.9 shows that not all subalgebras are compatible subalgebras.

Lemma 2.1.12. Let P be an orthomodular poset and S be a subalgebra of P. If P is regular and S is compatible, then S is regular.

Proof. Let $\{a, b, c\}$ be a set of elements which are pairwise compatible in S, and let $b \cdot c$ be the meet of b, c in S. By Proposition 2.1.7, it is enough to show that a and $b \cdot c$ are compatible in S. Note first that by Lemma 2.1.10 the elements of $\{a, b, c\}$ are pairwise compatible in P, and that the meet of b, c in P agrees with the meet of b, c in S. Then, as P is regular, it follows that a is compatible with $b \cdot c$ in P. Then, as S is a compatible subalgebra of P, it follows that a is compatible with $b \cdot c$ in S. ■

Lemma 2.1.13. Let P be an orthomodular poset and S be a subalgebra of P. Then the following are equivalent:

(i) For all x, y in S with x, y compatible in P, the meet of x, y in P belongs to S.

(ii) S is a compatible subalgebra of P.

Proof. (i) $\Rightarrow$ (ii) Let x, y be elements of S which are compatible in P. We must show that x, y are compatible in S. As x, y are compatible, they are contained in some Boolean subalgebra B of P. This implies that the meets $x \cdot y$, $x \cdot y'$, $x' \cdot y$, and $x' \cdot y'$ all exist in P and that their join in P equals 1. As x, y are compatible in P, we trivially have that x', y as well as x, y' and x', y' are compatible in P. It follows from our assumption that $x \cdot y$, $x \cdot y'$, $x' \cdot y$, and $x' \cdot y'$ all belong to S, and they are clearly orthogonal as well. As S is closed under orthogonal joins, it follows that their join in S equals 1. Therefore by Proposition 2.1.6, x, y are compatible in S.

(ii) $\Rightarrow$ (i). Let x, y be elements of S which are compatible in P. By assumption, x, y are compatible in S as well. Our result then follows from Lemma 2.1.10. ■

Corollary 2.1.14. Let P be a regular orthomodular poset and S be a subset of P which is closed under orthocomplementation. Assume further that for all $x, y \in S$ which are compatible in P we have that $x \cdot y$ is in S. Then S is a compatible subalgebra of P, and S itself is a regular orthomodular poset.

Proof. We first show S is a subalgebra of P. As S is closed under orthocomplementation, we must only show that S is closed under finite orthogonal joins. But if x, y are orthogonal elements in S, then x' and y' are compatible in P. By our assumption, $x' \cdot y'$ is in S, and as S is closed under complementation, we have that $x + y$ is in S. Having shown that S is a subalgebra of P, it follows directly from our assumption and Lemma 2.1.13 that S is a compatible subalgebra of P. That S itself is regular follows from Lemma 2.1.12. ■

2.2. Relation Algebras

A binary relation on a set X is a collection of ordered pairs of elements of X. Thus the collection of all binary relations on X, denoted RX, is simply the power set of $X \times X$. Partially ordered by set inclusion, $(RX, + \cdot, -, 0, 1)$ is a complete Boolean algebra where joins are given by unions, meets by intersections, and complementation is set complementation. Apart from these Boolean operations, there are several additional operations one frequently uses when working with relations. Relational multiplication ; is a binary operation, relational conversion $^{\smile}$ is a unary operation, and the identity relation $1'$ is a constant, where

$$R \,;\, S = \{(r, s) \,;\, \text{exists } t \text{ with } (r, t) \in R \text{ and } (t, s) \in S\}$$

$$R^{\smile} = \{(s, r) : (r, s) \in R\}$$

$$1' = \{(x, x): x \in X\}$$

Relation algebras are a generalization of the algebra *RX*. They were introduced by Tarski (1941). For general background on relation algebras, consult Jónsson (1982).

Definition 2.2.1. A relation algebra is a Boolean algebra $(B, +, \cdot, ^-, 0, 1)$ with an additional binary operation ; an additional unary operation $\smile$, and a constant $1'$, which satisfies the following identities:

(i) $(a ; b) ; c = a ; (b ; c)$.
(ii) $a ; 1' = a$.
(iii) $a ; (b + c) = a ; b + a ; c$.
(iv) $(a + b) ; c = a ; c + b ; c$.
(v) $a^{\smile\smile} = a$.
(vi) $(a + b)^{\smile} = a^{\smile} + b^{\smile}$.
(vii) $(a \cdot b)^{\smile} = a^{\smile} \cdot b^{\smile}$.
(viii) $(a ; b)^{\smile} = b^{\smile} ; a^{\smile}$.
(ix) $a^{\smile} ; (a ; b)^- + b^- = b^-$.

There are redundancies in the axiom system provided above. For instance, the fourth axiom easily follows from the others. Note also that the third and fourth axioms ensure that relational product ; is monotone in each argument.

This axiom system was chosen to be satisfied by any algebra *RX*. However, there are identities which hold in each algebra *RX* which are not implied by this axiom system (Jónsson, 1982). Therefore, relation algebras are a true generalization of the algebra of relations over a set. The following lemma, due to Chin and Tarski (1951), states that a fragment of modularity holds in any relation algebra. This is the starting point of the study of decompositions.

Lemma 2.2.2. Let a, b, c be elements of a relation algebra *R*.

(i) If $a ; c \leq a$ and $a ; c^{\smile} \leq a$, then $a \cdot (b ; c) = (a \cdot b) ; c$.
(ii) If $c ; a \leq a$ and $c^{\smile} ; a \leq a$, then $a \cdot (c ; b) = c ; (a \cdot b)$.

Proof. The first statement can be found in Chin and Tarski (1951), Corollary 2.19. The second statement follows from the first using (v), (vii), and (viii) of Definition 2.2.1. ∎

2.3. Relational Orthomodular Posets

In Harding (1996) it was shown that the collection of all decompositions of a set naturally carries the structure of an orthomodular poset. The key notion is that two decompositions are orthogonal precisely when they admit a common refinement. This construction was remarkably resilient to adaptations. It was shown that the decompositions of any algebraic, relational, or topological structure form an orthomodular poset in a natural manner. From this, one realizes that many of the methods used to construct orthomodular posets are special instances of this result. The most important such example is that the closed subspaces of a Hilbert space are in direct correspondence with decompositions of the Hilbert space.

Working with direct decompositions naturally leads one to consider certain equivalence relations. Therefore, it should not be surprising that the technique of building an orthomodular poset from the decompositions of a structure can be extended to constructing an orthomodular poset from an arbitrary relation algebra. Indeed, that was the approach taken in Harding (1996). This extra generality pays dividends in more powerful results; however, the applications of our results in the special case of the relation algebra RX are the most important. There will be little loss in understanding if one considers the remainder of the paper in this context.

Definition 2.3.1. For a relation algebra R, define

$$R^{(1)} = \{x \in R \,;\, x = x^{\smile} = x \,;\, x \text{ and } 1' \leq x\}.$$

We will call members of $R^{(1)}$ proper equivalence elements. The term equivalence element has historically been used without the requirement that $1' \leq x$.

Lemma 2.3.2. Let $x, y \in R^{(1)}$.

(i) $x \cdot y \in R^{(1)}$.
(ii) If $x \,;\, y \in R^{(1)}$, then $x \,;\, y = y \,;\, x$.
(iii) If $x \,;\, y = y \,;\, x$, then $x \,;\, y \in R^{(1)}$ and is the least upper bound of x, y in $R^{(1)}$.

Proof. (i) Surely $1' \leq x \cdot y$. Also $(x \cdot y)^{\smile} = x^{\smile} \cdot y^{\smile} = x \cdot y$. As ; is monotone in each argument, we have $(x \cdot y) \,;\, (x \cdot y) \leq x \,;\, x = x$ and $(x \cdot y) \,;\, (x \cdot y) \leq y \,;\, y = y$. So $(x \cdot y) \,;\, (x \cdot y) \leq x \cdot y$. But $1' \leq x \cdot y$, and hence $x \cdot y \leq (x \cdot y) \,;\, (x \cdot y)$, giving the desired equality.

(ii) Suppose that $x \,;\, y \in R^{(1)}$. Then $x \,;\, y = (x \,;\, y)^{\smile} = y^{\smile} \,;\, x^{\smile} = y \,;\, x$.

(iii) Note first that $(x \,;\, y)^{\smile} = y^{\smile} \,;\, x^{\smile} = y \,;\, x = x \,;\, y$. Then as ; is associative, we have $(x \,;\, y) \,;\, (x \,;\, y) = x \,;\, x \,;\, y \,;\, y = x \,;\, y$. Clearly $1' \leq x \,;\, y$, and therefore $x \,;\, y$ is in $R^{(1)}$. Suppose $z \in R^{(1)}$ is an upper bound of x, y. As ; is monotone in each argument, we have that $x \,;\, y \leq z \,;\, z = z$. Therefore $x \,;\, y$ is the least upper bound of x, y in $R^{(1)}$. ■

Definition 2.3.3. For a relation algebra R, we say that $B \subseteq R^{(1)}$ is a Boolean subsystem of R if the following conditions are satisfied:

(i) With the partial ordering inherited from R, B is a Boolean lattice.

(ii) $1'$ and 1 are the smallest and largest elements in B, respectively.
(iii) Finite meets in B agree with those in R.
(iv) Finite joins in B are given by relational product ;.

In view of Lemma 2.3.2, the fourth condition implies that $x ; y = y ; x$ for all x, y in B.

These definitions have special significance when interpreted in the relation algebra RX of all binary relations on a set X. $RX^{(1)}$ is the set of all equivalence relations on X. The importance of Boolean subsystems of RX is due to their close connection to direct decompositions of the set X, as explained in the following proposition.

Proposition 2.3.4. Given a sequence $\alpha_1, \ldots, \alpha_n$ of equivalence relations on a set X, let the map $\varphi: X \to X/\alpha_1 \times \cdots \times X/\alpha_n$ be defined by $\varphi(x) = (x/\alpha_1, \ldots, x/\alpha_n)$. Then the following are equivalent:

(i) $\varphi: X \to X/\alpha_1 \times \cdots \times X/\alpha_n$ is an isomorphism.
(ii) The members of the sequence $\alpha_1, \ldots, \alpha_n$ which are not equal to 1 are distinct and comprise the coatoms of a finite Boolean subsystem of RX.

Note that if $\alpha_i = 1$, then X/α_i is a singleton, and has no essential effect on the product.

Proof. (i) $\Rightarrow$ (ii). We may assume that the sequence $\alpha_1, \ldots, \alpha_n$ is arranged so that the members which differ from 1 are at the beginning. Let $k \leq n$ be such that $\alpha_1, \ldots, \alpha_k$ differ from 1 and $\alpha_{k+1}, \ldots, \alpha_n$ all equal 1. For $A \subseteq \{1, 2, \ldots, k\}$ let α_A be the meet of the α_i, where $i \in A$. Specifically,

$$\alpha_A = \{(x, y): x/\alpha_i = y/\alpha_i \text{ for all } i \in A\}$$

Note $\alpha_{A \cup B} = \alpha_A \cdot \alpha_B$, and therefore $A \subseteq B$ implies $\alpha_B \subseteq \alpha_A$.

Next we show that $\alpha_{A \cap B} = \alpha_A ; \alpha_B$. Clearly α_A, α_B are contained in $\alpha_{A \cap B}$, and as these are equivalence relations, $\alpha_A ; \alpha_B \subseteq \alpha_{A \cap B}$. Suppose (x, y) is in $\alpha_{A \cap B}$. As φ is an isomorphism, there is z in X with z/α_i equal to x/α_i for all i in A and z/α_i equal to y/α_i for all i which are not in A. Clearly (x, z) is in α_A. And as x/α_i equals y/α_i for all i in $A \cap B$, it follows that (z, y) is in α_B. Thus (x, y) is in $\alpha_A ; \alpha_B$.

Finally, we show that $\alpha_B \subseteq \alpha_A$ implies $A \subseteq B$. Suppose that i is in A, but not in B. Then as i is in A and the complement $\neg B$ of B, we have that α_A and $\alpha_{\neg B}$ are contained in α_i. If α_B is contained in α_A, then we have that both α_B and $\alpha_{\neg B}$ are contained in α_i. As these are equivalence relations, it follows that $\alpha_B ; \alpha_{\neg B}$ is contained in α_i. But we have just shown that $\alpha_B ; \alpha_{\neg B}$ equals $\alpha_\emptyset$, which is 1, contradicting our assumption $\alpha_i \neq 1$.

We have shown that the map ψ from the power set of $\{1, 2, \ldots, k\}$ to RX defined by $\psi(A) = \alpha_A$ is a dual order-isomorphism. Thus the image of ψ is a Boolean lattice under the partial ordering inherited from RX. As $\alpha_{A\cup B} = \alpha_A \cdot \alpha_B$, meets in this lattice agree with meets in RX. And as $\alpha_{A\cap B} = \alpha_A ; \alpha_B$, joins in this lattice are given by the relational product. Clearly $\alpha_\emptyset = 1$, and as φ is an embedding, $\alpha_{\{1,2,\ldots,k\}} = 1'$. So the image of ψ is a finite Boolean subsystem of RX. Finally, as $\{1\}, \ldots, \{k\}$ are the atoms of the power set of $\{1, 2, \ldots, k\}$, we have $\alpha_1, \ldots, \alpha_k$ are the coatoms of the image of ψ.

