# Algebraic Aspects of Orthomodular Lattices

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In this paper we try to give an up-to-date account of certain aspects of the theory of ortholattices (abbreviated OLs), orthomodular lattices (abbreviated OMLs) and modular ortholattices (abbreviated MOLs), not hiding our own research interests.

Since most of the questions we deal with have their origin in Universal Algebra, we start with a chapter discussing the basic concepts and results of Universal Algebra without proofs. In the next three chapters we discuss, mostly with proofs, the basic results and standard techniques of the theory of OMLs. In the remaining five chapters we work our way to the border of present day research, mostly no or only sketchy proofs. Chapter 5 deals with products and subdirect products, chapter 6 with free structures and chapter 7 with classes of OLs defined by equations. In chapter 8 we discuss embeddings of OLs into complete ones. The last chapter deals with questions originating in Category Theory, mainly amalgamation, epimorphisms and monomorphisms.

The later chapters of this paper contain an abundance of open problems. We hope that this will initiate further research.

## 1 Basic Universal Algebra

We assume a definition of natural numbers which makes every natural number $n$ the set of all natural numbers $k < n$. An $n$-ary operation on a set $A$ is a map $f$ of $A^n$ into $A$. An element $a \in A^n$ gives rise to a sequence $(a(0), a(1), \ldots, a(n-1))$ of $A$, more commonly written with indices $(a_0, a_1, \ldots, a_{n-1})$. The number $n$ is called the arity of the operation. It is important to allow the case $n = 0$. Since $A^0 = \{\emptyset\}$ a 0-ary or nullary operation is a map of $\{\emptyset\}$ into $A$. Since such a map is completely determined by the element $f(\emptyset)$ we usually specify 0-ary operations, also called constants, by giving the element $f(\emptyset)$. A similar remark applies to unary (1-ary) operations. We "identify" them with maps from $A$ to itself.

A closure system over a set $A$ is a set $\mathcal{C}$ of subsets of $A$ satisfying

If $\mathcal{B} \subseteq \mathcal{C}$ then $\bigcap \mathcal{B} \in \mathcal{C}$.

Here we make the convention that the intersection of the empty subset of $\mathcal{C}$ is $A$, so that $A \in \mathcal{C}$. If $\mathcal{C}$ is a closure system over $A$ and if $X \subseteq A$ we define the closure

$\Gamma X$ of $X$ with respect to the closure system $\mathcal{C}$ by $\Gamma X = \bigcap\{C | X \subseteq C \in \mathcal{C}\}$. $\Gamma X$ is obviously the smallest element of $\mathcal{C}$ containing $X$ as a subset. If for $\mathcal{B} \subseteq \mathcal{C}$ we define $\bigvee \mathcal{B} = \Gamma(\bigcup \mathcal{B})$ and $\bigwedge \mathcal{B} = \bigcap \mathcal{B}$ then $\mathcal{C}$ becomes a complete lattice. $\Gamma\emptyset$ is the smallest and $A$ is the largest element of this lattice.

An algebraic closure system is a closure system $\mathcal{C}$ satisfying

If $\mathcal{K} \subseteq \mathcal{C}$ is a chain, i.e. totally ordered by $\subseteq$, then $\bigcup \mathcal{K} \in \mathcal{C}$.

This definition, one of many equivalent ones, is particularly useful in proofs involving Zorn's lemma.

An algebra is a pair $\mathcal{A} = (A, (f_i)_{i \in I})$, where $A$ is a set, called the underlying set, or the universe of the algebra, and each $f_i$ is for some $n_i$, an $n_i$-ary operation on $A$. The family $(n_i)_{i \in I}$ is called the type of the algebra. Whereas every $n_i$ is finite it is necessary, in order to cover important examples, to admit infinitely many operations $f_i$.

A subuniverse of an algebra $\mathcal{A} = (A, (f_i)_{i \in I})$ is a set $B \subseteq A$ which is closed under all operations $f_i$, i.e. satisfies

If $a \in B^{n_i}$ then $f_i(a) \in B$,

or, in more common and more cumbersome notation

If $a_0, a_1, \ldots, a_{n_i - 1} \in B$ then $f_i(a_0, a_1, \ldots, a_{n_i - 1}) \in B$.

Let $Sub(\mathcal{A})$ be the set of all subuniverses of $\mathcal{A}$. The most important and in fact characteristic property is:

**Proposition 1.1** *If $\mathcal{A}$ is an algebra then $Sub(\mathcal{A})$ is an algebraic closure system.*

As a consequence of this we obtain that for any $X \subseteq A$ there exists a smallest subuniverse $\Gamma X$ containing $X$. $\Gamma X$ is said to be the subuniverse generated by $X$.

A subalgebra of an algebra $\mathcal{A} = (A, (f_i)_{i \in I})$ is an algebra $\mathcal{B} = (B, (g_i)_{i \in I})$ such that $B \subseteq A$ and $g_i$ is the restriction of $f_i$ to $B^{n_i}$, i.e.

If $a \in B^{n_i}$ then $g_i(a) = f_i(a)$.

Clearly this requires that $B$ is a subuniverse of $\mathcal{A}$. Conversely, if $B$ is a subuniverse of $\mathcal{A}$ and $g_i$ is the restriction of $f_i$ to $B^{n_i}$ then $(B, (g_i)_{i \in I})$ becomes a subalgebra of $\mathcal{A}$. Because of this one-one correspondence between subuniverses and subalgebras we will not be too fussy about distinguishing them. Thus, we refer to $Sub(\mathcal{A})$ as the closure system or lattice of subalgebras of $\mathcal{A}$. Many authors require the definition of algebra and subalgebra that the underlying set be not empty. We see no reason for this.

Let $\mathcal{A} = (A, (f_i)_{i \in I})$ and $\mathcal{B} = (B, (g_i)_{i \in I})$ be algebras of the same type. A homomorphism of $\mathcal{A}$ into $\mathcal{B}$ is a map $\varphi : A \to B$ satisfying for every $i \in I$ and $a \in A^{n_i}$

$$\varphi(f_i(a)) = g_i(\varphi \circ a)$$

or, in cumbersome notation,

$$\varphi(f_i(a_0, a_1, \ldots, a_{n_i-1})) = g_i(\varphi(a_0), \varphi(a_1), \ldots, \varphi(a_{n_1-1})).$$

$\mathcal{B}$ is said to be a homomorphic image of $\mathcal{A}$ iff there exists a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$. An embedding of $\mathcal{A}$ into $\mathcal{B}$ is a one-one homomorphism of $\mathcal{A}$ into $\mathcal{B}$. An isomorphism is an embedding which is onto.

Closely related to homomorphisms are congruence relations. A congruence relation, short: congruence, on an algebra $\mathcal{A} = (A, (f_i)_{i \in I})$ is an equivalence relation on $A$, i.e. a reflexive, symmetric and transitive relation $R$ satisfying

If $a, b \in A^{n_i}$ and $a(k)Rb(k)(0 \leq k < n_i)$ then $f_i(a)Rf_i(b)$.

We express this by saying that $R$ is compatible with the operation $f_i$. If $R$ is an equivalence relation on $A$ and $a \in A$ define the equivalence (congruence if $R$ is a congruence) class of $a$ modulo $R$ by $a/R = \{b | aRb\}$ and the quotient set of $A$ modulo $R$ by $A/R = \{a/R | a \in A\}$.

If $R$ is a congruence in $\mathcal{A}$ we may define operations $g_i$ in $A/R$. If $a \in A^{n_i}$ define $a/R \in (A/R)^{n_i}$ by $(a/R)(k) = a(k)/R$. Then define $g_i$ by

if $a \in A^{n_i}$ then $g_i(a/R) = f_i(a)/R$,

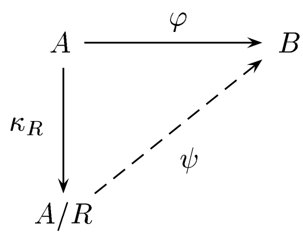
in cumbersome notation

$$g_i(a_0/R, a_1/R, \ldots, a_{n_i-1}/R) = f_i(a_0, a_1, \ldots, a_{n_i-1})/R.$$

With these operations $(A/R, (g_i)_{i \in I})$ becomes an algebra of the same type as $\mathcal{A}$ the so called quotient algebra $\mathcal{A}/R$ of $\mathcal{A}$ modulo $R$. The map $\kappa_R : A \to A/R$ defined by $\kappa_R(a) = a/R$ becomes a homomorphism of $\mathcal{A}$ onto $\mathcal{A}/R$, called the canonical homomorphism of $\mathcal{A}$ onto $\mathcal{A}/R$.

We are now in a position to establish the basic relationship between homomorphisms and congruences, known under various names.

**Homomorphism Theorem** *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of $\mathcal{A}$ into $\mathcal{B}$. Define a relation $R = \ker(\varphi)$ (the kernel of $\varphi$) by $aRb$ iff $\varphi(a) = \varphi(b)$. Then $R$ is a congruence in $\mathcal{A}$ and there exists a unique map $\psi : A/R \to B$ such that $\psi \circ \kappa_R = \varphi$. This map $\psi$ is a one-one homomorphism of $\mathcal{A}/R$ into $\mathcal{B}$.*



If $\varphi$ is onto then clearly $\psi$ is onto, hence an isomorphism. Thus every homomorphic image of $\mathcal{A}$ is isomorphic with a quotient algebra of $\mathcal{A}$.

It is of some importance that the set $Con(\mathcal{A})$ of all congruences in $\mathcal{A}$ is an algebraic closure system over $A \times A$.

Let $(A_k)_{k\in K}$ be a family of sets. The product $\prod_{k\in K} A_k$ of the family consists of all choice functions on the family, i.e. all maps $\alpha$ with domain $K$ such that $\alpha(k) \in A_k$ for all $k \in K$. With every product $\prod_{k\in K} A_k$ there come maps $pr_k$ of the product onto $A_k$ defined by $pr_k(\alpha) = \alpha(k)$. The map $pr_k$ is called the $k$-th projection. If the $\mathcal{A}_k = (A_k, (f_i^k)_{i\in I})$ are algebras we define operations $f_i$ in the product componentwise, i.e. by $f_i(\alpha)(k) = f_i^k(pr_k \circ \alpha)$. The resulting algebra $\prod_{k\in K} \mathcal{A}_k$ is called the product of the family $(\mathcal{A}_k)_{k\in K}$. In case of a product of algebras, the $pr_k$ are homomorphisms.