(ii) $\Rightarrow$ (i). As $\alpha_1 \ldots \alpha_n$ at least comprise the coatoms of a finite Boolean subsystem of RX, they meet to $1'$. It follows that φ is an embedding. Let $x_1, \ldots, x_n$ be elements of X and $m < n$. Suppose y is in X and y/α_i equals x_i/α_i for all $i \leq m$. Using the fact that $(\alpha_1 \cdot \alpha_2 \cdots \alpha_m) ; \alpha_{m+1} = 1$, we can find y' in X with y'/α_i equal to x_i/α_i for all $i \leq m + 1$. By an obvious induction, there is an element x in X with x/α_i equal to x_i/α_i for all $i \leq n$. Thus φ is onto. ∎

Corollary 2.3.5. Let $\alpha_1, \ldots, \alpha_n$ be a sequence of equivalence relations on a set X such that the natural map $\varphi: X \to X/\alpha_1 \times \cdots \times X/\alpha_n$ is an isomorphism. Let $A \subseteq \{1, \ldots, n\}$ and let α_A denote $\cap\{\alpha_i: i \in A\}$. Then the natural map from X/α_A to $\prod\{X/\alpha_i: i \in A\}$ is an isomorphism.

Proof. By the familiar isomorphism theorem, the interval $[\alpha_A, 1]$ of the lattice of equivalence relations on X is isomorphic to the lattice of equivalence relations on X/α_A via the map $\alpha \rightsquigarrow \hat{\alpha}$, where $\hat{\alpha} = \{(x/\alpha_A, y/\alpha_A): (x, y) \in \alpha\}$. Further, if α permutes with β, then $\hat{\alpha}$ permutes with $\hat{\beta}$. It follows that the members of $\{\hat{\alpha}_i: i \in A\}$ which differ from 1 comprise the coatoms of a finite Boolean subsystem of the algebra of all relations on X/α_A. Hence X/α_A is canonically isomorphic to $\prod\{X/\alpha_i: i \in A\}$. As $X/\hat{\alpha}_i$ is canonically isomorphic to X/α_i, our result follows. ∎

One might expect there to be some significance attached to infinite Boolean subsystems of RX, and that is indeed the case. We shall return to this point in Section 6.

Definition 2.3.6. For a relation algebra R, define

$$R^{(2)} = \{(x, x') \in R^{(1)} \times R^{(1)}: x \cdot x' = 1' \text{ and } x ; x' = 1\}$$

We call members of $R^{(2)}$ factor pairs of R. It is important to note that members of $R^{(2)}$ are ordered pairs and therefore $(x, x') \neq (x', x)$. We should also note that the condition $x ; x' = 1$ implies that $x ; x' = x' ; x$, since 1 is a proper equivalence element.

Remark 2.3.7. It is easy to see that $(x, x') \in R^{(2)}$ iff $\{1', x, x', 1\}$ is a Boolean subsystem of R. So an ordered pair of equivalence relations (α, α') on a set X is in $RX^{(2)}$ iff X is canonically isomorphic to $X/\alpha \times X/\alpha'$.

Definition 2.3.8. For a relation algebra R, define a binary relation $\sqsubseteq$ on $R^{(2)}$ by

$$(x, x') \sqsubseteq (y, y') \qquad \text{if} \quad x \leq y, \quad y' \leq x', \quad \text{and} \quad x \,;\, y' = y' \,;\, x$$

Then define a unary operation $\perp$: $R^{(2)} \to R^{(2)}$ by setting $(x, x')^{\perp} = (x', x)$, and define constants $\mathbb{\Omega} = (1', 1)$ and $\mathbb{\bot} = (1, 1')$.

The proofs of the following lemma and of the theorem which concludes this section are given in Harding (1996).

Lemma 2.3.9. Let R be a relation algebra and let (x, x'), (y, y') be in $R^{(2)}$. Then the following are equivalent:

(i) $(x, x') \sqsubseteq (y, y')$.
(ii) $x \leq y$ and $\{1', 1, x, x', y, y', (x' \cdot y), (x \,;\, y')\}$ is a Boolean subsystem of R.

Remark 2.3.10. In terms of the relation algebra RX, the above lemma shows that $(x, x') \sqsubseteq (y, y')$ if and only if X is canonically isomorphic to $X/y \times X/(x \,;\, y') \times X/x'$. This means that $(x, x') \sqsubseteq (y, y')$ if the decompositions $X \cong X/x \times X/x'$ and $X \cong X/y \times X/y'$ can be built in an obvious manner from a common ternary decomposition.

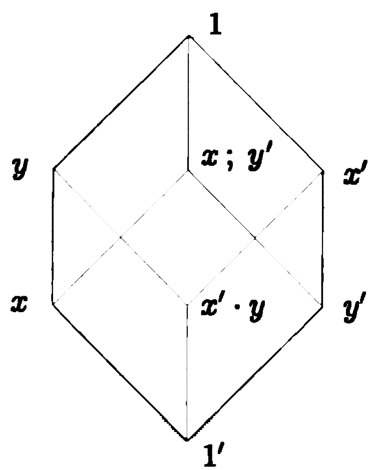

Fig. 1.

Theorem 2.3.11. For a relation algebra R, the system $(R^{(2)}, \subseteq, \underline{0}, \underline{1}, \perp)$ $\cap$ an orthomodular poset. Further, if $(x, x') \subseteq (y, y')^{\perp}$, then $(x, x') + (y, y') = ((x\ ;\ y), (x' \cdot y'))$.

3. COMPATIBILITY IN RELATIONAL OMPs

In this section we give a simple description of when a subset of a relational orthomodular poset $R^{(2)}$ is compatible.

Lemma 3.1. Let R be a relation algebra and (x_1, x_1') and (x_2, x_2') be elements of the orthomodular poset $R^{(2)}$. If (x_1, x_1') and (x_2, x_2') are compatible, then

$$(x_1, x_1') + (x_2, x_2') = (x_1\ ;\ x_2, x_1' \cdot x_2')$$

Note that the join of these elements exists, as they are compatible.

Proof. If (x_1, x_1') and (x_2, x_2') are compatible, then there is a Boolean subalgebra B of $R^{(2)}$ which contains (x_1, x_1') and (x_2, x_2'). Let

$$(z_1, z_1') = (x_1, x_1') \cdot (x_2, x_2')^{\perp},$$

$$(z_2, z_2') = (x_1, x_1') \cdot (x_2, x_2')$$

and

$$(z_3, z_3') = (x_2, x_1')^{\perp} \cdot (x_2, x_2').$$

Note that by Lemma 2.1.4, there is no need to specify whether these meets are taken in B or in $R^{(2)}$, as the two notions coincide. As our discussion can be considered to take place within a Boolean algebra, we have

$$(x_1, x_1') = (z_1, z_1') + (z_2, z_2') \tag{3.1}$$

and

$$(x_2, x_2') = (z_2, z_2') + (z_3, z_3') \tag{3.2}$$

Therefore

$$(x_1, x_1') + (x_2, x_2') = [(z_1, z_1') + (z_2, z_2')] + (z_3, z_3')$$

As (z_i, z_i'), $i = 1, 2, 3$, are pairwise orthogonal, and therefore $(z_1, z_1') + (z_2, z_2')$ is orthogonal to (z_3, z_3'), we may apply Theorem 2.3.11 to obtain

$$(x_1, x_1') + (x_2, x_2') = (z_1\ ;\ z_2\ ;\ z_3, z_1' \cdot z_2' \cdot z_3'). \tag{3.3}$$

Using Theorem 2.3.11, we find that equation (3.1) gives $x_1 = z_1\ ;\ z_2$ and (3.2) gives $x_2 = z_2\ ;\ z_3$. Therefore $x_1\ ;\ x_2$ equals $z_1\ ;\ z_2\ ;\ z_2\ ;\ z_3$, which in turn equals $z_1 ; z_2 ; z_3$. A similar argument shows that $x_1' \cdot x_2'$ equals $z_1' \cdot z_2' \cdot z_3'$. Substituting into (3.3) yields our result. ∎

Lemma 3.2. Let R be a relation algebra and (x_1, x_1') and (x_2, x_2') be elements of the orthomodular poset $R^{(2)}$. If (x_1, x_1') and (x_2, x_2') are compatible, then

$$(x_1, x_1') \cdot (x_2, x_2') = (x_1 \cdot x_2, x_1' ; x_2').$$

Note that the meet of these elements exists, as they are compatible.

Proof. If (x_1, x_1') and (x_2, x_2') are compatible, then so are their orthocomplements (x_1', x_1) and (x_2', x_2). Using the previous lemma, we have

$$(x_1, x_1')^{\perp} + (x_2, x_2')^{\perp} = (x_1' ; x_2', x_1 \cdot x_2)$$

Taking the orthocomplement of both sides of this equation yields our result. ∎

Lemma 3.3. Let R be a relation algebra and (x_1, x_1') and (x_2, x_2') be compatible elements of the orthomodular poset $R^{(2)}$. Then $(x_1, x_1') \sqsubseteq (x_2, x_2')$ iff $x_1 \leq x_2$.

Proof. If $(x_1, x_1') \sqsubseteq (x_2, x_2')$, then by definition $x_1 \leq x_2$. Conversely, if $x_1 \leq x_2$, then by the previous lemma

$$(x_1, x_1')^{\perp} + (x_2, x_2') = (x_1' ; x_2, x_1 \cdot x_2')$$

But $x_1 \leq x_2$ implies that $1 = x_1' ; x_1 \leq x_1' ; x_2$. The only factor pair having 1 in its first coordinate is the pair $(1, 1') = \perp$. As (x_1, x_1') is compatible with (x_2, x_2'), they are elements of some Boolean subalgebra of $R^{(2)}$. Since $(x_1, x_1')^{\perp} + (x_2, x_2') = \perp$, it follows that $(x_1, x_1') \sqsubseteq (x_2, x_2')$. ∎

Proposition 3.4. Let R be a relation algebra and B be a Boolean subalgebra of the orthomodular poset $R^{(2)}$. Then the image of the φ: $B \to R^{(1)}$, defined by $\varphi(x, x') = x$, is a Boolean subsystem of R. Further, considered as a map between Boolean algebras, φ is an isomorphism.

Proof. Lemma 3.3 shows that φ is an order embedding from the partially ordered set B to the partially ordered set R. Thus the image of φ is a Boolean lattice under the partial ordering inherited from R. Let x_1 and x_2 be elements in the image of φ. Then there must be elements x_1' and x_2' so that (x_1, x_1') and (x_2, x_2') are in B. As φ is an order embedding, it follows that the join of x_1 and x_2 in the image must be equal to the image of the join of (x_1, x_1') and (x_2, x_2'). By Lemma 3.1 we have that $\varphi((x_1, x_1') + (x_2, x_2')) = x_1 ; x_2$. Thus joins in the image of B are given by relational product. It follows by a similar argument using Lemma 3.2 that meets in the image of B agree with meets in R. Finally, as $(1', 1)$ and $(1, 1')$ are the bounds of B, it follows that $1'$ and 1 are the smallest and largest elements in the image of B. Thus the image of B is a Boolean subsystem of R. The further remark follows directly from what we have shown. ■

Proposition 3.5. Let R be a relation algebra and B be a Boolean subsystem of R. For each element $y \in B$, let y' denote the complement of y in the Boolean algebra B. Then the image of the map $\psi: B \rightarrow R^{(2)}$, defined by $\psi(y) = (y, y')$, is a Boolean subalgebra of the orthomodular poset $R^{(2)}$. Further, considered as a map between Boolean algebras, ψ is an isomorphism.

Proof. Note that as joins in B are given by relational product, the fact that y' is a complement of y in B ensures that (y, y') is a factor pair. Thus the map ψ is well defined. Also, as B is a Boolean algebra under the partial ordering inherited from R, we have that $x \leq y$ iff $y' \leq x'$. As all elements in B permute, it follows at once that $(x, x') \subseteq (y, y')$ iff $x \leq y$. Thus ψ is an order embedding from B into $R^{(2)}$. Clearly the image of ψ is closed-under orthocomplementation in $R^{(2)}$. We next show that the image of ψ is closed under finite orthogonal joins. Suppose that (x, x') and (y, y') are orthogonal elements in the image of ψ. Then $(x, x') + (y, y') = (x \,;\, y, x' \cdot y')$. But this last term is equal to $\psi(x \,;\, y)$. So the image of ψ is closed under orthogonal joins, and hence is a subalgebra of the orthomodular poset $R^{(2)}$ in the sense of Definition 2.1.3. As ψ is an order isomorphism between B and its image, it follows that the image of ψ is Boolean. The further remark follows directly from what we have shown. ■

Corollary 3.6. Let R be a relation algebra and A be a subset of $R^{(2)}$. Then A is compatible iff $\{a, a': (a, a') \in A\}$ is contained in a Boolean subsystem of R.

Proof. If A is compatible, then by Definition 2.1.5, A is contained in some Boolean subalgebra B of $R^{(2)}$. Let $\varphi: B \rightarrow R^{(1)}$ be the map given in Proposition 3.4. Then $\{a, a': (a, a') \in A\}$ is contained in the image of φ. But by Proposition 3.4, this image is a Boolean subsystem of R. Conversely, if $\{a, a': (a, a') \in A\}$ is contained in a Boolean subsystem B of R, then the image of the map $\psi: B \rightarrow R^{(2)}$ given in Proposition 3.5 is a Boolean subalgebra of $R^{(2)}$ which contains A. Hence A is compatible. ■

Definition 3.7. For a set X, let BooFin $(RX^{(2)})$ denote the collection of all finite Boolean subalgebras of $RX^{(2)}$. We define a decomposition of X to be an $n + 1$-tuple $(\varphi; X_1, \ldots, X_n)$ where φ is an isomorphism from X to $X_1 \times \ldots \times X_n$, and none of the X_i are one-element sets. Two decompositions $(\varphi \,;\, X_1, \ldots, X_n)$ and $(\varphi' \,;\, X_1', \ldots, X_n')$ are said to be equivalent if there is a permutation σ of $\{1, \ldots, n\}$ and isomorphisms $\phi_i: X_i \rightarrow X^1_{\sigma i}$ such that ϕ_i

$\circ\, \pi_i \circ \varphi = \pi_{\sigma i} \circ \varphi'$ for $i = 1, \ldots, n$. We then define DecompFin(X) to be the collection of all equivalence classes of decompositions of X.

Proposition 3.8. Let X be a set. Then there are mutually inverse isomorphisms Φ and Ψ between BooFin($RX^{(2)}$) and DecompFin(X) defined as follows:

(i) If the coatoms of B are enumerated (α_i, α_i'), $i = 1, \ldots, n$, then $\Phi(B)$ is the equivalence class of φ: $X \to X/\alpha_1 \times \cdots \times X/\alpha_n$.

(ii) $\Psi([\varphi: X \to X_1 \times \cdots \times X_n])$ is the finite Boolean subalgebra whose coatoms are $(\ker(\pi_i \circ \varphi), \ker(\pi_i' \circ \varphi))$, for $i = 1, \ldots, n$.

Here π_i is the projection onto X_i, and π_i' is the projection onto $\Pi\{X_j: j \neq i\}$.

Proof. Assume that (α_i, α_i'), $i = 1, \ldots, n$, are the coatoms of a finite Boolean subalgebra. Then by Proposition 3.4, $\alpha_1, \ldots, \alpha_n$ are the coatoms of a finite Boolean subsystem of RX. Then Proposition 2.3.4 shows that φ: $X \to X/\alpha_1 \times \cdots \times X/\alpha_n$ is a decomposition. Different enumerations of the coatoms would yield different, but equivalent, decompositions. So Φ is well defined.

If φ: $X \to X_1 \times \cdots \times X_n$ is a decomposition, then the natural map from X to $\Pi\{X/\ker(\pi_i \circ \varphi): i = 1, \ldots, n\}$ is an isomorphism. As the definition of a decomposition requires that each of the sets X_i have more than one element, each of the relations $\ker(\pi_i \circ \varphi)$ is distinct from 1. So by Proposition 2.3.4 the relations $\ker(\pi_i \circ \varphi)$, $i = 1, \ldots, n$, comprise the coatoms of a finite Boolean subsystem B of RX. But meets in B are given by set intersection, and $\ker(\pi_i' \circ \varphi) = \cap\{\ker(\pi_j \circ \varphi): j \neq i\}$. Therefore $\ker(\pi_i' \circ \varphi)$ is the complement of $\ker(\pi_i \circ \varphi)$ in B. It follows from Proposition 3.5 that there is a finite Boolean subalgebra of $RX^{(2)}$ whose coatoms are $(\ker(\pi_i \circ \varphi), \ker(\pi_i' \circ \varphi))$ $i = 1, \ldots, n$. Finally, the definition of equivalence ensures that any other member of this equivalence class will yield the same coatoms. So Ψ is well defined.

Given a subalgebra B of $RX^{(2)}$ whose coatoms are (α_i, α_i'), $i = 1, \ldots, n$, let φ: $X \to X/\alpha_1 \times \cdots \times X/\alpha_n$ be the natural isomorphism. Surely $\alpha_i = \ker(\pi_i \circ \varphi)$. But $\alpha_i' = \cap\{\alpha_j: j \neq i\}$ and $\ker(\pi_i') = \cap\{\ker(\pi_j \circ \varphi): j \neq i$. Therefore (α_i, α_i') is equal to $(\ker(\alpha_i \circ \varphi), \ker(\pi_i' \circ \varphi))$, showing that $\Psi \circ \Phi$ is the identity. Conversely, if φ: $X \to X_1 \times \cdots \times X_n$ is a decomposition, then there is a canonical isomorphism ϕ_i: $X_i \to X/\ker(\pi_i \circ \varphi)$. It follows that $\Phi \circ \Psi$ is the identity map as well. ■

4. REGULARITY IN RELATIONAL OMPs

In this section we show that any relational orthomodular poset $R^{(2)}$ is regular. Throughout, we will assume that R is a relation algebra and (x_i, X_i'), $i = 0, 1, 2$, are pairwise compatible elements in $R^{(2)}$.

Lemma 4.1. $x_2 \cdot (x'_0 \,;\, x'_1) = (x'_0 \cdot x_2) \,;\, (x'_1 \cdot x_2)$.

Proof. As (x_0, x'_0) is compatible with (x_2, x'_2), we have by Corollary 3.6 that x'_0 equals $(x'_0 \cdot x_2) \,;\, (x'_0 \cdot x'_2)$. And as (x_1, x'_1) is compatible with (x_2, x'_2), we have that x'_1 is equal to $(x'_1 \cdot x'_2) \,;\, (x'_1 \cdot x_2)$. Therefore

$$x_2 \cdot (x'_0 \,;\, x'^1) = x_2 \cdot [(x'_0 \cdot x_2) \,;\, (x'_0 \cdot x'^2) \,;\, (x'_1 \cdot x'_2) \,;\, (x'_1 \cdot x_2)]$$

Applying part (ii) of Lemma 2.2.2 to the right side of this equation with a equal to x_2, c equal to $x'_0 \cdot x_2$, and b equal to $(x'_0 \cdot x'_2) \,;\, (x'_1 \cdot x'_2) \,;\, (x'_1 \cdot x_2)$ gives

$$x_2'(x'_0 \,;\, x'_1) = (x'_0 \cdot x_2) \,;\, [x_2 \cdot ((x'_0 \cdot x'_2) \,;\, (x'_1 \cdot x'_2) \,;\, (x'_1 \cdot x_2))]$$

Applying part (i) of Lemma 2.2.2 to the term in square brackets with a equal to x_2, c equal to $x'_1 \cdot x_2$, and b equal to $(x'_0 \cdot x'_2) \,;\, (x'_1 \cdot x'_2)$ gives

$$x_2 \cdot (x'_0 \cdot x'_1) = (x'_0 \cdot x_2) \,;\, [x_2 \cdot ((x'_0 \cdot x'_2) \,;\, (x'_1 \cdot x'_2))] \,;\, (x'_1 \cdot x'_2)$$

But $(x'_0 \cdot x'_2) \,;\, (x'_1 \cdot x'_2) \leq x'_2$ and $x_2 \cdot x'_2 = 1'$. Our result follows easily. ■

Lemma 4.2. $x_2 \cdot (x'_0 \,;\, x'_1 \,;\, x'_2) = x_2 \cdot (x'_0 \,;\, x'_1)$

Proof. As (x_1, x'_1) is compatible with (x_2, x'_2), we have that

$$x_2 \cdot (x'_0 \,;\, x'_1 \,;\, x'_2) = x_2 \cdot (x'_0 \,;\, x'_2 \,;\, x'_1).$$

As (x_0, x'_0) is compatible with (x_2, x'_2), we have by Corollary 3.6 that $x'_0 \,;\, x'_2$ equals $(x'_0 \cdot x_2) \,;\, x'_2$. And as (x_1, x'_1) is compatible with (x_2, x'_2), we have that $x'_2 \,;\, x'_1$ is equal to $x'_2 \,;\, (x'_1 \cdot x_2)$. Thus

$$x_2 \cdot (x'_0 \,;\, x'_1 \,;\, x'_2) = x_2 \cdot [(x'_2 \cdot x_2) \,;\, x'_2 \,;\, (x'_1 \cdot x_2)]$$

As (x_1, x'_1) commutes with (x_2, x'_2) we have by Corollary 3.6 that x'_2 permutes with $(x'_1 \cdot x_2)$. This gives

$$x_2 \cdot (x'_0 \,;\, x'_1 \,;\, x'_2) = x_2 \cdot [(x'_0 \cdot x_2) \,;\, (x'_1 \cdot x_2) \,;\, x'_2].$$

Using part (ii) of Lemma 2.2.2 with a equal to x_2, c equal to $(x'_0 \cdot x_2)$; $(x'_1 \cdot x_2)$, and b equal to x'_2 gives

$$x_2 \cdot (x'_0 \,;\, x'_1 \,;\, x'_2) = (x'_0 \cdot x_2) \,;\, (x'_1 \cdot x_2)$$

Our result then follows from Lemma 4.1. ■

Lemma 4.3. $x_0 \cdot x_1 \cdot x_2 \cdot (x'_0 \,;\, x'_1 \,;\, x'_2) = 1$.

Proof. By Lemma 4.2 we have

$$x_0 \cdot x_1 \cdot x_2 \cdot (x'_0 \,;\, x'_1 \,;\, x'_2) = x_0 \cdot x_1 \cdot x_2 \cdot (x'_0 \,;\, x'_1)$$

But (x_0, x'_0) is compatible with (x_1, x'_1), and so by Corollary 3.6, $x_0 \cdot x_1 \cdot (x'_0 \,;\, x'_1)$ equals $1'$. Our result follows trivially. ■

Lemma 4.4. $x_2' ; (x_0 \cdot x_1) = (x_0 ; x_2') \cdot (x_1 ; x_2')$.

Proof. As (x_0, x_0') is compatible with (x_2, x_2'), we have by Corollary 3.6 that x_0 equals $(x_0 ; x_2) \cdot (x_0 \cdot x_2')$, and similarly x_1 equals $(x_1 ; x_2) \cdot (x_1; x_2')$. As meets are commutative,

$$x_2' ; (x_0 \cdot x_1) = x_2' ; [(x_0 ; x_2') \cdot (x_1 ; x_2') \cdot (x_0 ; x_2) \cdot (x_1 ; x_2)]$$

Using part (ii) of Lemma 2.2.2 with a equal to $(x_0 ; x_2') \cdot (x_1 ; x_2')$, c equal to x_2', and b equal to $(x_0; x_2) \cdot (x_1; x_2)$ yields

$$x_2' ; (x_0 \cdot x_1) = [x_2' ; ((x_0 ; x_2) \cdot (x_1 ; x_2))] \cdot (x_0 ; x_2') \cdot (x_1 ; x_2').$$

But $(x_0 ; x_2) \cdot (x_1 ; x_2) \geq x_2$ and $x_2', x_2 = 1$. Our result follows easily. ∎

Lemma 4.5. $x_2' ; (x_0 \cdot x_1 \cdot x_2) = (x_0 ; x_2') \cdot (x_1 ; x_2')$ and therefore x_2' permutes with $x_0 \cdot x_1 \cdot x_2$.

Proof. As (x_0, x_0') is compatible with (x_2, x_2'), it follows from Corollary 3.6 that $x_0 \cdot x_2$ equals $(x_0 ; x_2') \cdot x_2$. Similarly as (x_1, x_1') is compatible with (x_2, x_2'), we have that $x_1 \cdot x_2$ equals $(x_1 ; x_2') \cdot x_2$. Therefore

$$x_2' ; (x_0 \cdot x_1 \cdot x_2) = x_2' ; [(x_0 ; x_2') \cdot (x_1 ; x_2') \cdot x_2]$$

Using part (ii) of Lemma 2.2.2 with a equal to $(x_0 ; x_2') \cdot (x_1 ; x_2')$, c equal to x_2', and b equal to x_2 gives that $x_2' ; (x_0 \cdot x_1 \cdot x_2) = (x_0 ; x_2') \cdot (x_1 ; x_2')$. As the term on the right side of this last equality is a proper equivalence element, it follows that x_2 permutes with $x_0 \cdot x_1 \cdot x_2$. ∎

Lemma 4.6. $x_0' ; x_1' ; x_2' ; (x_0 \cdot x_1 \cdot x_2) = 1$.