The cartesian product $A \times B$ of two sets is defined in basic set theory as $\{(a,b)|a \in A, b \in B\}$. If $\mathcal{A} = (A,(f_i)_{i\in I})$, $\mathcal{B} = (B,(g_i)_{i\in I})$ we may define operations $h_i$ in $A \times B$ by $h_i(a) = (f_i(pr_A(a)), g_i(pr_B(a)))$. This gives a new algebra $\mathcal{A} \times \mathcal{B}$. This can be subsumed under the product defined before. We leave out the details.

An important property of products which, in more general settings, can be used to define products is the

**Extension Property of Products** *If $\mathcal{A}$, $\mathcal{A}_k$ $(k \in K)$ are algebras and for every $k \in K$, $\varphi_k$ is a homomorphism of $\mathcal{A}$ into $\mathcal{A}_k$ then there exists a unique homomorphism of $\mathcal{A}$ into $\prod_{k\in K} \mathcal{A}_k$ satisfying $pr_k \circ \varphi = \varphi_k$ for every $k$.*

A subdirect product of a family $(\mathcal{A}_k)_{k\in K}$ of algebras is a subalgebra $\mathcal{A}$ of the product $\prod_{k\in K} \mathcal{A}_k$ such that every $pr_k$ maps $\mathcal{A}$ onto $\mathcal{A}_k$. A subdirect representation of an algebra $\mathcal{A}$ is an embedding $\varphi$ of $\mathcal{A}$ into a product $\prod_{k\in K} \mathcal{A}_k$ such that the maps $pr_k \circ \varphi$ map $\mathcal{A}$ onto $\mathcal{A}_k$. An algebra $\mathcal{A}$ is said to be subdirectly irreducible iff for every subdirect representation $\varphi : \mathcal{A} \to \prod_{k\in K} \mathcal{A}_k$ one of the maps $pr_k \circ \varphi$ is one-one, hence an isomorphism. The fundamental result about subdirectly irreducibles is

**Birkhoff's Subdirect Representation Theorem** *Every algebra is isomorphic with a subdirect product of subdirectly irreducible algebras.*

Subdirect irreducibility of an algebra has a neat and very useful characterization in terms of congruences. The smallest congruence of an algebra $\mathcal{A} = (A, (f_i)_{i\in I})$ is obviously the diagonal $\Delta_A = \{(a,a) | a \in A\}$. An algebra $\mathcal{A} = (A, (f_i)_{i\in I})$ is subdirectly irreducible iff there is a smallest congruence different from $\Delta_A$.

Let $\mathcal{K}$ be a class of algebras of the same type. We define $H(\mathcal{K})$ to be the class of all homomorphic images of algebras in $\mathcal{K}$, $S(\mathcal{K})$ the class of all subalgebras of algebras in $\mathcal{K}$, and $P(\mathcal{K})$ the class of all products of algebras in $\mathcal{K}$. An equational class or variety of algebras is a class $\mathcal{K}$ with $H(\mathcal{K}) \subseteq \mathcal{K}$, $S(\mathcal{K}) \subseteq \mathcal{K}$ and $P(\mathcal{K}) \subseteq \mathcal{K}$. If $\mathcal{K}$ is any class of algebras of the same type then $HSP(\mathcal{K})$ is the smallest variety containing $\mathcal{K}$.

In our general considerations so far we have distinguished corresponding operations in algebras of the same type by giving them the same index. This is practically never done in concrete cases. In these cases one identifies corresponding operations by denoting them by the same symbol. For illustration let us consider the case of groups. In order to get subgroups as a special case of our general notion of subalgebras we have to consider a group as an algebra with the group multiplication as a binary operation, the forming of inverses as a unary operation and the unit as a constant. In order to make this fit our general development so far we would have to introduce operations (say) $f_0, f_1, f_2$. Instead of doing this we introduce special symbols $\cdot$ for the group multiplication, $^{-1}$ for the forming of inverses and $e$ for the unit. We then say a group is an algebra $(G, (\cdot, ^{-1}, e))$ of type $(2,1,0)$, indicating the arities of the operations in the given order.

# 2 The basics of ortholattices and orthomodular lattices

We assume that the reader is familiar with the basics of lattice theory, it's description as a partially ordered set and it's representation by Hasse diagrams.

A bounded lattice is an algebra $(L, (\vee, \wedge, 0, 1))$ where $(L, (\vee, \wedge))$ is a lattice, $0$ is a lower bound of $L$ and $1$ is an upper bound of $L$. An orthocomplementation on a bounded lattice is a unary operation $'$ satisfying

$$a \vee a' = 1 \quad a \wedge a' = 0$$

$$a \leq b \Rightarrow b' \leq a'$$

$$a'' = a.$$

An easy consequence of this are the DeMorgan laws

$$(a \vee b)' = a' \wedge b', \quad (a \wedge b)' = a' \vee b'.$$

Each of these can replace the second condition in the description, thus defining an orthocomplementation by equations. An ortholattice (abbreviated: OL) is an algebra $(L, (\vee, \wedge, ', 0, 1))$ where $(L, (\vee, \wedge, 0, 1))$ is a bounded lattice and $'$ is an orthocomplementation on it. If one interchanges the binary operations $\vee, \wedge$ and the constants $0, 1$ one again obtains an OL, called the dual of the original OL. If the operations are clear we speak simply of the OL $L$.

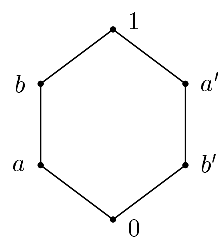
An orthomodular lattice (abbreviated: OML) is an OL satisfying the orthomodular law

if $a \leq b$ then $a \vee (a' \wedge b) = b$.

This law can again be relaced by the equation

$$a \vee (a' \wedge (a \vee b)) = a \vee b.$$

The prime example of an OL which is not orthomodular is the benzene ring (see diagram).




In fact we have

**Proposition 2.1** *Let $L$ be an OL. These are equivalent*

**(1)** *$L$ is an OML*

**(2)** *if $a \leq b$ and $a' \wedge b = 0$ then $a = b$*

**(3)** *the benzene ring is not a subalgebra of $L$.*

The proof of this is straightforward.

A fundamental concept in OLs is commutativity. An element $a$ is said to commute with an element $b$, in symbols $aCb$, if $a = (a \wedge b) \vee (a \wedge b')$. Clearly $a \leq b$ implies $aCb$ and $aCb$ implies $aCb'$. The commutator $\gamma(a, b)$ of elements $a, b$ of an OL is defined by $\gamma(a, b) = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$.

**Proposition 2.2** *In an OL $L$ the following are equivalent.*

**(1)** *$L$ is an OML*

**(2)** $aCb \Rightarrow bCa$

**(3)** $aCb \Rightarrow a'Cb$

**(4)** $aCb \Rightarrow a \vee (a' \wedge b) = a \vee b$

**(5)** $aCb \Leftrightarrow \gamma(a, b) = 0.$

**Proof.** $1 \Rightarrow 5$. Assume $aCb$. Then $\gamma(a,b) = (a' \vee b) \wedge (a \vee b) \wedge (a' \vee b') \wedge (a \vee b') = (a' \vee b) \wedge ((a \wedge b) \vee (a \wedge b') \vee b) \wedge (a' \vee b') \wedge ((a \wedge b) \vee (a \wedge b') \vee b') = (a' \vee b) \wedge ((a \wedge b') \vee b) \wedge (a' \vee b') \wedge ((a \wedge b) \vee b') = b \wedge b' = 0$, using the dual of the orthomodular law. Assume conversely that $\gamma(a, b) = 0$. Clearly $a \geq (a \wedge b) \vee (a \wedge b')$. But $a \wedge (a' \vee b') \wedge (a' \vee b) \leq \gamma(a, b) = 0$ which gives $aCb$ by condition 2 of (2.1).

$5 \Rightarrow 2$. $aCb \Rightarrow \gamma(a, b) = 0 \Rightarrow \gamma(b, a) = 0 \Rightarrow bCa$.

$2 \Rightarrow 3$. $aCb \Rightarrow bCa \Rightarrow bCa' \Rightarrow a'Cb$.

$3 \Rightarrow 4$. $a \vee (a' \wedge b) \leq a \vee b \Rightarrow a \vee (a' \wedge b) C a \vee b \Rightarrow a' \wedge (a \vee b') C a \vee b \Rightarrow a' \wedge (a \vee b') = (a' \wedge (a \vee b') \wedge (a \vee b)) \vee (a' \wedge (a \vee b') \wedge a' \wedge b') = (a' \wedge ((a' \wedge b) \vee (a' \wedge b'))') \vee (a' \wedge b') = (a' \wedge a) \vee (a' \wedge b') = a' \wedge b'$. Thus $a \vee (a' \wedge b) = a \vee b$.

$4 \Rightarrow 1$. Obvious since $a \leq b$ implies $aCb$. ∎

In an OML $aCb$ holds iff $a \vee (a' \wedge b) = a \vee b$. If this last equation holds we have $(a \vee b) \wedge (a \vee b') = (a \vee (a' \wedge b)) \wedge (a \vee b') = a$, hence $aCb$.