Proof. Combining Lemma 4.5 and Lemma 4.4, we have

$$x_0' ; x_1' ; x_2' ; (x_0 \cdot x_1 \cdot x_2) = x_0' ; x_1' ; x_2' ; (x_0 \cdot x_1)$$

But (x_2, x_2') is compatible with (x_1, x_1') as well as with (x_0, x_0'). Therefore x_2' permutes with x_1' and x_0'. So

$$x_0' ; x_1' ; x_2' ; (x_0 \cdot x_1 \cdot x_2) = x_2' ; x_0' ; x_1' ; (x_0 \cdot x_1)$$

But (x_0, x_0') is compatible with (x_1, x_1'), so $x_0' ; x_1' ; (x_0 \cdot x_1)$ equals 1. Our result then follows trivially. ∎

Lemma 4.7. $(x_0 \cdot x_1 \cdot x_2, x_0' ; x_1' ; x_2')$ is a factor pair and is the meet in $R^{(2)}$ of the set $\{(x_i, x_i'): i = 0, 1, 2\}$.

Proof. Clearly $x_0 \cdot x_1 \cdot x_2$ is a proper equivalence element. But the elements (x_i, x_i'), $i = 0, 1, 2$, being pairwise compatible implies that x_i', $i = 0, 1, 2$, are pairwise permuting. So by Lemma 2.3.2(iii), we have that $x_0' ; x_1' ; x_2'$ is also a proper equivalence element. It then follows by Lemma 4.3 and Lemma 4.6 that $(x_0 \cdot x_1 \cdot x_2, x_0' ; x_1' ; x_2')$ is in $R^{(2)}$. For each $i = 0, 1, 2$ we clearly have that $x_0 \cdot x_1 \cdot x_2 \leq x_i$ and $x_0' ; x_1' ; x_2' \geq x_i'$. But by Lemma 4.5 we have that $x_0 \cdot x_1 \cdot x_2$ permutes with x_2'. Clearly the proofs of all the above lemmas are valid under permutation of the indices, and therefore $x_0 \cdot x_1 \cdot x_2$ permutes with each x_i', $i = 0, 1, 2$. By Definition 2.3.8, $(x_0 \cdot x_1 \cdot x_2, x_0' ; x_1' ; x_2')$ is a lower bound of $\{(x_i, x_i'): i = 0, 1, 2\}$. We need only show that this is the greatest lower bound of this set. Suppose (y, y') is a lower bound of $\{(x_i, x_i'): i = 0, 1 2\}$. Then as $y \leq x_i$, $i = 0, 1, 2$, it follows that $y \leq x_0 \cdot x_1 \cdot x_2$. And as $y' \geq x_i'$, $i = 0, 1, 2$, it follows that $y' \geq x_0' ; x_1' ; x_2'$. But as $(y, y') \subseteq (x_i, x_i')$, $i = 0, 1, 2$, we have that y permutes with x_i', $i = 0, 1, 2$. Therefore y permutes with $x_0' ; x_1' ; x_2'$. This shows that (y, y') is below the element $(x_0 \cdot x_1 \cdot x_2, x_0' ; x_1' ; x_2')$. ■

Lemma 4.8. $(x_0 \cdot x_1 \cdot x_2) ; (x_0 \cdot x_1 \cdot x_2') = x_0 \cdot x_1$.

Proof. Applying part (ii) of Lemma 2.2.2 with a equal to $x_0 \cdot x_1$, c equal to $x_0 \cdot x_1 \cdot x_2$, and b equal to x_2', we have

$$(x_0 \cdot x_1 \cdot x_2) ; (x_0 \cdot x_1 \cdot x_2') = [(x_0 \cdot x_1 \cdot x'_2) ; x_2'] \cdot x_0 \cdot x_1$$

Applying Lemma 4.5, we have that

$$(x_0 \cdot x_1 \cdot x_2) ; (x_0 \cdot x_1 \cdot x_2') = (x_1 ; x_3') \cdot (x_2 ; x_3') \cdot x_1 \cdot x_2$$

Then as $x_1 ; x_3' \geq x_1$ and $x_2 ; x_3' \geq x_2$, our result follows at once. ■

Theorem 4.9. For a relation algebra R, the orthomodular poset $R^{(2)}$ is regular.

Proof. By Corollary 2.1.8 it is enough to show that every pairwise compatible subset of $R^{(2)}$ with three elements is compatible. Let (x_i, x_i'), $i = 0, 1, 2$, be pairwise compatible elements in $R^{(2)}$. We must show that the set $\{(x_i, x_i'): i = 0, 1, 2\}$ is compatible. By Proposition 2.1.6 it is enough to show that for each $\alpha \in 2^3$ the family $\{(x_i, x_i')^{\alpha(i)}: i = 0, 1, 2\}$ has a meet in $R^{(2)}$ and

$$\sum_{\alpha \in 2^3} \prod_{i=0}^{2} (x_i, x_i')^{\alpha(i)} = 1 \tag{4.1}$$

Note that (x_i, x_i') being compatible with (x_j, x_j') implies that (x_i, x_i') is compatible with the orthocomplement of (x_j, x_j'), namely (x_j', x_j). Therefore the above lemmas remain valid if we interchange occurrences of x_i with x_i' everywhere. Thus Lemma 4.7 shows that for each $\alpha \in 2^3$, the family $\{(x_i, x_i')^{\alpha(i)}: i = 0, 1, 2\}$ has a meet in $R^{(2)}$. To verify condition (4.1), note first that we are guaranteed that the join in question exists, as it is an orthogonal join. To show that this join is equal to $\perp$, it is sufficient to show that the first component of this join equals 1, as there is only one factor pair having 1 in its first coordinate. Then as we have an orthogonal join, we may use the description of orthogonal joins in Theorem 2.3.11. Namely, the first coordinate of this join is the relational product of the elements in the set $\{x_0^{\alpha(0)} \cdot x_1^{\alpha(1)} \cdot x_2^{\alpha(2)} : \alpha \in 2^3\}$. Here we are using x_i^0 to denote x_i and x_i^1 to denote x_i'. Our result now follows easily from Lemma 4.8. ■

5. REGULARITY OF STRUCTURAL DECOMPOSITION

We have seen that the collection of all decompositions of a nonempty set X naturally forms an orthomodular poset, $RX^{(2)}$. In the presence of some additional type of structure on the set X, one may be interested only in the decompositions which are compatible with this additional structure. For instance, if X carries a group structure, then we may be concerned only with decompositions of X which are compatible with the group operations. In Harding (1996) it was shown that if X is equipped with any algebraic, topological, or relational structure, then the decompositions of X which are compatible with this additional structure form a subalgebra of the orthomodular poset $RX^{(2)}$. With this extra flexibility, many of the common constructions of orthomodular posets can be realized as the decompositions of certain types of structures.

In this section we wish to address the question of whether the structure-preserving decompositions of an algebra, relational structure, or topological space form a regular orthomodular poset. As shown by Example 2.1.9, a subalgebra of a regular orthomodular poset need not be regular. So regularity in this more general setting does not follow *a priori* from the results of the previous section.

Proposition 5.1. Let R be a relation algebra and S be a subset of $R^{(2)}$ which is closed under orthocomplementation. Assume further that for all (x_i, x_i'), $i = 1, 2$, in S which are compatible in $R^{(2)}$ we have that $(x_1 \cdot x_2, x_1' ; x_2')$ is in S. Then S is a compatible subalgebra of $R^{(2)}$ and S itself is a regular orthomodular poset.

Proof. This follows directly from Corollary 2.1.14 and Theorem 4.9. ■

Definition 5.2. An algebraic structure is a set X equipped with a family $(f_i)_I$ of finitary, or infinitary, operations. A relational structure is a set X with a nonempty binary relation R. And a topological structure is a set X together with a topology τ. The notions of product and isomorphism for structures of the same type will have the usual meaning. For an algebraic, relational, or topological structure X we define Fact X to be all those decompositions $(\alpha,$

α') in $RX^{(2)}$ for which there exist structures on X/α and X/α' making the structure X canonically isomorphic to their product.

Lemma 5.3. Let X be a structure and $(\alpha_1, \alpha_2) \in RX^{(2)}$. If there are structures on X/α_1 and X/α_2 making X structurally isomorphic to $X/\alpha_1 \times X/\alpha_2$, then these structures are unique. If X is an algebraic structure, the structures on X/α_1 and X/α_2 are given by the usual quotient construction. If X is a relational structure with relation R, the relations R_1 and R_2 on X/α_1 and X/α_2 are given by

$$R_i = \{(x/\alpha_i, y/\alpha_i): (x, y) \in R\}$$

And if X is a topological structure with topology τ, then the topologies τ_1 and τ_2 on X/α_1 and X/α_2 are given by

$$\tau_i = \{\pi_i[A]: A \text{ is open in } \tau\}$$

Here τ_i is the projection onto X/α_i.

Proof. For algebraic structures this result is very well known. To verify our result for relational structures it is enough to show that

$$(X_1 \times X_2, R) = (X_1, R_1) \times (X_2, R_2)$$

implies that $R_1 = \{(x_1, x'_1)$: exist x_2, x'_2 with $((x_1, x_2), (x'_2, x'_2)) \in R\}$. But this follows readily, as R_2 is nonempty. Finally, to verify our result for topological structures, it is enough to show that

$$(X_1 \times X_2, \tau) = (X_1, \tau_1) \times (X_2, \tau_2)$$

implies that $\tau_1 = \{\pi_1[A]: A \in \tau\}$. As we are dealing with the product of topological spaces, the projection π_1 is both open and continuous (Kelly, 1955). So for $A \in \tau$ we have $\pi_1[A] \in \tau_1$, as τ_I is open. But if $B \in \tau_1$, then $\pi_1^{-1}[B]$ is in τ as π_1 is continuous, and $\pi_1[\pi_1^{-1}[B]]$ is equal to B. ■

The above result shows that Fact X fully describes all structure-preserving decompositions of X, up to isomorphisms. We must now establish several technical results about products of relational and topological structures. As an aid, we introduce the following notation.

Definition 5.4. If τ_1 and τ_2 are topologies on X_1 and X_2, we let $\tau_1 \times \tau_2$ denote the product topology on $X_1 \times X_2$. Similarly, if R_1 and R_2 are relations on X_1 and X_2, we define $R_1 \times R_2$ to be the product relation on $X_1 \times X_2$. Specifically,

$$R_1 \times R_2 = \{((x_1, x_2), (x'_1, x'_2)): (x_1, x'_1) \in R_1 \text{ and } (x_2, x'_2) \in R_2\}$$

If τ is a topology on the product of sets $A \times B$, we define $\tau_A = \{\pi_A[U]$:

$U \in \tau\}$. The previous lemma shows that in particular circumstances τ_A will be a topology, but that is not in general the case. Similarly, if R is a relation on $A \times B$, we define

$$R_A = \{(a, a'): \text{exists } b, b' \text{ with } ((a, b), (a', b')) \in R\}$$

Finally, we warn the reader that this notation will be routinely abused when considering n-fold products. Thus if R is a relation on $A \times B \times C \times D$, we feel free to write $R_{A\times C}$ with the obvious meaning.

Lemma 5.5. If $(A \times B \times C \times D, R)$ is naturally isomorphic to both $(A \times B, R_{A\times B}) \times (C \times D, R_{C\times D})$ and $(A \times C, R_{A\times C}) \times (B \times D, R_{B\times D})$, then it is naturally isomorphic to $(A, R_A) \times (B, R_B) \times (C, R_C) \times (D, R_R)$.

Proof. We must show that $R = R_A \times R_B \times R_C \times R_D$. Surely R is contained in this product relation. For the converse, let $((a_1, b_2, c_3, d_4), (a'_1, b'_2, c'_3, d'_4))$ be an element of the product relation. Then there exist

$$((a_i, b_i, c_i, d_i), (a'_i, b'_i, c'_i, d'_i)) \in R \text{ for } i = 1, \ldots, 4$$

This gives us that $((a_1, b_1), (a'_1, b'_1))$ is in $R_{A\times B}$ and that $((c_3, d_3), (c'_3, d'_3))$ is in $R_{C\times D.}$ As we have assumed that R is equal to the product of the relations $R_{A\times B} \times R_{C\times D}$, it follows that

$$((a_1, b_1, c_3, d_3), (a'_1, b'_1, c'_3, d'_3)) \in R \tag{5.1}$$

and similarly

$$((a_2, b_2, c_4, d_4), (a'_2, b'_2, c'_4, d'_4)) \in R \tag{5.2}$$

As we have assumed R is equal to $R_{A\times C} \times R_{B\times D}$, it follows from (5.1) and (5.2) that

$$((a_1, b_2, c_3, d_4), (a'_1, b'_2, c'_3, d'_4)) \in R$$

This establishes our result. ■

Lemma 5.6. Let τ be a topology on the product of sets $X \times Y$ and suppose that there exist topologies on X, Y with τ the product topology. Then for any $U \in \tau$ and any point (x, y) in U there is some $V \in \tau$ with

(i) (x, y) in V
(ii) $V \subseteq U$
(iii) $V = \pi_X[V] \times \pi_Y[V]$.

Proof. From Lemma 5.3, we have that τ_X and τ_Y are topologies and that $\tau = \tau_X \times \tau_Y$. As $\{A \times B: A \in \tau_X, B \in \tau_Y\}$ is a basis for the product topology

τ, there must be sets V_1, V_2 in τ so that (x, y) is in $\pi_X[V_1] \times \pi_Y[V_2]$ and $\pi_X[V_1] \times \pi_Y[V_2]$ is contained in U. Take V equal to $\pi_X[V_1] \times \pi_Y[V_2]$. ■

Lemma 5.7. Let $(A \times B \times C \times D, \tau)$ be a topological space such that $\tau_{A\times B}$, $\tau_{C\times D}$, $\tau_{A\times C}$, and $\tau_{B\times D}$ are topologies. If the given space is canonically isomorphic to both $(A \times B, \tau_{A\times B}) \times (C \times D, \tau_{C\times D})$ and $(A \times C, \tau_{A\times C}) \times (B \times D, \tau_{B\times D})$, then τ_A, τ_B, τ_C, τ_D are topologies and the given space is canonically isomorphic to $(A, \tau_A) \times (B, \tau_B) \times (C, \tau_C) \times (D, \tau_D)$.

Proof. First we must show that $\tau_A = \{\pi_A[U]: U \in \tau\}$ is a topology. Clearly it is closed under arbitrary unions; we must show it is closed under finite intersections. Suppose U_1 and U_2 are in τ and set U_3 to be the intersection of $\pi_A[U_1] \times B \times C \times D$ and $\pi_A[U_2] \times B \times C \times D$. Clearly $\pi_A[U_3]$ is equal to $\pi_A[U_1] \cap \pi_A[U_2]$. We need only show that U_3 is in τ. Once we have established the claim

$$\pi_A[U] \times B \times C \times D \text{ is in } \tau \text{ for all } U \text{ in } \tau \tag{5.3}$$

it will follow that U_3 is the intersection of two open sets, and hence is open. To establish this claim, suppose that $U \in \tau$. As we have assumed $\tau = \tau_{A\times C} \times \tau_{B\times D}$, it follows that $U' = \pi_{A\times C}[U] \times B \times D$ is in τ. A simple calculation shows that

$$\pi_A[U] \times B \times C \times D = \pi_{A\times B}[U'] \times C \times D$$

Then, as $\tau = \tau_{A\times B} \times \tau_{C\times D}$, we have that $\pi_A[U] \times B \times C \times D$ is in τ.

Next, we must show that τ is equal to the product topology $\tau_A \times \tau_B \times \tau_C \times \tau_D$. Our claim (5.3) has just established that $\pi_A[U] \times B \times C \times D$ is in τ for all $U \in \tau$. Clearly this holds for the projections π_B, π_C, π_D as well. But the collection of all such sets generates the product topology. Thus τ is finer than the product topology.

Conversely, suppose $U \in \tau$ and that $(a, b, c, d) \in U$. As τ is equal to $\tau_{A\times B} \times \tau_{C\times D}$, Lemma 5.6 provides the existence of a set V in τ such that (i) (a, b, c, d) is in V, (ii) $V \subseteq U$, and (iii) V is equal to $\pi_{A\times B}[V] \times \pi_{C\times D}[V]$. Then as τ is equal to $\tau_{A\times C} \times \tau_{B\times D}$, Lemma 5.6 provides the existence of a set W in τ such that (i) (a, b, c, d) is in W, (ii) $W \subseteq V$, and (iii) W is equal to $\pi_{A\times C}[W] \times \pi_{B\times D}[W]$. Let

$$P = \pi_A[W] \times \pi_B[W] \times \pi_C[W] \times \pi_D[W]$$

Clearly P is an open neighborhood of (a, b, c, d) in the product topology. If we can show that $P \subseteq U$, it will follow that the product topology is finer than τ. Suppose that (a_1, b_2, c_3, d_4) is in P. Then there are elements a_i, b_i, c_i, d_i for $i = 1, \ldots, 4$ with (a_i, b_i, c_i, d_i) in W for $i = 1, \ldots, 4$. As W equals $\pi_{A\times C}[W] \times \pi_{B\times D}[W]$ we have that

$$(a_1, b_2, c_1, d_2) \quad \text{and} \quad (a_3, b_4, c_3, d_4) \text{ are in } W$$

Then as $W \subseteq V$ and $V = \pi_{A \times B}[V] \times \pi_{C \times D}[V]$, we have that (a_1, b_2, c_3, d_4) is in V. As $V \subseteq U$, our result follows. ■

Theorem 5.8. If X is an algebraic, relational, or topological structure, then Fact X is a compatible subalgebra of $RX^{(2)}$. Further, the orthomodular poset Fact X is regular.

Proof. We rely on Proposition 5.1. Surely Fact X is closed under orthocomplementation. If (α_1, α_1') and (α_2, α_2') are elements of Fact X which are compatible in $RX^{(2)}$, we must show $(\alpha_1 \cdot \alpha_2, \alpha_1'; \alpha_2')$ is in Fact X. This amounts to showing that there are structures on $X/\alpha_1 \cdot \alpha_2$ and $X/\alpha_1'; \alpha_2'$ making X structurally isomorphic to the product.

For algebraic structures we must show that $\alpha_1 \cdot \alpha_2$ and $\alpha_1'; \alpha_2'$ are congruences of X. The intersection $\alpha_1 \cdot \alpha_2$ of two congruences is always a congruence. But the compatibility of (α_1, α_1') and (α_2, α_2') provides that α_1' permutes with α_2', and the relational product $\alpha_1'; \alpha_2'$ of permuting congruences is a congruence.

For relational and topological structures, the key is provided by Lemmas 5.5 and 5.7. As (α_1, α_1') and (α_2, α_2') are compatible, Proposition 3.4 shows that $\alpha_1, \alpha_2, \alpha_1', \alpha_2'$ lie in a Boolean subsystem B of RX. As B is generated as a Boolean algebra by α_1 and α_2, an element such as $\alpha_1; \alpha_2$ or $\alpha_1; \alpha_2'$ must be either the unit or a coatom of B. As all the coatoms arise in this manner, we have by Proposition 2.3.4 that as sets

$$X \cong X/\alpha_1; \alpha_2 \times X/\alpha_1; \alpha_2' \times X/\alpha_1'; \alpha_2 \times X/\alpha_1'; \alpha_2'$$

Note that Corollary 2.3.5 shows that X/α_1 is canonically isomorphic to $X/\alpha_1; \alpha_2 \times X/\alpha_1; \alpha_2'$, with similar statements holding for X/α_2, etc. Therefore, Lemmas 5.5 and 5.7 show that we can equip the factors $X/\alpha_1; \alpha_2$, etc., with structures so that the above isomorphism is a structural isomorphism. Our result then follows, as $X/\alpha_1 \cdot \alpha_2$ is canonically isomorphic to $X/\alpha_1; \alpha_2 \times X/\alpha_1; \alpha_2' \times X/\alpha_1'; \alpha_2$. ■

Definition 5.9. For an algebraic, relational, or topological structure X, we define BooFin (Fact X) to be the collection of all finite Boolean subalgebras of Fact X. We define a structural decomposition of X to be a decomposition $(\varphi; X_1, \ldots, X_n)$ of the set X, where the X_i can be endowed with structures of the same type as X, making φ a structural isomorphism from X to $X_1 \times \cdots \times X_n$. Two decompositions of the structure X are said to be equivalent if they are equivalent as decompositions of the set X (see Definition 3.7). We then define SDecompFin(X) to be the collection of all equivalence classes of structural decompositions of X.

Obviously BooFin(Fact X) is a subset of BooFin($RX^{(2)}$). Noticing that any set decomposition of X which is equivalent to a structural decomposition of X must itself be a structural decomposition, we then have that SDecomp-Fin(X) is a subset of DecompFin(X). In Proposition 3.8 we showed that there were mutually inverse isomorphisms Φ and Ψ between BooFin($RX^{(2)}$) and DecompFin(X). Thus we can consider the restrictions of Φ, Ψ to BooFin (Fact X) and SDecompFin(X).

Theorem 5.10. Let X be any algebraic, relational, or topological structure. Then the restrictions of Φ and Ψ are mutually inverse isomorphisms between BooFin(Fact X) and SDecompFin(X).

Proof. We have only to show that the restriction of Φ to BooFin(Fact X) is a map into SDecompFin(X) and that the restriction of Ψ to SDecomp-Fin(X) is a map into BooFin(Fact X). The reader should recall the definitions of these maps given in Proposition 3.8.

Suppose B is a Boolean subalgebra of Fact X with coatoms (α_i, α_i'), $i = 1, \ldots, n$. We know there is a set isomorphism $\varphi: X \to X/\alpha_1 \times \cdots \times X/\alpha_n$. We must show that there are structures on the X/α_i making this a structural isomorphism. If X is an algebraic structure, this follows immediately, as our assumption that each (α_i, α_i') is in Fact X provides that each α_i is a congruence.

To verify our result for relational structures, it is sufficient to show that if R is a relation on a set $A_1 \times \cdots \times A_n$ with $R = R_i \text{ x } R_i'$ for $i = 1, \ldots, n$, then R is equal to the product relation $R_1 \times \cdots \times R_n$. Here we are using R_i for the projection R_{A_i} of R onto A_i and R_i' for the projection of R onto $\prod\{A_j: j \neq i\}$. Surely R is contained in the product relation. Suppose that $((a^1_1, \ldots, a^n_n), (b^1_1, \ldots, b^n_n))$ is an element of this product relation. Then as (a^i_i, b^i_i) is in R_i for $i \leq n$, there exist elements $((a^i_1, \ldots, a^i_n), (b^i_1, \ldots, b^i_n))$ in R for $i \leq n$. From the assumption that $R = R_2 \times R_2'$ we have

$$((a^1_1, a^2_2, a^1_3, \ldots, a^1_n), (b^1_1, b^2_2, b^1_3, \ldots, b^1_n)) \in R$$

Then as $R = R_3 \times R_3'$ we have

$$((a^1_1, a^2_2, a^3_3, a^1_4, \ldots, a^1_n), (b^1_1, b^2_2, b^3_3, b^1_4, \ldots, b^1_n)) \in R$$

Continuing in this fashion, we have $((a^1_1, \ldots, a^n_n), (b^1_1, \ldots, b^n_n))$ is in R. This shows that R is equal to the product relation.

To verify our result for topological spaces, we assume that τ is a topology on a set $A = A_1 \times \cdots \times A_n$. We must show that if τ_i and τ_i' are topologies with $\tau = \tau_i \times \tau_i'$ for $i = 1, \ldots, n$, then τ is equal to the product topology $\tau_1 \times \cdots \times \tau_n$. Here $\tau_i = \{\pi[G]: G \in \tau\}$ and $\tau_i' = \{\pi_i'[G]: G \in \tau\}$, where π_i, is the projection onto A_i and π_i' is the projection onto $\prod\{A_j: j \neq i\}$.

It follows easily that $\{\pi_i[G] \times \pi_i'[A]: i \leq n, G \in \tau\}$ is a subbasis for the product topology $\tau_1 \times \cdots \times \tau_n$. As we are assuming that $\tau = \tau_i \times \tau_i'$, this subbasis is contained in τ. So τ is finer than the product topology. Conversely, let $\bar{a} = (a_1, \ldots, a_n)$ be a point in A and let $E \in \tau$ be an open neighborhood of a. As we have assumed that $\tau = \tau_i \times \tau_i'$ for $i \leq n$, we can find sets G_i, G_i', in τ such that each $\pi_i[G_i] \times \pi_i'[G_i']$ is an open neighborhood of $\bar{a}$ in τ which is contained in E. Further, these sets can be chosen so that for each $i < n$.

$$\pi_{i+1}[G_{i+1}] \times \pi_{i+1}'[G_{i+1}'] \subseteq \pi_i[G_i] \times \pi_i'[G_i']$$

Our claim is that $\pi_1[G_1] \times \cdots \pi_n[G_n]$ is an open neighborhood of $\bar{a}$ in the product topology which is contained in E. Surely this set is open in the product topology and contains $\bar{a}$. Suppose that $\bar{b} = (b_1, \ldots, b_n)$ and that $\bar{b} \in \pi_1[G_1] \times \cdots \pi_n[G_n]$. As $a \in \pi_n[G_n] \times \pi_n'[G_n']$ and $b_n \in \pi_n[G_n]$, we have that

$$(a_1, \ldots, a_{n-1}, b_n) \in \pi_n[G_n] \times \pi_n'[G_n']$$

Then, as $\pi_n[G_n] \times \pi_n'[G_n'] \subseteq \pi_{n-1}[G_{n-1}] \times \pi_{n-1}'[G_{n-1}']$ and $b_{n-1} \in \pi_{n-1}[G_{n-1}]$, we have

$$(a_1, \ldots, a_{n-2}, b_{n-1}, b_n) \in \pi_{n-1}[G_{n-1}] \times \pi_{n-1}'[G_{n-1}']$$

Proceeding in this manner, we have that $\bar{b}$ is in $\pi_1[G_1] \times \pi_i'[G_i']$ and hence in E. This shows that the product topology is finer than τ, and hence the two are equal.