**Proposition 2.3** *Let $L$ be an OML, $a, x_i \in L$ $(i \in I)$. If $aCx_i$ $(i \in I)$ and if $\bigvee_I x_i$ exists then $\bigvee_{i \in I}(a \wedge x_i)$ exists and $a \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I}(a \wedge x_i)$, and dually.*

**Proof.** Clearly $a \wedge \bigvee_I x_i$ is an upper bound of $\{a \wedge x_i | i \in I\}$. Let $v$ be any upper bound of this set and put $u = v \wedge a \wedge \bigvee_{i \in I} x_i$. Then $a \wedge x_j \leq u \leq a \wedge \bigvee_{i \in I} x_i$ and $u' \wedge a \wedge \bigvee_{i \in I} x_i \leq (a' \vee x_j') \wedge a \wedge \bigvee_I x_i = a \wedge x_j' \wedge \bigvee_{i \in I} x_i$ $(j \in I)$, hence $u' \wedge a \wedge \bigvee_{i \in I} x_i \leq a \wedge \bigwedge_{i \in I} x_i' \wedge \bigvee_{i \in I} x_i = 0$, hence $u = a \wedge \bigvee_{i \in I} x_i$ by 2 of (2.1). It follows that $a \wedge \bigvee_{i \in I} x_i$ is the least upper bound of $\{a \wedge x_i | i \in I\}$. ∎

If $a$ is an element of an OML $L$ we define $C(a) = \{x \in L | aCx\}$. If $A \subseteq L$ we define $C(A) = \{x \in L | aCx$ holds for all $a \in A\}$.

**Proposition 2.4** *If $a$ is an element of an OML $L$, if $x_i \in C(a)$ $(i \in I)$ and if $\bigvee_{i \in I} x_i$ exists then $\bigvee_{i \in I} x_i \in C(a)$, and dually. In particular $C(a)$ is a subalgebra of $L$.*

**Proof.** We have $(a \wedge \bigvee_{i \in I} x_i) \vee (a' \wedge \bigvee_{i \in I} x_i) = \bigvee_{i \in I}((a \wedge x_i) \vee (a' \wedge x_i)) = \bigvee_{i \in I} x_i$, hence $\bigvee_{i \in I} x_i Ca$, hence $\bigvee_{i \in I} x_i \in C(a)$. ∎

An element $c$ of an OML is said to be central if it commutes with all elements of $L$, and the set of all central elements $C(L)$ is called the centre of $L$.

For elements $a \leq b$ in an OL $L$ we define $[a, b] = \{x | a \leq x \leq b\}$ and speak of the interval $[a, b]$.

**Proposition 2.5** *Let* $a \leq b$ *be central in an OML* $L$. *(1)* $([a,b],(\vee,\wedge,*,a,b))$ *is an OL where* $*$ *is defined by* $x^* = a \vee (b \wedge x')$. *(2) The map* $f : L \to [a,b]$ *defined by* $f(x) = a \vee (b \wedge x)$ *is an onto homomorphism.*

**Proof.** (1) Let $a \leq x \leq b$. Then $x \vee x^* = x \vee a \vee (b \wedge x') = x \vee (b \wedge x')$ and as $x \leq b$ orthomodularity gives $x \vee x^* = b$. As $xCa, b \wedge x'$ (2.3) yields $x \wedge x^* = x \wedge (a \vee (b \wedge x')) = (x \wedge a) \vee (x \wedge b \wedge x') = a$. Note $x^{**} = a \vee (b \wedge (a \vee (b \wedge x'))') = a \vee (b \wedge a' \wedge (b' \vee x))$. Then as $b \geq x$ the dual of the orthomodular law gives $x^{**} = a \vee (a' \wedge x)$ and as $a \leq x$ the orthomodular law gives $x^{**} = x$. Therefore $*$ is an orthocomplementation. (2) Obviously $f(x') = x^*$. In view of the DeMorgan laws it suffices to show $f(x \vee y) = f(x) \vee f(y)$. As $b$ is central (2.3) provides $f(x \vee y) = a \vee (b \wedge (x \vee y)) = a \vee (b \wedge x) \vee (b \wedge y) = (a \vee (b \wedge x)) \vee (a \vee (b \wedge y)) = f(x) \vee f(y)$. Therefore $f$ is a homomorphism and trivially onto. ∎

**Corollary 2.6** *Every interval* $[a,b]$ *in an OML* $L$ *is an OML under the orthocomplementation* $x^* = a \vee (b \wedge x')$. *Further, this OML is a homomorphic image of a subalgebra of* $L$.

**Proof.** Suppose $[a,b]$ is an interval in $L$. Consider the subalgebra $S$ of $L$ consisting of all elements which commute with both $a, b$. Then $S$ contains the interval $[a,b]$ and $a, b$ are central in the OML $S$. From the previous proposition, $[a,b]$ is an OL under the orthocomplementation $x^* = a \vee (b \wedge x')$, and this OL is a homomorphic image of the OML $S$, hence is orthomodular. ∎

**Lemma 2.7** *If* $c$ *is central in an OML* $L$, *then* $f : L \to [0,c] \times [0,c']$ *defined by* $f(x) = (x \wedge c, x \wedge c')$ *is an isomorphism. Conversely, if* $f : L \to A \times B$ *is an isomorphism, then there is a central element* $c$ *in* $L$ *with* $A \cong [0,c]$ *and* $B \cong [0,c']$.

**Proof.** Assume $c$ is central in $L$. By (2.5) the map $f : L \to [0,c] \times [0,c']$ is a homomorphism. As $(x \wedge c) \vee (x \wedge c') = x$, $f$ is one-one. For $x \leq c$, $y \leq c'$ commutativity gives $f(x \vee y) = (x,y)$, hence $f$ is onto. For the converse, take $c$ in $L$ with $f(c) = (1,0)$. ∎

The most important tool for computations in OMLs is the

**Foulis-Holland Theorem** *If one of the elements* $a, b, c$ *of an OML commutes with the other two then the sublattice (not subalgebra) generated by* $\{a,b,c\}$ *is distributive.*

**Proof.** Assume $a, bCc$. Then $c$ is central in $\Gamma\{a,b,c\}$ and hence the map $x \mapsto (x \wedge c, x \wedge c')$ is an isomorphism of $\Gamma\{a,b,c\}$ with the product $[0,c] \times [0,c']$. Clearly $\{a \wedge c, b \wedge c, c\}$ generate a distributive sublattice of $[0,c]$ and $\{a \wedge c', b \wedge c', c', 0\}$ generate a distributive sublattice of $[0,c']$. The sublattice generated by $\{a,b,c\}$ is clearly isomorphic with a sublattice of the product of these two sublattices. ∎

**Proposition 2.8** *Let* $X$ *be a subset of an OML* $L$. *The subalgebra* $\Gamma X$ *generated by* $X$ *is Boolean iff any two elements of* $X$ *commute.*

**Proof.** Clearly in a Boolean algebra any two elements commute. Assume that any two elements of $X$ commute. Then $X \subseteq C(X)$, hence $\Gamma X \subseteq C(X)$, hence $X \subseteq C(\Gamma X)$, hence $\Gamma X \subseteq C(\Gamma X)$. Thus any two elements of $\Gamma X$ commute, which implies that $\Gamma X$ is Boolean by the Foulis-Holland Theorem. ∎

# 3 Subalgebras

We begin with Boolean subalgebras. Note that every element $x$ of an OL $L$ is contained in a Boolean subalgebra of $L$, namely the subalgebra $\{0, x, x', 1\}$. Also, as the union of a chain of Boolean subalgebras is again a Boolean subalgebra, a direct application of Zorn's lemma yields that each Boolean subalgebra of $L$ is contained in a maximal Boolean subalgebra of $L$, also termed a block of $L$. Therefore

**Proposition 3.1** *Every OL is the union of its blocks.*

Suppose $L$ is an OL. If $L$ is an OML then for any $a \leq b$ in $L$ we have $aCb$, hence by (2.8) $\Gamma\{a, b\}$ is Boolean, so $a, b$ are elements of some block of $L$. Conversely, if $L$ is not an OML then by (2.1) there are $a \leq b$ in $L$ which generate a benzene ring, hence are not elements of any Boolean subalgebra of $L$. Therefore

**Proposition 3.2** *An OL $L$ is an OML iff the partial ordering on $L$ is the union of the partial ordering on the blocks of $L$.*

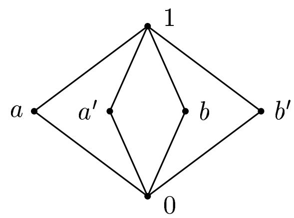
This shows that each OML is determined by its blocks. We remark that there are a number of results, such as Greechie's Loop Lemma [25], Dichtl's astroids [20], and Kalmbach's bundle lemma describing conditions on a family of Boolean algebras $(B_i)_I$ which ensure their union is an OML. See [32] for a complete account. We next collect a few basic and well known properties of blocks, all of which are easily proved.

**Proposition 3.3** *Let $L$ be an OML. (1) The centre $C(L)$ is the intersection of the blocks of $L$. (2) If $M$ is a subalgebra of $L$, then the blocks of $M$ are exactly $\{B \cap M | B$ is a block of $L\}$. (3) If $(L_i)_I$ is a family of OMLs, then the blocks of $\prod_I L_i$ are $\{\prod_I B_i | B_i$ is a block of $L_i\}$.*

For OMLs $L, M$ and a homomorphism $\varphi : L \to M$ the image $\varphi[B]$ of any block of $L$ is obviously a Boolean subalgebra of $M$, but need not be a block (consider the identical embedding of 2 into $2^2$). We are not aware of such an example with the homomorphism $\varphi$ onto.

An important class of OLs are those of height two, i.e. in which the maximal number of elements in a chain is three. Clearly such an OL is determined by specifying its cardinality, which, if finite, must be some even number at least four. Define $MO_k$ for $\kappa \geq 1$ to be the OL of height two having cardinality $2 \times \kappa + 2$. Extend this by setting $MO_0$ to be a two element Boolean algebra. Clearly each

$MO_\kappa$ is a modular ortholattice (abbreviated: MOL). A diagram of $MO_2$ is given below.




It is a simple, but useful observation that for any elements $x, y$ in an OL $L$, $\Gamma\{x, y\}$ is $MO_2$ iff $\gamma(x, y) = 1$ and $L$ is non-trivial. This yields

**Lemma 3.4** *Let $L$ be an OML generated by the elements $x, y$. Then $[0, \gamma(x, y)]$ is either trivial or $MO_2$, $[0, \gamma(x, y)']$ is Boolean, and $L \cong [0, \gamma(x, y)] \times [0, \gamma(x, y)']$.*

**Proof.** As $\gamma(x, y)$ commutes with both $x, y$, it is central in $L$. So there are homomorphisms $f : L \to [0, \gamma(x, y)]$ and $g : L \to [0, \gamma(x, y)']$ defined by setting $f(z) = z \wedge \gamma(x, y)$ and $g(z) = z \wedge \gamma(x, y)'$. Note $\gamma(f(x), f(y)) = f(\gamma(x, y))$ is the unit of $[0, \gamma(x, y)]$ so this interval is either $MO_2$ or trivial. Similarly $\gamma(g(x), g(y)) = 0$ so $g(x)$ commutes with $g(y)$ and by (2.8) the interval $[0, \gamma(x, y)']$ is Boolean. ∎

As any finitely generated Boolean algebra is finite we then have

**Corollary 3.5** *For $x, y$ elements of an OML, $\Gamma\{x, y\}$ is finite.*

An as an OML is Boolean iff all elements commute (2.8) we have

**Corollary 3.6** *If $L$ is a non-Boolean OML, then $MO_2$ is a homomorphic image of a subalgebra of $L$.*

A variety $V$ is called locally finite if for every $A \in V$ and every finite subset $S \subseteq A$ the subalgebra $\Gamma S$ generated by $S$ is finite. The first of these two corollaries may raise some false expectations—none of the varieties of MOLs, OMLs, or OLs is locally finite. To see this consider the MOL $L$ of all subspaces of a three dimensional vector space over the reals with orthocomplementation being given by orthogonal subspaces. One can easily find three elements of $L$ that generate an infinite subalgebra.

There is a useful weakening of the notion of locally finite.

**Definition 3.7** *A variety $V$ has the finite embedding property (f.e.p.) if for every $A \in V$ and every finite $S \subseteq A$ there is a finite $B \in V$ and a one-one set mapping $\varphi : S \to B$ such that for each basic operation $f_i$ and each $s_0, \ldots, s_{n_i - 1} \in S$, if $f_i(s_0, \ldots, s_{n_i-1}) \in S$ then $\varphi(f_i(s_0, \ldots, s_{n_i-1})) = f_i(\varphi(s_0), \ldots, \varphi(s_{n_i-1}))$.*

It follows from consideration of MacNeille completions of orthocomplemented posets (section 8) that the variety of OLs has the f.e.p., and it follows from results on finite MOLs (section 7) that the variety of MOLs does not have f.e.p. There is a natural hope that one can combine the fact that Boolean algebras are locally finite with various techniques to produce OMLs by "gluing" together Boolean algebras to show that the variety of OMLs has the f.e.p. Such attempts have not so far been successful. Due to the connection to the word problem, the following is one of the basic outstanding problems in the theory of OMLs.

**Problem.** Does the variety of OMLs have the f.e.p.Γ

# 4 Congruences and homomorphisms

Given an OL $(L, (\vee, \wedge, ', 0, 1))$ one calls the bounded lattice $(L, (\vee, \wedge, 0, 1))$ the bounded lattice reduct of the OL. Clearly any homomorphism between two OLs is also a homomorphism between their bounded lattice reducts. For Boolean algebras $(B, (\vee, \wedge, ', 0, 1))$ and $(C, (\vee, \wedge, ', 0, 1))$ it is well known that a map $f : B \to C$ is a homomorphism between the Boolean algebras iff it is a homomorphism between their bounded lattice reducts. But it is a simple matter to find an automorphism of the bounded lattice reduct of $MO_2$ that is not an automorphism of $MO_2$. However,

**Proposition 4.1** *If $L$ is an OML then every congruence on the bounded lattice reduct of $L$ is a congruence on $L$.*

**Proof.** Let $R$ be a congruence on the lattice reduct of $L$. We must show that $aRb$ implies $a'Rb'$. We first show this in the special case that $a \leq b$. In this case, orthomodularity gives $a' = b' \vee (b \wedge a')$. So $a' = b' \vee (b \wedge a')Rb' \vee (a \wedge a') = b'$. In the general case $aRb$ implies $(a \wedge b)R(a \vee b)$, hence $(a \wedge b)'R(a \vee b)'$, and as $a', b'$ lie between $(a \wedge b)'$ and $(a \vee b)'$ the result follows. ∎

We next examine more closely the structure of congruences on an OML.

**Lemma 4.2** *Let $R$ be a congruence on an OML $L$. For $a, b \in L$ these are equivalent. (1) $aRb$, (2) $(a \vee b) \wedge (a' \vee b')R0$, (3) $a \vee x = b \vee x$ for some $x$ with $xR0$.*

**Proof.** Assuming (1) $(a \vee b) \wedge (a' \vee b')R(a \vee a) \wedge (a' \vee a') = 0$. Assuming (2) set $x = (a \vee b) \wedge (a' \vee b')$. Assuming (3) $a = (a \vee 0)R(a \vee x) = (b \vee x)R(b \vee 0) = b$. ∎

An algebra $A$ is called congruence regular if every congruence on $A$ is determined by any one of its equivalence classes. In other words, $A$ is congruence regular if for any $a$ in $A$ and any congruences $R, S$ on $A$ we have $a/R = a/S$ implies that $R = S$. Groups and Boolean algebras are examples of congruence regular algebras, while distributive lattices are not. The following is a special case of a well known result that applies to any lattice with 0 where each interval $[0, x]$ is complemented.

**Proposition 4.3** *Every OML is congruence regular.*

**Proof.** Suppose $R, S$ are congruences on an OML $L$ and that $a/R = a/S$ for some $a$ in $L$. If $xR0$ then for $y$ a complement of $x$ in the interval $[0, a \vee x]$ we have that $xR0$ implies $yR(a \vee x)$, hence $yS(a \vee x)$, giving $xS0$. Thus $a/R = a/S$ implies $0/R = 0/S$. Suppose then that $cRd$. Let $e$ be a complement of $c$ in the interval $[0, c \vee d]$. As $cR(c \vee d)$ we have $eR0$, hence $eS0$, so $cS(c \vee d)$, and $cSd$. ∎

As is customary with congruence regular algebras, we often choose to work with a particular equivalence class of a congruence. For groups we choose the equivalence class containing the group identity, which is a normal subgroup of the group. For OMLs, we choose the equivalence class containing 0. The following formula for recovering a congruence from its 0 equivalence class is implicit in the previous proof.

**Proposition 4.4** *If $R$ is a congruence on an OML $L$, then $aRb$ iff $a \vee x = b \vee x$ for some $xR0$.*

An algebra $A$ is called congruence permutable if for any congruences $R, S$ on $A$ we have $R \circ S = S \circ R$. Note that groups, rings and Boolean algebras are congruence permutable while distributive lattices are not. The following is a special case of the well known result that any relatively complemented lattice is congruence permutable [18, pg. 93].

**Proposition 4.5** *Every OML is congruence permutable.*

Recall, for an OL $L$ the collection $Con(L)$ of all congruences on $L$ is an algebraic closure system over $L \times L$, hence forms complete lattice under set inclusion. Meets in this lattice are given by set intersection and upwardly directed joins are given by unions. If $L$ is an OML, then as $L$ is congruence permutable, binary joins in this lattice are given by relational product. The collection $Id(L)$ of all ideals of the lattice $L$ also forms a complete lattice under set inclusion. Meets in this lattice are given by set intersection. Upwardly directed joins are given by unions, and binary joins are given by $I \vee J = \{x | x \leq a \vee b \text{ for some } a \in I, b \in J\}$.

**Proposition 4.6** *For an OML $L$ the map $F : Con(L) \to Id(L)$ defined by $F(R) = 0/R$ is a bounded lattice embedding which preserves arbitrary joins and meets.*

**Proof.** For a congruence $R$ clearly $0/R$ is an ideal. As $L$ is congruence regular $F$ is one-one. As meets in both $Con(L)$ and $Id(L)$ are given by intersections, $F$ preserves arbitrary meets, and similarly $F$ preserves upwardly directed joins. It remains only to show that $F$ preserves binary joins. Let $R, S$ be congruences. Note that their join in $Con(L)$ is $R \circ S$. If $z$ belongs to $F(R \circ S)$ there is some $xR0$ and some $yS0$ with $x \vee y = x \vee z$, hence $z \leq x \vee y$. Conversely, if $xR0$ and $yS0$, then $(x \vee y)(R \circ S)0$ as $R \circ S$ is a congruence. So $z \leq x \vee y$ implies that $z(R \circ S)0$. This shows that $F(R \circ S) = F(R) \vee F(S)$. ∎

It is of interest to characterize those ideals that arise as the zero equivalence classes of congruences on an OML $L$, much the way we distinguish normal subgroups of a group.

**Proposition 4.7** *Let $I$ be an ideal of an OML $L$. Then $I$ is the zero equivalence class of some congruence on $L$ iff $x \in I$ and $y \in L$ implies $y \wedge (y' \vee x) \in I$.*

**Proof.** Obviously if $I = 0/R$ for some congruence $R$, then $x \in I$, $y \in L$ implies $y \wedge (y' \vee x) R y \wedge (y' \vee 0)$, so $y \wedge (y' \vee x)$ belongs to I. Conversely, assume that $I$ is closed under the given condition. Set $R = \{(a, b) | a \vee x = b \vee x \text{ for some } x \in I\}$. As 0 is in $I$ $R$ is reflexive. By the symmetry of the definition $R$ is symmetric. As $I$ is closed under finite joins it follows that $R$ is transitive, and further that $R$ is compatible with joins. It remains only to show that $R$ is compatible with orthocomplementation. Suppose $aRb$. Then $a \vee x = b \vee x$ for some $x$ in $I$. So $a' \wedge (a \vee x)$ and $b' \wedge (b \vee x)$ belong to $I$, hence $a' \wedge (a \vee b \vee x)$ and $b' \wedge (a \vee b \vee x)$ belong to $I$, and as $I$ is a downset $a' \wedge (a \vee b)$ and $b' \wedge (a \vee b)$ belong to $I$. As $I$ is closed under joins and $a \vee b$ commutes with both $a', b'$ we have $(a \vee b) \wedge (a' \vee b')$ belongs to $I$. But $a' \vee ((a \vee b) \wedge (a' \vee b')) = a' \vee b'$ and $b' \vee ((a \vee b) \wedge (a' \vee b')) = a' \vee b'$. Thus $a' R b'$. ∎

Among the most useful results about the congruence lattice of an OL follows below. This will open the door to such powerful techniques as Jónsson's Theorem [17, pg. 147].