Having shown that the restriction of Φ to BooFin(Fact X) is a mapping into SDecompFin(X) for any algebraic, relational, or topological structure X, we must establish a similar result for the restriction of Ψ to SDecompFin (X). Namely, if $\varphi: X \to X_1 \times \cdots \times X_n$ is a structural decomposition of X, we must show that the Boolean subalgebra B of $RX^{(2)}$ with coatoms (ker($\pi_i \circ \varphi$), (ker($\pi_i' \circ \varphi$)) is entirely contained in Fact X. As X_i is isomorphic to $X/\ker(\pi_i \circ \varphi)$ and $\prod\{X_j: j \neq i\}$ is isomorphic to $X/\ker(\pi_i' \circ \varphi)$, it follows that each of these coatoms is in Fact X. But every element of B is a meet of coatoms, and Fact X is a compatible subalgebra of $RX^{(2)}$. It follows from Lemma 2.1.13 that B is a subset of Fact X. ■

For any algebraic, relational, or topological structure X, Theorem 5.8 shows that Fact X is a regular orthomodular poset, and Theorem 5.10 shows that finite Boolean subalgebras of Fact X correspond to finitary direct product decompositions of X. In the remainder of this section we will establish similar results for other orthomodular posets which arise from decompositions. Before proceeding further, we first establish the converse to a well-used result from topological algebra.

Lemma 5.11. Let (A, τ_A) and (B, τ_B) be topological spaces and let f_A and f_B be n-ary operations on A and B, respectively. If the associated n-ary operation on $A \times B$ is continuous with respect to the product topology on $A \times B$, then the maps f_A and f_B are continuous.

Proof. Let φ be the natural homeomorphism from $(A \times B, \tau_A \times \tau_B)^n$ to the product $(A, \tau_A)^n \times (B, \tau_B)^n$. A routine calculation shows that for any $U \subseteq A$

$$f_A^{-1}[U] = \pi_{A^n}[\varphi[f^{-1}[U \times B]]]$$

If U is open in τ_A, then $U \times B$ is open in $\tau_A \times \tau_B$. As f is continuous, φ is a homeomorphism, and the projection π_{A_n} is open, we have that $f_A^{-1}[U]$ is open in $(A, \tau_A)^n$. Thus f_A is continuous. ■

We have shown that the decompositions of any algebra, relational structure, or topological space form a regular orthomodular poset. These results can also be combined to apply to many other situations. Rather than proceed in the fullest possible generality, we choose a generic case to illustrate the point.

Definition 5.12. A partially ordered topological group $(X, \cdot, -, 1, \leq, \tau)$ consists of a group $(X, \cdot, -, 1)$ equipped with a partial ordering $\leq$ and a topology τ such that:

(i) $x \leq y$ implies that $axb \leq ayb$ for all $a, b, x, y \in$ X.
(ii) The group operations are continuous with respect to the topology τ.

For a partially ordered topological group X, we define Fact X to be the set of all pairs of equivalence relations (α, α') in $RX^{(2)}$ such that the sets X/α and X/α' can be endowed with partially ordered topological group structures making X structurally isomorphic to their product.

As a topological group can be separately considered as an algebra, a relational structure, and as a topological space, our earlier results show that a decomposition of a partially ordered topological group is completely determined by the decomposition of its underlying set. Thus Fact X fully describes the decompositions of X up to isomorphism.

Proposition 5.13. Let X be a partially ordered topological group. Then Fact X is a compatible subalgebra of $RX^{(2)}$. Further, Fact X is a regular orthomodular poset.

Proof. Assume that (α, α') and (β, β') are elements of Fact X which are compatible in $RX^{(2)}$. In view of Proposition 5.1, we must show that $(\alpha \cdot \beta, \alpha'; \beta')$ is in Fact X. From the assumption that (α, α') and (β, β') are in Fact X, it follows that they are also elements of Fact $(X, \cdot -, 1)$ and Fact $(X, \leq)$

and Fact (X, τ). By Theorem 5.8, we have the existence of operations $\cdot_i$, $-_i$, 1_i, relations $\leq_i$, and topologies τ_i so that

$$(X, \cdot, -, 1, \leq, \tau) \cong (X/\alpha \cdot \beta, \cdot_1, -_1, 1_1, \leq_1, \tau_1) \times (X/\alpha'; \beta', \cdot_2, -_2, 1_2, \leq_2, \tau_2)$$

We need to show that $(X/\alpha \cdot \beta, \cdot_1, -_1, 1_1, \leq_1, \tau_1)$ and $(X/\alpha'; \beta', \cdot_2, -_2, 1_2, \leq_2, \tau_2)$ are partially ordered topological groups. Surely they are partially ordered groups. This is easy to verify, as a failure of any the group identities, a failure of any of the defining conditions of a partial ordering, or a failure of condition (i) in Definition 5.12 would translate directly into a failure of the same condition in the product. It remains to check that operations $\cdot_i$, $-_i$, 1_i are continuous with respect to the topologies τ_i. But this is the content of Lemma 5.11. ■

Proposition 5.14. Let X be a partially ordered topological group. Then the restrictions of Φ and Ψ are mutually inverse isomorphisms between BooFin (Fact X) and SDecomp Fin (X).

Proof. We first show the restriction of Φ is a mapping into SDecompFin (X). Suppose B is a finite Boolean subalgebra of Fact X with coatoms (α_i, α_i'), $i = 1, \ldots, n$. Then Theorem 5.10 provides that φ: $X \to X/\alpha_1 \times \cdots \times X/\alpha_n$ is separately an algebraic, relational, and topological decomposition of X. That these structures on the X/α_i jointly make X/α_i a partially ordered topological group follows from the assumption that (α_i, α_i') is in Fact X. Conversely, showing that the restriction of Ψ is a mapping into BooFin (Fact X) follows as Fact X is a compatible subalgebra of $RX^{(2)}$, exactly as in the proof of Theorem 5.10. ■

Remark 5.15. Flachsmeyer (1982) and Katrnoška (1990) introduced a method to construct an orthomodular poset $\mathscr{L}\mathscr{A}$ from the idempotents of a ring $\mathscr{A}$ with unit. For idempotents e and f, set $e \leq f$ if $ef = e = fe$, and put $e^{\perp} = 1 - e$. As idempotents of $\mathscr{A}$ correspond to direct decompositions of the left $\mathscr{A}$-module $\mathscr{A}_A$, it follows that $\mathscr{L}\mathscr{A}$ is isomorphic to Fact $\mathscr{A}_A$. In view of Theorem 5.8, we have that $\mathscr{L}\mathscr{A}$ is regular. See Harding (1996), Remarks 4.9, and Remark 5.8, for further comments on the relationship between $\mathscr{L}\mathscr{A}$ and decompositions of modules.

Remark 5.16. It is well known that the complex algebra of a group G is a relation algebra G^+ Jónsson (1982). The elements of G^+ are subsets of G, the relational product of two subsets A and B is the usual product AB of two subsets of a group, $A^{\smile}$ is given by A^{-1}, and $1'$ is $\{e\}$. As noted in Harding (1996), Remark 4.10 $(G^+)^{(1)}$ consists of all subgroups of G and $(G^+)^{(2)}$ consists of all pairs of subgroups (A, B) with $A \cap B$ equal to $\{e\}$ and

AB equal to G. It follows directly from Theorem 4.9 that $(G^+)^{(2)}$ is a regular orthomodular poset.

Remark 5.17. Mushtari (1989) demonstrated a method to construct an orthomodular poset from a bounded modular lattice M. Let $M^{(2)}$ be the collection of all ordered pairs of complementary elements of M, i.e., pairs (x, x') with $x \cdot x' = 0$ and $x + x' = 1$. Define a relation $\sqsubseteq$ on $M^{(2)}$ by setting $(x, x') \sqsubseteq (y, y')$ if $x \leq y$ and $y' \leq x'$. Define a unary operation $\perp$ on $M^{(2)}$ by setting $(x, x')^{\perp}$ to be (x', x), and define constants $\underline{0} = (0, 1)$ and $\underline{1} = (1, 0)$. With these operations, $M^{(2)}$ is an orthomodular poset. As shown in Harding (1996), Theorem 4.12, for each modular lattice M, there is a relation algebra R_M with $M^{(2)}$ equal to $(R_M)^{(2)}$. By Theorem 4.9 we have that $M^{(2)}$ is a regular orthomodular poset.

Definition 5.18. Let G be a group with operators in the sense of van der Waerden (1949, p. 138), i.e., G is a group together with a family $\mathcal{F}$ of endomorphisms of G. A map $\|\cdot\|$ from G to the positive reals is called a norm if (i) $\|x\| = 0$ iff $x = 0$, (ii) $\|x\| = \|-x\|$, and (iii) $\|x+y\| \leq \|x\| + \|y\|$. If $(G_i, \|\cdot\|_i)$, $i = 1, \ldots n$, is a family of normed groups with operators of the same type, we define their product to be the product group together with a norm $\|\cdot\|$ defined by $\|(g_1, \ldots, g_n)\|^2 = \Sigma_1^n \|g_i\|_i^2$. Given a normed group with operators G, we define Fact G to be all those (α, α') in $RG^{(2)}$ such that G/α and G/α' can be equipped with the structure of normed groups with operators making G structurally isomorphic to their product.

Lemma 5.19. Let $G_1, \ldots, G_n$ be groups with operators of the same type and $\|\cdot\|$ be a norm on their product G. Then there are norms on the G_i making $(G, \|\cdot\|)$ equal to the product if and only if for all $(g_1, \ldots, g_n) \in G$

$$\|(g_1, \ldots, g_n)\|^2 = \sum_{i=1}^{n} \|(0, \ldots, 0, g_i, 0, \ldots, 0)\|^2 \tag{5.4}$$

If such norms exist, they are uniquely determined by $\|g_i\|_i = \|(0, \ldots, 0, g_i, 0, \ldots, 0)\|$.

Proof. Suppose such norms $\|\cdot\|_i$ do exist. Then by the definition of the product of normed groups,

$$\|(0, \ldots, 0, g_i, 0, \ldots, 0)\|^2 = \|g_i\|_i^2 + \sum_{j \neq i} \|0\|_j^2$$

Therefore $\|g_i\|_i = \|(0, \ldots, 0, g_i, 0, \ldots, 0)\|$, and as $\|(g_1, \ldots, g_n)\|^2 = \Sigma_i^n \|g_i\|_i^2$, condition (5.4) follows directly. Conversely, define $\|\cdot\|_i$ by $\|g_i\|_i = \|(0, \ldots, 0, g_i, 0, \ldots, 0)\|$. Then $\|\cdot\|_i$ is a norm on G_i and (5.4) implies $\|(g_1, \ldots, g_n)\|^2 = \Sigma_1^n \|g_i\|_i^2$. ∎

In particular, this lemma shows that Fact G completely determines the decompositions of the normed group with operators G up to isomorphisms.

Proposition 5.20. Let G be a normed group with operators. Then Fact G is a compatible subalgebra of $RG^{(2)}$ and hence is a regular orthomodular poset.

Proof. We rely on Proposition 5.1. Surely Fact G is closed under orthocomplementation. Suppose (α_1, α'^1) and (α_1, α_2') are elements of Fact G which are compatible in $RG^{(2)}$. Then by Theorem 5.10 there exist groups with operator structures on $G/\alpha_1; \alpha_2$, etc., so that as groups with operators

$$G \cong G/\alpha_1; \alpha_2 \times G/\alpha_1; \alpha_2' \times G/\alpha_1'; \alpha_2 \times G/\alpha_1'; \alpha_2'$$

Let $G_1, \ldots, G_4$ denote the factors $G/\alpha_1; \alpha_2$, etc., and G' denote their product. Then as G is isomorphic to G' as a group with operators, there is a norm $\|\cdot\|$ on G' making this an isomorphism of normed groups with operators. Our assumption that (α_1, α_1') is in Fact G, together with Lemma 5.19, gives that for all $(g_1, g_2, g_3, g_4) \in G'$

$$\|(g_1, g_2, g_3, g_4)\|^2 = \|(g_1, g_2, 0,0)\|^2 + \|(0, 0, g_3, g_4)\|^2$$

And similarly, as (α_2, α_2') is in Fact G, we have for all $(g_1, g_2, g_3, g_4) \in G'$

$$\|(g_1, g_2, g_3, g_4)\|^2 = \|(g_1, 0, g_3, 0)\|^2 + \|(0, g_2, 0, g_4)\|^2$$

Applying the second identity to both terms on the right of the first yields that $\|(g_1, g_2, g_3, g_4)\|^2$ is equal to

$$\|(g_1, 0, 0, 0)\|^2 + \|(0, g_2, 0, 0)\|^2 + \|(0, 0, g_3, 0)\|^2 + \|(0, 0, 0, g_4)\|^2$$

So by Lemma 5.19, there exist norms $\|\cdot\|_i$ on G_i making the normed group with operators G isomorphic to the product. It follows that $(\alpha_1 \cdot \alpha_2, \alpha_1'; \alpha_2')$ is in Fact G as required. ■

Proposition 5.21. Let G be a normed group with operators. Then the restrictions of Φ and Ψ are mutually inverse isomorphisms between the sets BooFin (Fact G) and SDecompFin (G).