**Proposition 4.8** *For an OL $L$, $Con(L)$ is distributive.*

**Proof.** It is well known that the congruence lattice of any lattice is distributive [18, pg. 75]. Our result then follows from the fact that the congruence lattice of an algebra $A$ is a sublattice of the congruence lattice of any reduct of $A$. ∎

To summarize, we have shown that OMLs are congruence regular, congruence permutable, and congruence distributive. It seems to be an open question to completely characterize those lattices which are isomorphic to the congruence lattice of some OML. As a final remark we note that matters are much worse in the absence of orthomodularity. Ortholattices are not in general congruence regular, or congruence permutable, but, as shown above, are congruence distributive.

# 5 Products, directly and sub-directly irreducibles

A congruence $R$ on an OML $L$ is called a factor congruence if $R$ has a complement in the congruence lattice of $L$. Note, as $Con(L)$ is distributive $R$ will then have exactly one complement, which we denote by $R'$. For readers familiar with the definition of factor congruences for general algebras [17, pg. 52] we recall that every OML is congruence permutable.

**Lemma 5.1** *If $R$ is a factor congruence on an OML $L$, then the natural map $f : L \to L/R \times L/R'$ is an isomorphism. Conversely, if $f : L \to A \times B$ is an isomorphism, then the kernels of $pr_1 \circ f$ and $pr_2 \circ f$ are complementary factor congruences.*

**Proof.** By general considerations $f$ is a homomorphism. If $f(x) = f(y)$, then $(x, y)$ belongs to both $R$ and $R'$, hence $x = y$. Given $x, y$ in $L$, the fact that $R, R'$ permute and join to the largest congruence of $L$ gives the existence of $z$ with $xRz$ and $zR'y$. Then $f(z) = (x/R, y/R')$. Therefore the map $f$ is one-one and onto. Conversely, if $R$ and $S$ are the kernels of the natural projections of $A \times B$ onto $A$ and $B$, then $R, S$ intersect to the identical relation on $A \times B$, and for any $(a, b)$, $(c, d)$ in $A \times B$, $(a, b)R(a, d)$ and $(a, d)S(c, d)$. Thus $R \circ S$ is the universal relation on $A \times B$. ∎

The exact nature of the correspondence between central elements and factor congruences on an orthomodular lattice $L$ is made precise by the following result. We leave the proof to the reader.

**Proposition 5.2** *The map $c \mapsto \{(x, y)|x \vee c = y \vee c\}$ is a lattice isomorphism between $C(L)$ and the Boolean subalgebra of $Con(L)$ of factor congruences.*

**Definition 5.3** *An ortholattice $L$ is called directly irreducible if for every isomorphism $f : L \to L_1 \times \cdots \times L_n$ there is an index $k$ so that the projection $pr_k \circ f : L \to L_k$ is an isomorphism.*

In view of the above remarks we have the following result.

**Proposition 5.4** *For an OML $L$, these are equivalent. (1) $L$ is directly irreducible, (2) $C(L)$ consists of exactly two elements, (3) $L$ has exactly two factor congruences.*

**Lemma 5.5** *If $R$ is a congruence on an OML $L$ and $0/R$ has a largest element $c$, then $c$ is central in $L$ and $R$ is a factor congruence.*

**Proof.** As $cR0$ it follows that $x \wedge (x' \vee c)$ belongs to $0/R$ for each $x$ in $L$. Hence $x \wedge (x' \vee c) \leq c$ so $x \wedge (x' \vee c) = x \wedge c$ for all $x$ in $L$, showing that $c$ is central. Therefore $c'$ is central and there is a congruence $R'$ with $0/R'$ equal to $[0, c']$. Then $R, R'$ are complements in the congruence lattice of $L$. ∎

Recall, an OL $L$ is subdirectly irreducible if it has a least non-zero congruence, and simple if it has exactly two congruences. Obviously any simple OL is subdirectly irreducible, and every subdirectly irreducible OL is directly irreducible. From the preceding lemma we have the following.

**Proposition 5.6** *A finite OML is directly irreducible iff it is simple. Therefore every finite OML is isomorphic to a finite direct product of simple OMLs.*

**Proof.** Every finite OML is isomorphic to a finite direct product of directly irreducible OMLs. ∎

This result can be easily generalized to hold for any OML in which all chains are finite. More generally, it is known to hold for any chain finite relatively complemented lattice [18, pg. 94]. The first step towards a different generalization of (5.6) is given by the following [27].

**Theorem 5.7** *The notions of directly irreducible and simple coincide in any variety $V$ generated by a class of OMLs with a finite upper bound on the lengths of their chains.*

It is hopeless to expect that each OML in a variety such as $V$ will be isomorphic to a direct product of simple algebras. In the Boolean case, this would amount to having each Boolean algebra $B$ isomorphic to a power $2^X$ for some set $X$, and by cardinality considerations alone this is impossible for any countable Boolean algebra. However, Stone's theorem provides that any Boolean algebra $B$ is isomorphic to the collection of all continuous functions in $2^X$ for some Boolean space $X$ (where 2 is given the discrete topology). We obtain a weaker, but useful, analogue of Stone's theorem.

**Theorem 5.8** *Let $L$ be in a variety $V$ generated by a class of OMLs with a finite upper bound on the lengths of their chains. Then there is a family of OMLs $L_x$ indexed by the elements $x$ of a Boolean space $X$, and a topology $\tau$ on $\bigcup\{L_x|x \in X\}$ such that (i) the subspace topology on each $L_x$ discrete, (ii) $L_x$ simple for all $x$ in a dense open subset of $X$, and (iii) $L \cong \{f \in \prod L_x | f \text{ is continuous}\}$.*

While this result might seem ungainly, there are effective tools for working with such a representation—one can essentially lift many first order properties from the $L_x$ to $L$. For further details of this result see [19, 28]. We remark that the reader familiar with the notion of discriminator varieties [17, pg. 165] will have seen representation theorems very similar to the one above. However,

**Proposition 5.9** *The only varieties of OMLs which are discriminator varieties are the trivial variety and the variety of Boolean algebras.*

**Proof.** Let $V$ be a non-Boolean variety of OMLs. Then $V$ contains $MO_2$ (3.6) which is simple and has a subalgebra which is not simple. So by [17, lemma 9.2] $V$ is not a discriminator variety. ∎

We mention a generalization of (5.6) in another direction [39]. It seems not unreasonable to hope that the following result might be extended to a variety generated by OMLs having at most $n$ blocks.

**Theorem 5.10** *An OML with finitely many blocks is directly irreducible iff it is simple. Therefore an OML with finitely many blocks is isomorphic to a finite direct product of a Boolean algebra and simple OMLs.*

Likely the most useful result concerning representations of OMLs by direct products remains Birkhoff's subdirect representation theorem which states that every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras. Unfortunately, when working with the full variety of OMLs, the subdirectly irreducibles are difficult to narrow down. In fact

**Proposition 5.11** *Every OML is a subalgebra of a simple, hence subdirectly irreducible, OML.*

**Proof.** Given an OML $L$, construct an OML $M$ by "gluing" $L$ and a four element Boolean at their bounds. ∎

Still, there are many varieties of OMLs where one has very good control over the subdirectly irreducibles. As ortholattices are congruence distributive (4.8), one may apply Jónsson's theorem [17, pg. 146] and Los' Theorem [17, pg. 210] to any variety $V$ generated by a class $\mathcal{K}$ of OMLs to gain insight into the first order properties of the subdirectly irreducibles in $V$. For example, if every member of $\mathcal{K}$ has at most $n$ elements in each of its chains, then the same is true of every subdirectly irreducible, and, in view of (5.7), of every directly irreducible member of $V$.

# 6 Free ortholattices

**Definition 6.1** *Given a class $\mathcal{K}$ of algebras of the same type, $F \in \mathcal{K}$ is $\mathcal{K}$-freely generated by a set $X$ if (i) $X \subseteq F$, (ii) $X$ generates $F$, and (iii) every set map $f : X \to A$ with $A \in \mathcal{K}$ extends uniquely to a homomorphism $\hat{f} : F \to A$.*

By a standard argument two algebras $\mathcal{K}$-freely generated by $X$ are isomorphic. We next show the existence of a $\mathcal{K}$-freely generated algebra over $X$ where $\mathcal{K}$ is the class of all algebras of a given type. Such algebras are called absolutely freely generated.

**Definition 6.2** *Given a set $X$ and a type $\tau = (n_i)_I$ let $\Sigma$ be the set of all finite strings of symbols from $X \cup I$. Define the set of terms of type $\tau$ over $X$ to be the smallest subset $S$ of $\Sigma$ such that (i) $X \subseteq S$, and (ii) if $i \in I$ and $p_0, \ldots, p_{n_i - 1} \in S$, then the string $ip_0 \cdots p_{n_1 - 1}$ is in $S$.*

We use $T(X)$ to denote the set of terms of type $\tau$ over $X$ and use the common convention of writing $f_i(p_0, \ldots, p_{n_1 - 1})$ in place of the string $ip_0 \cdots p_{n_i - 1}$. For each index $i \in I$ let $\bar{f}_i$ be the $n_i$-ary operation on $T(X)$ defined by setting $\bar{f}_i(p_0, \ldots, p_{n_i - 1}) = f_i(p_0, \ldots, p_{n_i - 1})$. Then $(T(X), (\bar{f}_i)_{i \in I})$ is an algebra of type $\tau$ called the term algebra of type $\tau$ over $X$. The following result is well known [17, pg. 66] and easily proved.

**Proposition 6.3** *The term algebra $T(X)$ is absolutely freely generated by $X$.*

Next we show the existence of $\mathcal{K}$-freely generated algebras over a set $X$, at least under mild assumptions on $\mathcal{K}$. For any set $X$ we define $\Theta_{\mathcal{K}}(X)$ to be the intersection of all congruences $\phi \in Con(T(X))$ such that $T(X)/\phi$ belongs to $IS(\mathcal{K})$.

**Theorem 6.4** *If $\mathcal{K}$ is closed under $I, S, P$, then the algebra $T(X)/\Theta_{\mathcal{K}}(X)$ is $\mathcal{K}$-freely generated by $X/\Theta_{\mathcal{K}}(X)$.*

Note, if $V$ is a variety containing an algebra with more than one element, one can easily show that $X/\Theta_V(X)$ is in bijective correspondence with $X$, and it follows that there is an algebra $V$-freely generated by $X$. We denote this (essentially unique) algebra by $F_V(X)$. For $V$ the variety of one element algebras we let $F_V(X)$ be a one element algebra. In either case there is an obvious homomorphism $\alpha : T(X) \to F_V(X)$. The reader should consult [17, pg. 66] for a proof of above result.

**Definition 6.5** *An equation, or identity, of type $\tau$ over $X$ is an ordered pair $(p, q)$ where $p, q \in T(X)$. An algebra $A$ satisfies the equation, written $A \models p \approx q$, if $f(p) = f(q)$ for every homomorphism $f : T(X) \to A$, and a class of algebras $\mathcal{K}$ satisfies the equation, written $\mathcal{K} \models p \approx q$, if $A \models p \approx q$ for each $A \in \mathcal{K}$.*

For example, the pair $(x \vee y, y')$ is an equation in the type of OLs over the set $X = \{x, y, z\}$. This equation will be valid in some algebras (in any one element algebra for instance), but is not valid in any non-trivial ortholattice. The following result is well known [17, pg. 73].

**Proposition 6.6** *For a variety $V$ and terms $p, q$ in $T(X)$ the following are equivalent (i) $V \models p \approx q$, (ii) $F_V(X) \models p \approx q$, (iii) $(p, q) \in \Theta_V(X)$, (iv) $\alpha(p) = \alpha(q)$.*

Recall, $\alpha : T(X) \to F_V(X)$ is the natural homomorphism.

**Definition 6.7** *A variety $V$ has a solvable free word problem over $X$ if there is an algorithm to determine for any terms $p, q$ in $T(X)$ whether $\alpha(p) = \alpha(q)$.*

In view of the above proposition, a solvable free word problem over $X$ gives an algorithm to determine whether an equation $p \approx q$ holds for all algebras in $V$.

**Theorem 6.8** *The variety of lattices has solvable free word problem over any set.*

**Proof.** While we do not provide a complete proof of this well known theorem [18, pg. 163], it is worthwhile to sketch its features. Define $\leq$ to be the smallest binary relation on $T(X)$ satisfying (i) $x \leq x$ for all $x$ in $X$, (ii) $a \leq c$ and $b \leq c$ implies $a \vee b \leq c$, (iii) $a \leq b$ and $a \leq c$ implies $a \leq b \wedge c$, (iv) $a \leq b$ or $a \leq c$ implies $a \leq b \vee c$, and (v) $a \leq c$ or $b \leq c$ implies $a \wedge b \leq c$. One can show that $\leq$ is a quasi-order on $T(X)$. Setting $\phi$ to be the usual equivalence relation associated with a quasi-order, one then shows $T(X)/\phi$ is freely generated in the variety of lattices by $X/\phi$. As $\leq$ can be effectively computed, the word free word problem for lattices is solvable. ∎

**Theorem 6.9** *The variety of OLs has solvable free word problem over any set.*

**Proof.** Again, the reader is directed to [8] for a complete proof, but we sketch the details. Given a set $X$, take another set $X'$ in bijective correspondence with $X$ and disjoint from $X$. Consider the term algebra $T(X \cup X')$ of the type of lattices and define the relation $\leq$ on $T(X \cup X')$ as above. As $T(X \cup X')$ is absolutely free, the obvious map $' : X \cup X' \to X \cup X'$ extends to a homomorphism from $T(X \cup X')$ to its dual. Define $R$ to be the smallest subset of $T(X \cup X')$ satisfying (i) $X \cup X'$ is contained in $R$, (ii) $a, b \in R$ and $a', b' \not\leq a \vee b$ implies $a \vee b \in R$, and (iii) $a, b \in R$ and $a \wedge b \not\leq a', b'$ implies $a \wedge b \in R$. One can show that "adding" a top and bottom element to $R/\phi$ yields an ortholattice freely generated by $X/\phi$. ∎

Various useful results about free lattices and free ortholattices are collected in the following. Here Whitman's condition refers to the property that $a \wedge b \leq c \vee d$ iff one of $a \wedge b \leq c$, $a \wedge b \leq d$, $a \leq c \vee d$, $b \leq c \vee d$.

**Proposition 6.10** *(1) Every free lattice satisfies Whitman's condition. (2) A lattice freely generated by a three element set contains a sublattice freely generated by a countable set. (3) Every free ortholattice satisfies Whitman's condition. (4) An ortholattice freely generated by a two element set contains a subalgebra freely generated by a countable set.*

The first two statements can be found in [18, pg. 166]. The third is easily seen from the above construction of free ortholattices. The fourth is found in [8]. Another very useful fact, easily proved along the lines of (3.4) is the following.

**Proposition 6.11** $MO_2 \times 2^4$ *is freely generated by a two element set in the variety of OMLs and the variety of MOLs. Therefore the free word problem on two generators is solvable in the variety of OMLs and the variety of MOLs.*

This gives an extremely simple procedure to determine if an equation involving only two variables is valid in every OML—one simply checks to see if it is valid in $MO_2$. See [37] for a discussion of how this simple observation could greatly simplify many proofs in the literature. For more than two generators the situation is nearly completely open. Some of the few known facts are collected below.

**Proposition 6.12** *If $X$ has at least three elements, then an OML freely generated by $X$ contains a free lattice on countably many generators as a sublattice of its lattice reduct.*

**Proof.** Kalmbach [26, 31] has shown that any lattice $L$ can be embedded into the lattice reduct of some OML $K(L)$. If $L$ is a lattice freely generated by $X$, then there is an OL homomorphism from the free OML $F$ on $X$ onto $K(L)$. So there is a lattice homomorphism from the lattice reduct of $F$ onto $K(L)$, and as $L$ is projective in the variety of lattices (see section 9) $L$ is isomorphic to a sublattice of the lattice reduct of $F$. The result then follows as $L$ contains a sublattice freely generated by a countable set. ∎

**Proposition 6.13** *Let $X$ be a set with at least three elements and let $L$ be freely generated by $X$ in the variety of OMLs or MOLs. (1) $L$ contains an infinite chain and has infinitely many blocks. (2) $L$ does not contain an uncountable chain.*

**Proof.** (1) The previous result shows a free orthomodular lattice over $X$ has an infinite chain. There is an example in [14] of a 3-generated MOL with infinite chains and infinitely many blocks. This provides the other assertions in this claim. (2) As noticed by several authors, this is generally true of free algebras in any variety of algebras having a semilattice reduct [16]. ∎

While we do not wish to develop the notion of word problems for finitely presented algebras, we do want to mention one of the very significant results in the area. The reader is directed to [41] for general background and the proof of the following result.

**Theorem 6.14** *There is a finitely presented MOL with unsolvable word problem.*

There remain many unsolved problems in this area. The first is of paramount importance, the others less important but still of considerable interest.

**Problems** 1. Is the free word problem for OMLs (MOLs) on three or more generators solvableΓ 2. Can a freely generated OML have an uncountable blockΓ 3. If $a, b$ are complements in a freely generated ortholattice are $b \vee a'$ and $b \wedge a'$ complements of $a$Γ 4. Characterize the finite subalgebras of a freely generated ortholattice (OML).

# 7 Varieties of ortholattices

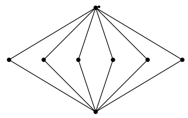
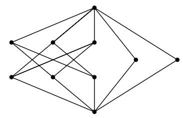
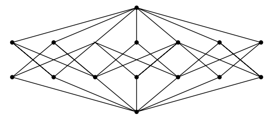
For an ortholattice $L$ let $[L]$ be the variety of ortholattices generated by $L$, and for a class $\mathcal{K}$ of ortholattices let $[\mathcal{K}]$ be the variety generated by $\mathcal{K}$. Note that the class of all one element OLs is a variety often called the trivial variety.

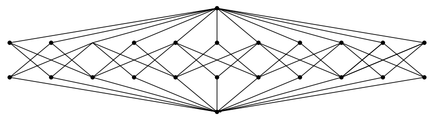
**Proposition 7.1** *(1) The trivial variety is the smallest variety of OLs. (2) Every non-trivial variety of OLs contains the variety of Boolean algebras. (3) Every non-Boolean variety of OLs contains either $[MO2]$ or $[Benzene]$.*

**Proof.** (1) Obvious. (2) Every ortholattice with more than one element contains a two element Boolean algebra as a subalgebra, and the two element Boolean algebra generates the variety of Boolean algebras. (3) By (2.1) every ortholattice which is not orthomodular contains a subalgebra isomorphic to Benzene, and in (3.6) we showed that every non-Boolean variety of OMLs contains $MO_2$. ∎

For varieties of OMLs somewhat more is known [12].

**Proposition 7.2** *Let $V$ be a variety of OMLs that is generated by its finite members. If $V$ is not contained in $[MO2]$, then $V$ contains a variety generated by one of the four OMLs shown below.*



In each of these figures orthocomplementary elements are directly above and below one another, or directly beside one another for the middle elements. In the final figure the two elements on the left end are to be "identified" with the two on the right end.