Proof. We first show that the restriction of Φ is a mapping into SDecompFin (G). Assume B is a finite Boolean subalgebra of Fact G with coatoms (α_i, α_i'), $i = 1, \ldots, n$. Let G_i denote G/α_i and G' denote the product $G_1 \times \cdots \times G_n$. Theorem 5.10 provides the existence of group with operator structures on the G_i making the product G' isomorphic to G as a group with operators. Therefore, there is a norm $\|\cdot\|$ on G' making G isomorphic to G' as a normed group with operators. We need to show that there are norms $\|\cdot\|_i$ on the G_i so that $(G', \|\cdot\|)$ is equal to the product. Our assumption that (α_i, α_i') is in Fact G, together with Lemma 5.19, gives

$$\|(g_1, \ldots, g_n)\|^2 = \|(0, \ldots, 0, g_i, 0, \ldots, 0)\|^2 + \|(g_1, \ldots, g_{i-1}, 0, g_{i+1}, \ldots, g_n)\|^2$$

for all $i = 1, \ldots, n$. Then condition (5.4) of Lemma 5.19 follows easily, establishing the existence of the required norms on the G_i.

Showing that the restriction of Ψ is a mapping into BooFin (Fact G) follows from compatibility, as in the proof of Theorem 5.10. ■

Any vector space is a group with operators. Here the operators are the unary operations of multiplication by a scalar, one such operation for each scalar. Clearly then any normed vector space, and in particular any inner product space, is a normed group with operators. Recall that a subspace S of an inner product space is a *splitting* subspace if $S + S^{\perp}$ is equal to the entire space. For Hilbert spaces every closed subspace is a splitting subspace, but this is not true of inner product spaces in general. It has long been known that the splitting subspaces of an inner product space form an orthomodular poset (Dvurečenskij, 1993). As we shall see, they form a regular orthomodular poset.

Proposition 5.22. Let E be an inner product space. Then the orthomodular poset $(S(E), \subseteq, 0, E, \perp)$ of splitting subspaces of E is isomorphic to Fact $(E, \|\cdot\|)$.

Proof. For any vector space V there is a bijection between Fact V and the collection of all ordered pairs of subspaces of V which intersect trivially and together span V. To each such ordered pair of subspaces (A, B) we associate the ordered pair of congruences (α_A, α_B), where $\alpha_C = \{(u, v): u - v \in C\}$ for any subspace C of V. Note that $V/\alpha_A \cong B$ and $V/\alpha_B \cong A$.

We claim that for an ordered pair of subspaces (A, B) of E, (α_A, α_B) is in Fact $(E, \|\cdot\|)$ if and only if A is a splitting subspace and $B = A^{\perp}$. From this it will follow that the map $S \rightsquigarrow (\alpha_S, \alpha_{S^{\perp}})$ is an isomorphism from $(S(E), \subseteq, 0, E, \perp)$ to Fact $(E, \|\cdot\|)$.

Suppose A is a splitting subspace of E. We must show that there are norms on E/α_A and $E/\alpha_{A^{\perp}}$ making $(E, \|\cdot\|)$ isomorphic to the product. But this follows easily as $E/\alpha_A \cong A^{\perp}$ and $E/\alpha_{A^{\perp}} \cong A$ and for any $a \in A$ and $a' \in A^{\perp}$ we have,

$$\|a + a'\|^2 = (a + a')\cdot(a + a') = a\cdot a + a'\cdot a' = \|a\|^2 + \|a'\|^2$$

Conversely, assume that there are norms on E/α_A and E/α_B making $(E, \|\cdot\|)$ isomorphic to the product. This implies that there are norms on A and B making $(a, b) \rightsquigarrow a + b$ an isomorphism from the normed group $A \times B$ to $(E, \|\cdot\|)$, and these norms on A and B can be none other than the restrictions of $\|\cdot\|$. Thus $\|a + b\|^2 = \|a\|^2 + \|b\|^2$ for all $a \in A$ and $b \in B$. We must show that $B = A^{\perp}$. But the subspace lattice of E is modular, and we know that A and B are complements in this lattice, so it is sufficient to show that $B \subseteq A^{\perp}$. Let $a \in A$ and $b \in B$. Then as $\|a + b\|^2 = \|a\|^2 + \|b\|^2$ we have

$$(a + b)\cdot(a + b) = a\cdot a + a\cdot b + b\cdot a + b\cdot b = a\cdot a + b\cdot b$$

Therefore

$$a\cdot b + b\cdot a = 0 \qquad \text{for all} \quad a \in A \quad \text{and} \quad b \in B$$

If we are working with a real inner product space, we conclude directly that $a\cdot b = 0$ for all $a \in A$ and $b \in B$, and hence $B \subseteq A^{\perp}$. If our inner product space is complex, the above shows that $a\cdot b$ is purely imaginary for all $a \in A$ and $b \in B$. But A is closed under scalar multiplication, so $(ia)\cdot b = i(a\cdot b)$ is also purely imaginary. Thus $a\cdot b = 0$ for all $a \in A$ and $b \in B$ in the complex case as well. ■

6. INFINITE BOOLEAN SUBALGEBRAS

In Section 3 we established a bijective correspondence between Boolean subsystems of a relation algebra R and Boolean subalgebras of the orthomodular poset $R^{(2)}$. This was used to establish a correspondence between the finite Boolean subalgebras of $RX^{(2)}$ and equivalence classes of finite direct decompositions of the set X. In Section 5 we extended this result to relate the finite Boolean subalgebras of Fact X to equivalence classes of finitary decompositions of X for any type of algebraic, relational, or topological structure. But we have avoided any mention of infinite Boolean subalgebras. The obvious generalization of the above results to infinite Boolean subalgebras will not work — a nonatomic Boolean subalgebra of $RX^{(2)}$ will certainly not yield an infinite direct product decomposition of X. The key is to replace the notion of an infinite direct product with the more general notion of a continuously varying product, i.e., a sheaf. We begin by briefly reviewing the fundamentals of sheaves and describing the notation we will use.

A sheaf is a triple $\tilde{S} = (S, Z, \pi)$ where S and Z are topological spaces, and π: $S \to Z$ is a local homeomorphism. This means that every point in S has an open neighborhood which is mapped homeomorphically onto an open subset of Z. For each $z \in Z$, the set $\pi^{-1}[\{z\}]$ is called the stalk over z and is denoted S_z. For any $U \subseteq Z$, a section of $\tilde{S}$ over U is a map f: $U \to S$ which is continuous with respect to the subspace topology on U and satisfies $\pi \circ f = id_U$. A section over Z is called a global section. We use $\Gamma_U\tilde{S}$ to denote the collection of all sections over U, and $\Gamma\tilde{S}$ for the collection of all global sections. We use $|_U$ to denote the natural restriction map from $\Gamma\tilde{S}$ to $\Gamma_U\tilde{S}$. For further background on sheaves consult Swan (1964).

Next, we describe an important method for constructing sheaves due essentially to Pierce (1967; see also Arens and Kaplansky, 1949; Burris and Sankappanavar, 1981; Comer, 1971; Davey, 1973; Macintyre, 1973). Let X be a set and B be a Boolean subsystem of RX. We use βB to denote the Stone space of B, i.e., the collection of all prime ideals P of B topologized by taking $\{\beta\alpha: \alpha \in B\}$ as a basis, where $\beta\alpha = \{P: \alpha \in P\}$. For each $P \in \beta B$, we have in particular that P is an up-directed family of equivalence relations on X. So $\cup P$ is an equivalence relation on X which we denote by α_P. We then use S_P to denote S/α_P and S for the disjoint union of the sets S_P. Finally, as a topology on S we take the smallest topology containing all sets of the form $\{x/\alpha_P: P \in K\}$, where $x \in X$ and K is a clopen subset of βB. Then $\tilde{S} = (S, \beta B, \pi)$ is a sheaf, which we call the Pierce sheaf of X over B. The map $x \rightsquigarrow \underset{\sim}{x}$, where $\underset{\sim}{x}(P) = x/\alpha_P$ for all $P \in \beta B$, is an isomorphism from X to $\Gamma\tilde{S}$. Further, for $x, y \in X$ and $\alpha \in B$, we have that $\underset{\sim}{x}$ and $\underset{\sim}{y}$ agree on the clopen subset $\beta\alpha$ of βB if and only if $(x, y) \in \alpha$. All the above results are well known. A proof which uses notation similar to the above can be found in (Harding, 1993).

Definition 6.1. We say $(\varphi; S, Z, \pi)$ is a Boolean sheaf representation of a set X if (i) Z is a Boolean space, (ii) $\tilde{S} = (S, Z, \pi)$ is a sheaf, (iii) $\varphi: X \to \Gamma\tilde{S}$ is an isomorphism, and (iv) for each nonempty clopen subset $K \subseteq Z$, there is some $z \in K$ with the stalk S_z having more than one element. We say that two Boolean sheaf representations $(\varphi; S, Z, \pi)$ and $(\varphi'; S', Z', \pi')$ are equivalent if there is a homeomorphism $\sigma: Z \to Z'$ and an isomorphism $\phi: S \to S'$ such that (i) ϕ maps the stalk S_z isomorphically to $S'_{\sigma z}$, and (ii) $\phi \circ \varphi(x) \circ \sigma^{-1} = \varphi'(x)$ for each $x \in X$. We then define BooSh (X) to be the collection of all equivalence classes of Boolean sheaf representations of X. We also define Boo $(RX^{(2)})$ to be the collection of all Boolean subalgebras of the orthomodular poset $RX^{(2)}$.

There are many ways to formulate the definition of equivalence, and we have chosen the one most convenient for our purposes. In the following lemma we shall show that every element of the sheaf space S is in the range of some global section. Using this, it is easy to see that the definition given above is equivalent to the existence of a homeomorphism $\sigma: Z \to Z'$ such that $\varphi(x)(z) = \varphi'(x)(\sigma(z))$ for all $x \in X$ and $z \in Z$. The map ϕ in the above definition, then, is uniquely determined. In fact, one can show that if such a map ϕ does exist, then not only is it an isomorphism, but a homeomorphism as well. But we will not need these facts, and work only with the definition we have given above.

Lemma 6.2. Let $(\varphi; S, Z, \pi)$ be a Boolean sheaf representation of X and let T denote $\{\ker(|_K \circ \varphi): K \subseteq Z \text{ clopen}\}$.

(i) If f, g are global sections and $K \subseteq Z$ is clopen, then there exists a global section h which agrees with f on K and agrees with g on $\neg K$.

(ii) Each element of S is in the range of some global section.

(iii) If $(\varphi; S, Z, \pi)$ is equivalent to $(\varphi'; S', Z', \pi')$ via $\sigma: Z \to Z'$ and $\phi: S \to S'$, then $\ker(|_K \circ \varphi) = \ker(|_{\sigma[K]} \circ \varphi')$.

(iv) T is a Boolean subsystem of RX and $K \rightsquigarrow \ker(|_{\neg K} \circ \varphi)$ is a Boolean algebra isomorphism.

(v) $z \rightsquigarrow \{\ker(|_K \circ \varphi): z \in K \text{ clopen}\}$ is a homeomorphism from Z to βT.

(vi) For all $x, y \in X$ and $z \in Z$ we have $\varphi(x)(z) = \varphi(y)(z)$ iff $(x, y) \in \ker(|_K \circ \varphi)$ for some clopen neighborhood K of z.

Proof. (i) Trivial.

(ii) Suppose $s \in S$. Choose an open neighborhood A of s on which π is a homeomorphism. Then there is a unique local section whose range is exactly A. If necessary, we can restrict the domain of this section to produce a local section with clopen domain whose range contains s. Using a compactness argument, and an idea similar to the one expressed in part (i), we can produce a global section whose range contains s.

(iii) As ϕ is an isomorphism, $\varphi(x)(z) = \varphi(y)(z)$ iff $\phi(\varphi(x)(z)) = \phi(\varphi(y)(z))$. But using equivalence, $\phi(\varphi(x)(z)) = \phi(\varphi(y)(z))$ iff $\varphi'(x)(\sigma(z)) = \varphi'(y)(\sigma(z))$.

(iv) For a clopen subset K of Z, let $\theta(K)$ denote the relation $\ker(|_K \circ \varphi)$. Note first that $\theta(K \cup M) = \theta(K) \cap \theta(M)$. This provides a certain monotonicity that shows $\theta(K)$ and $\theta(M)$ are contained in $\theta(K \cap M)$, and as these are all equivalence relations, $\theta(K) ; \theta(M) \subseteq \theta(K \cap M)$. Suppose that $(x, y) \in \theta(K \cap M)$. Then $\varphi(x)$ and $\varphi(y)$ agree on $K \cap M$. Define h to agree with $\varphi(x)$ on K and with $\varphi(y)$ on $\neg K$. As K is clopen, h is continuous, and therefore is equal to $\varphi(w)$ for some $w \in X$. Then $(x, w) \in \theta(K)$ and $(w, y) \in \theta(M)$, showing that $\theta(K \cap M) = \theta(K); \theta(M)$. Therefore T is closed under finite intersections and relational products. It follows that T is a lattice under set inclusion with meets given by intersections and joins given by relational products. Then from the above descriptions of $\theta(K \cap M)$ and $\theta(K \cup M)$, it follows that the map $K \rightsquigarrow \theta(\neg K)$ is a lattice homomorphism onto T. As Clopen Z is a Boolean algebra, T is also Boolean. Therefore T is a Boolean subsystem of RX.