For varieties of MOLs the situation becomes very interesting. We remind the reader that a subdirectly irreducible (ortho) complemented modular lattice of height three is called a (orthocomplemented) projective plane.

**Theorem 7.3** *The varieties of MOLs generated by their finite members are exactly the* $[MO_\kappa]$ *where* $\kappa$ *is a cardinal.*

**Proof.** This is a difficult theorem, but we can outline the steps in the proof. Suppose $L$ is a finite subdirectly irreducible MOL. If $L$ is of height two or less, then $L$ is equal to $MO_n$ for some $n < \omega$. Otherwise $L$ contains an element $a$ of height 3. By a theorem of Bruns [9] the interval $[0, a]$ of $L$ is an orthocomplemented projective plane. But Baer showed [3] that every involution on a finite projective plane has a fixed point, hence no finite projective plane admits an orthocomplementation. Thus every finite subdirectly irreducible MOL is an $MO_n$ for some $n < \omega$.

Suppose $V$ is a variety generated by a class $\mathcal{K}$ of finite MOLs. As every finite MOL is a direct product of simple, hence subdirectly, MOLs (5.6) we may assume each member of $\mathcal{K}$ is subdirectly irreducible, hence equal to $MO_n$ for some $n < \omega$. If $\{m | MO_m \in \mathcal{K}\}$ is finite, then it has a maximum $n$, and clearly $V = [MO_n]$. Suppose that $\{m | MO_m \in \mathcal{K}\}$ is infinite. We claim that $MO_\omega$ belongs to $V$, hence $V = [MO_\omega]$. But this follows as $V$ is an equational class and any equation in $n$ variables failing in $MO_\omega$ must fail in some $n$ generated subalgebra of $MO_\omega$, hence in $MO_n$. Finally, note that $[MO_\kappa] = [MO_\omega]$ for each infinite cardinal $\kappa$ as $MO_\kappa$ and $MO_\omega$ satisfy the same equations. ∎

Note that it is an easy consequence of Jónsson's theorem that $[MO_n]$ is covered by $[MO_{n+1}]$ for each $n < \omega$, hence the varieties $[MO_\kappa]$ form a chain of order type $\omega + 1$. But these are not the only varieties of MOLs. Let $P$ be an orthocomplemented projective plane, such as the lattice of subspaces of a three dimensional vector space over the reals with the orthocomplement of a subspace $S$ being its orthogonal subspace $S^-$. Clearly there are equations valid in all $MO_\kappa$, such as $\gamma(x,\gamma(y,z)) \approx 0$, which are not valid in $P$. Thus $[P]$ is distinct from all $[MO_\kappa]$. However, it is a simple matter to show that $MO_\omega$ is a subalgebra of an interval of $P$, hence $[P]$ contains $MO_\omega$. The following two theorems summarize the remaining facts known of varieties of MOLs. The first is due to Bruns [9] and the second to Roddy [40].

**Theorem 7.4** *If $L$ is a subdirectly irreducible MOL containing an atom, then either $[L] = [MO_\kappa]$ for some cardinal $\kappa$ or $[L]$ contains $[P]$ for some orthocomplemented projective plane $P$.*

**Theorem 7.5** *Every variety of MOLs distinct from $[MO_\kappa]$ for all cardinals $\kappa$ contains $[MO_\omega]$.*

We are left with the following open problem sometimes referred to as Bruns' conjecture. We consider it a basic open problem in the theory of OMLs.

**Problem.** Does every variety of MOLs which is different from $[MO_\kappa]$ for all cardinals $\kappa$ contain an orthocomplemented projective plane$\Gamma$

# 8 Completions

A lattice $L$ is called complete if every subset of $L$ has a greatest lower bound and a least upper bound. A completion of $L$ is a lattice embedding of $L$ into a complete lattice $C$. A completion of $L$ is called regular if the embedding preserves all existing joins and meets from $L$, and is called join (meet) dense if every element of $C$ is the join (meet) of images of elements of $L$. It is well known that an embedding that is both join and meet dense is regular.

**Theorem 8.1** *Every lattice $L$ can be join densely embedded into a complete lattice $C$ which satisfies exactly the same equations as $L$.*

This well known theorem [18, pg. 68] is proved by considering the mapping of $L$ into the ideal lattice $Id(L)$ of $L$ which takes an element $a$ of $L$ to the principal ideal $a\downarrow$ generated by $a$. One easily checks that this embedding preserves all existing meets, but destroys all but essentially finite joins.

**Theorem 8.2** *Every lattice can be join and meet densely embedded, hence regularly embedded, into a complete lattice $C$.*

**Proof.** We provide a sketch, for complete details see [35]. Given a lattice $L$, let $P$ be the power set of $L$. Define maps $L, U : P \to P$ by setting, for each $A \subseteq L$, $L(A) = \{x | \forall a \in A, x \leq a\}$ and $U(A) = \{x | \forall a \in A, a \leq x\}$. One easily checks that the composite $LU$ is a closure operator on $P$. Therefore the closed sets form a complete lattice $C$ under set inclusion. Consider the map $\varphi : L \to P$ defined by setting $\varphi(a) = a\downarrow$. Obviously $\varphi$ is a lattice embedding of $L$ into $C$. If $A = LU(A)$

it follows that $A = \bigcap\{u\downarrow | u \in U(A)\}$ and, as $A$ is a downset, $A = \bigcup\{a\downarrow | a \in A\}$. Therefore $\varphi$ is both join and meet dense. ∎

In [5] it was shown that up to isomorphism there is only one join and meet dense completion of a lattice $L$. We call this the MacNeille completion of $L$. Unfortunately MacNeille completions of lattices are poorly behaved when it comes to preserving identities. In fact, the variety of all lattices and the variety of one element lattices are the only varieties of lattices which are closed under MacNeille completions [29]. One might hope to find a completion which is both regular and preserves identities. This is not possible [4, pg. 233].

**Proposition 8.3** *There is a distributive lattice which can not be regularly embedded into any complete distributive lattice.*

We next turn our attention to Boolean algebras. Recall the classic result of Stone that for each Boolean algebra $B$ there is a zero dimensional compact Hausdorff space $X$, called the Stone space of $B$, with $B$ isomorphic to the Boolean algebra of clopen subsets of $X$. Stone's representation theorem provides two natural completions for Boolean algebras.

**Theorem 8.4** *Let $B$ be a Boolean algebra with Stone space $X$. Then the collection $Reg(X)$ of all regular open subsets of $X$ is a complete Boolean algebra, and the natural embedding of $B$ into $Reg(X)$ is both join and meet dense, hence regular.*

For a proof of this well known theorem see [4, pg. 157]. In view of the characterization of MacNeille completions of lattices as join and meet dense completions [5] we call this the MacNeille completion of $B$ and denote it $B^*$. Obviously taking the full power set of $X$ will also provide a completion of $B$, which we call the canonical completion of $B$ and denote by $B^\sigma$. An abstract characterization of this completion follows below.

**Theorem 8.5** *Up to isomorphism there is a unique embedding $e : B \to C$ of a Boolean algebra $B$ into a complete Boolean algebra $C$ such that (i) each element of $C$ is a join of meets and a meet of joins of elements of $e[B]$, and (ii) if $S, T \subseteq B$ with $\bigwedge e[S] \leq \bigvee e[T]$ then there are finite $S' \subseteq S$, $T' \subseteq T$ with $\bigwedge e[S'] \leq \bigvee e[T']$.*

We remark that the most useful of lattice completions, the ideal lattice, cannot be applied to Boolean algebras as $Id(B)$ is only complemented if $B$ is finite. However, the MacNeille completion $B^*$ does provide even a strengthening of (8.1) in the Boolean case. We next turn our attention to completions of ortholattices. Again, we can not use ideal lattices to obtain completions as they will not be (ortho) complemented. In fact, it is not at first apparent that there are any general methods to complete ortholattices. To the best of our knowledge, there are two.

**Theorem 8.6** *Up to isomorphism, there is a unique embedding $e : L \to C$ of an OL $L$ into a complete OL $C$ which is both join and meet dense, hence regular.*

We call this the MacNeille completion $L^*$ of $L$. Existence was proved by MacLaren [34] by taking all subsets $A \subseteq L$ which are equal to the lower bounds of their upper bounds, and defining the orthocomplementation $A^- = \{u' : u$ is an upper bound of $A\}$. Uniqueness follows from [5].

**Theorem 8.7** *Up to isomorphism, there is a unique embedding $e : L \to C$ of an OL $L$ into a complete OL $C$ such that (i) every element of $C$ is a join of meets and a meet of joins of elements of $e[L]$, and (ii) if $S, T \subseteq B$ with $\bigwedge e[S] \leq \bigvee e[T]$ then there are finite $S' \subseteq S$, $T' \subseteq T$ with $\bigwedge e[S'] \leq \bigvee e[T']$.*

We call this the canonical completion $L^\sigma$ of $L$. We remark that the corresponding theorem holds for bounded lattices as well. Methods of obtaining such a completion have been around for some time. For lattices one uses Urquhart's [42] stable sets and for ortholattices Goldblatt's filter space [22]. However, the first abstract characterization and detailed study of this completion is in [21]. Unfortunately, neither completion behaves well with respect to preserving equations.

**Proposition 8.8** *There is an OML $L$ with neither $L^*$ nor $L^\sigma$ orthomodular.*

To produce an OML whose MacNeille completion is not orthomodular take an incomplete inner product space $E$. Let $L$ be the OML of all subspaces $S$ of $E$ which are either finite dimensional, or whose orthogonal subspace $S^-$ is finite dimensional. Then the MacNeille completion of $L$ is the ortholattice $L(E, -)$ of all subspaces $S$ of $E$ which satisfy $S = S^{--}$. But by a theorem of Amemiya and Araki [2] $L(E, -)$ is orthomodular iff $E$ is complete. More elementary examples are given in [26] using a technique to construct an orthomodular lattice from a given lattice due to Kalmbach [31]. An example of an OML whose canonical completion is not orthomodular is given in [30]. This example is also based on the Kalmbach construction. Taking the direct product of these two counterexamples yields an OML with neither $L^*$ nor $L^\sigma$ orthomodular. We remark that co! ! mpletions can be found for these examples (based on completing the underlying inner product space or the underlying lattice used in the Kalmbach construction), but in general the following remains one of the major open problems in the area.