To show that the map $K \rightsquigarrow \theta(\neg K)$ is an isomorphism, it is sufficient to show that if $K \neq \emptyset$, then $\theta(\neg K)$ is nontrivial. As $(\varphi; S, Z, \pi)$ is a Boolean sheaf representation, by definition there is some stalk over K with at least two elements. Therefore there are global sections which differ at some point in K, and these global sections may be chosen to agree on $\neg K$. Therefore $\theta(\neg K)$ is nontrivial.

(v) It is well known that the map $z \rightsquigarrow \{\neg K: z \in K\}$ is a homeomorphism from Z to the Stone space of Clopen Z. As $K \rightsquigarrow \theta(\neg K)$ is an isomorphism of Boolean algebras, it follows that $z \rightsquigarrow \{\theta(K): z \in K\}$ is a homeomorphism from Z to βT.

(vi) Surely if $(x, y) \in \ker(|_K \circ \varphi)$ for some clopen neighborhood K of z, then $\varphi(x)$ and $\varphi(y)$ agree at the point z. Conversely, if $\varphi(x)$ and $\varphi(y)$ agree at z, choose an open neighborhood A of $\varphi(x)(z)$ on which π is a homeomorphism. Then $\varphi(x)^{-1}[A] \cap \varphi(y)^{-1}[A]$ is an open neighborhood of z and hence contains a clopen neighborhood K of z. As π is one to one on A, it follows that $\varphi(x)$ and $\varphi(y)$ agree on K. ■

Theorem 6.3. Let X be a set. Then there are mutually inverse isomorphisms Φ and Ψ between Boo $(RX^{(2)})$ and BooSh (X) defined as follows:

(i) $\Phi(B)$ is the equivalence class of the Pierce sheaf representation of X over the Boolean subsystem $\{\alpha: (\alpha, \alpha') \in B\}$.
(ii) $\Psi([\varphi: X \to \Gamma\tilde{S}])$ is $\{(\ker(|_K \circ \varphi), \ker(|_{\neg K} \circ \varphi)): K \subseteq Z \text{ clopen}\}$.

Proof. Let $(\varphi; S, Z, \pi)$ be a Boolean sheaf representation of X and let T denote $\{\ker(|_K \circ \varphi): K \subseteq Z \text{ clopen}\}$. Part (iii) of Lemma 6.2 shows that the value of Ψ is independent of the particular representative of the equivalence class chosen, hence Ψ is well defined. Part (iv) of Lemma 6.2 gives that T is a Boolean subsystem of RX, and clearly $\ker(|_{\neg K} \circ \varphi)$ is the complement of $\ker(|_K \circ \varphi)$ in T. It follows from Proposition 3.5 that $\{(\ker(|_K \circ \varphi), \ker(|_{\neg K} \circ \varphi)): K \subseteq Z \text{ clopen}\}$ is a Boolean subalgebra of $RX^{(2)}$. Therefore Ψ is a well-defined map into Boo $(RX^{(2)})$, and our earlier discussion gives reference to the fact that Φ is a well-defined map into BooSh (X).

Next, suppose B is a Boolean subalgebra of $RX^{(2)}$. Let $C = \{\alpha: (\alpha, \alpha') \in B\}$, and let $(\sim; S, \beta C, \pi)$ be the Pierce sheaf representation of X over C. Then each clopen subset of βC is of the form $\beta\alpha$ for some $\alpha \in C$. In our earlier discussion of the Pierce sheaf we gave reference to the fact that $\tilde{x}$ agrees with $\tilde{y}$ on $\beta\alpha$ iff $(x, y) \in \alpha$. This shows that $\alpha = \ker(|_{\beta\alpha} \circ \sim)$. It follows that $\Psi \circ \Phi(B) = B$, and hence $\Psi \circ \Phi$ is the identity.

Now we establish that $\Phi \circ \Psi$ is the identity. Let $(\varphi; S, Z, \pi)$ be a Boolean sheaf representation of X and let $T = \{\ker(|_K \circ \varphi): K \subseteq Z \text{ clopen}\}$. Then $\Phi \circ \Psi$ applied to the equivalence class of this sheaf representation is the equivalence class of the Pierce sheaf representation $(\sim; S', \beta T, \pi')$ of X over T. Part (v) of Lemma 6.2 provides a homeomorphism $\sigma: Z \to \beta T$. As every element of S is in the range of some global section $\varphi(x)$, part (vi) of Lemma 6.2 shows we can define a map $\phi: S \to S'$ by setting $\phi(\varphi(x)(z)) = \tilde{x}(\sigma(z))$. Indeed, if $\varphi(x)(z)$ is equal to $\varphi(y)(z)$, then $(x, y) \in \ker(|_K \circ \varphi)$ for some clopen neighborhood K of z. But this relation $\ker(|_K \circ \varphi)$ is an element of the prime ideal $\sigma(z)$, so $\tilde{x}(\sigma(z))$, equals $\tilde{y}(\sigma(z))$, showing that ϕ is well defined. Reversing this argument shows that ϕ is one to one. Indeed, if $\phi(\varphi(x)(z))$ equals $\phi(\varphi(y)(z))$, then $(x, y) \in \ker(|_K \circ \varphi)$ for some clopen neighborhood K of z. In particular, $\varphi(x)(z) = \varphi(y)(z)$. Clearly this map ϕ takes the stalk S_z to the stalk $S'_{\sigma z}$, and as every element of S' is in the range of some global section x, ϕ is an isomorphism from S_z to $S'_{\sigma z}$. By definition, $\phi \circ \phi(x) = \tilde{x} \circ \sigma$ for each $x \in X$. Therefore $(\varphi; S, Z, \pi)$ is equivalent to $(\sim; S', \beta T, \pi')$, showing that $\Phi \circ \Psi$ is the identity. ■

As finite direct product decompositions correspond to Boolean sheaf representations over finite, hence discrete, Boolean spaces, Theorem 6.3 can be viewed as an extension of Proposition 3.8.

Definition 6.4. A sheaf of algebras of type $(n_i)_I$ is a sheaf $\tilde{S} = (S, Z, \pi)$ where each stalk S_z has a family of operations $(f_i^z)_I$. Each map f_i^z must have arity n_i, and we require that the natural map f_i: $\cup_Z S_z^{n_i} \to S$ be continuous with respect to the subspace topology on $\cup_Z S_z^{n_i}$ inherited from S^{n_i}. It is well known that the global sections $\Gamma\tilde{S}$ of a sheaf of algebras form a subalgebra of the product of the stalks.

A Boolean sheaf representation of an algebra X is a Boolean sheaf representation φ: $X \to \Gamma\tilde{S}$ of the set X where the stalks of $\tilde{S}$ can be equipped with operations making $\tilde{S}$ a sheaf of algebras of the same type of X and φ a structure-preserving isomorphism. If such operations on the stalks do exist, it is easily seen that they must make the natural projection from X to S_z a homomorphism, and hence they are uniquely determined.

Let X be an algebra and φ: $X \to \Gamma\tilde{S}$ and φ': $X \to \Gamma\tilde{S}'$ be equivalent Boolean sheaf representations of the set X. It is easily seen that if one of these is a Boolean sheaf representation of the algebra X, then so is the other. We may then define SBooSh (X) to be all equivalence classes of Boolean sheaf representations of the algebra X. Note that SBooSh (X) is a subset of BooSh (X). We also define Boo (Fact X) to be the collection of all Boolean subalgebras of Fact X. Clearly Boo (Fact X) is a subset of Boo $(RX^{(2)})$.

Proposition 6.5. Let X be an algebra. Then the restrictions of Φ and Ψ are mutually inverse isomorphisms between Boo (Fact X) and SBooSH (X).

Proof. We need only show that the restrictions of Φ and Ψ are maps between Boo (Fact X) and SBooSh (X). If B is a Boolean subalgebra of Fact X, then $B' = \{\alpha\colon (\alpha, \alpha') \in B\}$ is a Boolean sublattice of the congruence lattice of X consisting of pairwise permuting congruences. It is well known that the Pierce sheaf representation of X over B' is a representation of the algebra X. A complete proof of this using notation similar to ours is given in Harding (1993). So Φ is a map into SBooSH (X). Also, if φ: $X \to \Gamma\tilde{S}$ is a Boolean sheaf representation of the algebra X over a Boolean space Z, then

$\ker(\mathrm{l}_K \circ \varphi)$ is a congruence of X for each $K \subseteq Z$. Therefore Φ is a map into Boo (Fact X). ■

Definition 6.6. A sheaf of binary relational structures is a sheaf $\tilde{S} = (S, Z, \pi)$ where each stalk S_z is equipped with a nonempty binary relation R_z so that $\cup\{R_z: z \in Z\}$ is an open subset of S^2. For sheaves over Boolean spaces, this condition on the relations R_z can be stated in a more usable form. If f, g are global sections and $f(z)R_z g(z)$, we require that there be a clopen neighborhood K of z such that $f(v)R_v g(v)$ for all $v \in K$. Given such a sheaf of relational structures, the global sections ΓS also are equipped with a binary relation $\tilde{R}$. Here $f\,\tilde{R}g$ iff $f(z)\ R_z g(z)$ for all $z \in Z$.

A Boolean sheaf representation of a relational structure X is a Boolean sheaf representation φ: $X \to \Gamma\tilde{S}$ of the set X where the stalks of $\tilde{S}$ can be equipped with binary relations making $\tilde{S}$ a sheaf of relational structures and φ a structure-preserving isomorphism. If such relations do exist, they are uniquely determined as $\varphi(x)(z)R_z\varphi(y)(z)$ iff there are $x'Ry'$ so that $\varphi(x)$ agrees with $\varphi(x')$ and $\varphi(y)$ agrees with $\varphi(y')$ on some clopen neighborhood of z. If we have a pair of equivalent Boolean sheaf representations of the set X, and one is a representation of the structure X, then both are representations of the structure X. So we may define SBooSh (X) to be all equivalence classes of Boolean sheaf representations of the structure X. Note that SBooSh (X) is a subset of BooSh (X). Finally, we let Boo (Fact X) denote the collection of all Booleaan subalgebras of Fact X. Note that Boo (Fact X) is a subset of Boo ($RX^{(2)}$).

Proposition 6.7. Let X be a relational structure. Then the restrictions of Φ and Ψ are mutually inverse isomorphisms between Boo (Fact X) and SBooSh (X).

Proof. We need only show that the restrictions are maps between these sets.

Let B be a Boolean subalgebra of Fact X and $B' = \{\alpha: (\alpha, \alpha') \in B\}$. Define relations on the stalks of the Pierce sheaf $\tilde{S} = (S, \beta B', \pi)$ of X over B' by setting $R_P = \{(\tilde{x}(P), \tilde{y}(P)): (x, y) \in R\}$. Surely these relations make $\tilde{S}$ a sheaf of relational structures. To show that $\sim$ is a structural isomorphism, we must show that xRy iff $\tilde{x}(P)R_p\tilde{y}(P)$ for all $P \in \beta B'$. That the first condition implies the second follows directly from the definition of the relations R_p. Conversely, assume the second condition. Then for each P there is $(x_p, y_p) \in R$ with $\tilde{x}(P) = \tilde{x}_p(P)$ and $\tilde{y}(P) = \tilde{y}_p(P)$. But $\tilde{x}(P) = \tilde{x}_p(P)$ implies that $\tilde{x}$ and $\tilde{x}_p$ agree on a clopen neighborhood of P, and similarly for $\tilde{y}$ and $\tilde{y}_p$. Using compactness, we can find a pairwise disjoint clopen cover $\beta\alpha_1, \ldots, \beta\alpha_n$ of $\beta B'$ and elements $(x_i, y_i) \in R$ such that (x_i, x) and (y_i, y) are in α_i for all $i = 1, \ldots, n$. Then $\alpha_1, \ldots, \alpha_n$ are the coatoms of a Boolean subalgebra of

B. So by theorem 5.10, $X \to X/\alpha_1 \times \cdots \times X/\alpha_n$ is a decomposition of the relational structure X. Therefore xRy, showing that $\sim$ is a structural isomorphism. So Φ is a mapping into SBooSh (X).

Suppose φ: $X \to \Gamma\tilde{S}$ is a Boolean sheaf representation of the structure X over a Boolean space Z, and let K be a clopen subset of Z. Setting $\alpha =$ ker($|_K \circ \varphi$) and $\alpha' = X/\alpha$ is canonically isomorphic to $\Gamma_K\tilde{S}$ and X/α' is canonically isomorphic to $\Gamma_{\neg K}\tilde{S}$. Therefore X/α and X/α' can be equipped with relations making X structurally isomorphic to their product. So (α, α') $\in$ Fact X, and therefore Ψ is a mapping into Boo (Fact X). ■

Remark 6.8. While we have succesfully characterized the Boolean subalgebras of Fact X for any algebraic or relational structure X, we do not have such results for topological structures, or structures involving such analytical features as a norm.

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