**Problem.** Can every OML be embedded into a complete OML.

There are partial results known. In the presence of a finiteness condition we obtain the following generalization of the well known fact that the MacNeille completion of a Boolean algebra is Boolean.

**Theorem 8.9** *Let $V$ be a variety generated by a class of OMLs with a finite upper bound on the lengths of their chains. Then $V$ is closed under MacNeille completions.*

The proof of this theorem [28] relies heavily on the representation theorem for such varieties outlined in theorem (5.8). It is a minor open problem whether this theorem would apply to a variety generated by a class $\mathcal{K}$ of OMLs with a finite upper bound on the number of their commutators. The following results are useful in setting limits on what one can hope to obtain in a completion. See [30] for a proof of the following.

**Proposition 8.10** *Every regular completion of an OML factors through the MacNeille completion. Therefore there is an OML which cannot be regularly embedded into a complete OML.*

**Theorem 8.11** *There is a MOL which cannot be embedded into a complete MOL.*

**Proof.** A deep theorem of Kaplansky [32, pg. 178] shows every complete MOL is a continuous geometry, and therefore has a dimension function. Let $M$ be the MOL of all subspaces $S$ of a Hilbert space $H$ for which either $S$ or $S^-$ is finite dimensional. As $M$ contains a countable set of pairwise perspective atoms, $M$ cannot admit a dimension function, hence cannot be embedded into a complete MOL. ∎

We conclude this section with a positive result which may eventually be helpful in solving the completion problem. This result has a long history, and we honestly do not know who to credit for it. See [11] for an outline of a proof and description of the history.

**Proposition 8.12** *Every OML can be embedded into an OML in which each element is a join of two or fewer atoms.*

# 9 Categorical properties

Every variety of algebras naturally forms a category whose objects are the algebras in the variety and whose morphisms are the homomorphisms between these algebras (not necessarily onto homomorphisms). There are a large number of categorical questions one can ask of such varieties. We content ourselves with but a few, namely questions relating to monomorphisms, epimorphisms, injectives and projectives. The survey article [33] is excellent source of information on categorical issues relating to varieties of algebras.

**Definition 9.1** *Let $V$ be a variety and $h : B \to C$ be a homomorphism between members of $V$. We say $h$ is a monomorphism if for all algebras $A$ in $V$ and all homomorphisms $f, g : A \to B$ we have $h \circ f = h \circ g$ implies $f = g$. Similarly $h : B \to C$ is an epimorphism if for all algebras $D$ in $V$ and all homomorphisms $f, g : C \to D$ we have $f \circ h = g \circ h$ implies $f = g$.*

One easily sees that one-one homomorphisms are monomorphisms and onto homomorphisms are epimorphisms. The question arises whether there are any others. For monomorphisms the answer is easily found.

**Proposition 9.2** *Let $V$ be a variety of ortholattices. Then the monomorphisms in $V$ are exactly the one-one homomorphisms.*

This is a well known result which holds for any variety of algebras. The proof follows by noting that for any $B$ in $V$ and any $x \neq y$ in $B$ there are homomorphisms $f, g$ from the free algebra on one generator (a four element Boolean algebra in our setting) with $f$ mapping the generator to $x$ and $g$ mapping the generator to $y$. The dual question whether every epimorphism is onto poses much greater difficulty. Before describing the known results, we introduce an additional notion [23, pg. 252] which is also of considerable interest.

**Definition 9.3** *Let $\mathcal{K}$ be a class of algebras of the same type. A $V$-formation in $\mathcal{K}$ is a quintuplet $(B, L_1, L_2, f_1, f_2)$ where $B, L_1, L_2$ are algebras in $\mathcal{K}$ and $f_i : B \to L_i$ $(i = 1, 2)$ are embeddings. An amalgamation of the $V$-formation in $\mathcal{K}$ is a triple $(C, g_1, g_2)$ where $C$ is an algebra in $\mathcal{K}$ and $g_i : L_i \to C$ $(i = 1, 2)$ are embeddings with $g_1 \circ f_1 = g_2 \circ f_2$. The amalgamation is called strong if $g_1[L_1] \cap g_2[L_2] = g_1[f_1[B]]$. The class $\mathcal{K}$ is said to have the (strong) amalgamation property if every $V$-formation in $\mathcal{K}$ has a (strong) amalgamation.*

The connection between amalgamations and epimorphisms is given by the following well known result [33].

**Lemma 9.4** *If a variety $V$ has the strong amalgamation property, then the epimorphisms in $V$ are exactly the onto homomorphisms.*

**Proof.** Suppose $h : B \to C$ is not onto. If $(D, f, g)$ is a strong amalgamation of the $V$-formation $(h[B], C, C, id, id)$ then $f \circ h = g \circ h$ but $f \neq g$. ∎

**Proposition 9.5** *The variety of OLs has the strong amalgamation property, therefore the epimorphisms in this variety are exactly the onto homomorphisms.*

**Proof.** A more detailed treatment is given in [10] but we can outline the idea. Suppose $L_1$ and $L_2$ are ortholattices and that $B = L_1 \cap L_2$ is a subalgebra of both. Define a relation $\leq$ on $L_1 \cup L_2$ by setting $x \leq y$ iff one of the following occurs (i) $x, y$ belong to the same $L_i$ $(i = 1, 2)$ and $x \leq_i y$, or (ii) $x$ belongs to $L_i$, $y$ belongs to $L_j$ and there is some $b$ in $B$ with $x \leq_i b \leq_j y$. One easily checks that $(L_1 \cup L_2, \leq)$ is a partially ordered set and that the union of the orthocomplementations on $L_1, L_2$ is an orthocomplementation on $L_1 \cup L_2$. The result then follows from the well known [34] and easily proved fact that the MacNeille completion of a orthocomplemented poset is an ortholattice. ∎

The following well known result [33] is an interesting exercise.

**Proposition 9.6** *The variety of Boolean algebras has the strong amalgamation property, so epimorphisms in this variety are exactly the onto homomorphisms.*

The situation for orthomodular lattices is not so fortunate [10].

**Proposition 9.7** *Neither the variety of OMLs, nor the variety of MOLs, have the amalgamation property.*

There are however a number of special cases where $V$-formations can be amalgamated in OML. The first result below was established in [10], the second is a reformulation of Greechie's celebrated "paste job" [24].

**Theorem 9.8** *In the variety of OMLs, any $V$-formation $(B, L_1, L_2, f_1, f_2)$ with $B$ Boolean has a strong amalgamation.*

**Theorem 9.9** *Let $(B, L_1, L_2, f_1, f_2)$ be a $V$-formation in the variety of OMLs such that there is an element $a$ in $B$ with $f_i[B]$ the union of the principal ideal $[0, f_i(a)]$ and the principal filter $[f_i(a'), 1]$ in $L_i$. Then there is a strong amalgamation $(C, g_1, g_2)$ of this $V$-formation with $L = g_1[L_1] \cup g_2[L_2]$.*

Unfortunately, the above considerations have left the following open questions, which we consider to be basic open problems in the area.

**Problem.** In the variety of OMLs (MOLs) are the epimorphisms exactly the onto homomorphismsΓ

We remark that in [11] an effective procedure is given to determine if epimorphisms coincide with onto homomorphisms in any variety generated by a finite number of finite OMLs. We next turn our attention to injective and projective algebras.

**Definition 9.10** *An algebra $C$ in a variety $V$ is called injective if for every monomorphism $f : A \to B$ and every homomorphism $g : A \to C$ there exists a homomorphism $h : B \to C$ with $h \circ f = g$. Dually, $C$ is called projective if for every epimorphism $f : B \to A$ and every homomorphism $g : C \to A$ there exists a homomorphism $h : C \to B$ with $f \circ h = g$. The notions of weakly injective and weakly projective are formed by replacing monomorphisms and epimorphisms with one-one and onto homomorphisms.*

Note that injectives and weakly injectives coincide in any variety of algebras as the monomorphisms in any variety are exactly the one-one homomorphisms. Characterizing injectives in certain varieties of OLs will pose little difficulty.

**Theorem 9.11** *In the variety of Boolean algebras, an algebra is injective iff it is complete. In the variety of OLs, the variety of OMLs and the variety of MOLs, an algebra is injective iff it has exactly one element.*

**Proof.** The result for Boolean algebras is well known [4, pg. 113]. Suppose $V$ is one of the varieties of OLs, OMLs, MOLs and $C$ is a member of $V$ having more than one element. As each of these varieties has simple algebras of arbitrarily large cardinality ($MO_\kappa$) there is a simple algebra $B$ in $V$ with cardinality greater than $C$. For $f : 2 \to B$ and $g : 2 \to C$ the obvious embeddings, there is no homomorphism $h : B \to C$ with (or without) $h \circ f = g$. ∎

Note that projectives and weakly projectives need not coincide in a variety where epimorphisms are not exactly the onto homomorphisms. We consider only weakly projectives. The following well known result [4, pg. 36] provides an abstract characterization of the weakly projectives in any variety.

**Theorem 9.12** *For $C$ an algebra in a variety $V$ these are equivalent. (1) $C$ is weakly projective in $V$. (2) There is a free algebra $F$ in $V$ and homomorphisms $f : F \to C$ and $g : C \to F$ with $f \circ g$ the identity on $C$.*

In particular any free algebra in $V$ is weakly projective in $V$, and any weakly projective in $V$ must be a subalgebra of a free algebra. However, it can be difficult to provide a more direct characterization of weakly projectives. Even for the variety of Boolean algebras, no satisfactory description is known. But we do have the following sufficient condition.

**Proposition 9.13** *In the variety of Boolean algebras, every at most countable algebra with more than one element is weakly projective.*

It is perhaps surprising that there are complete descriptions of the Boolean algebras that are weakly projective in the varieties of OLs and OMLs.

**Proposition 9.14** *In the variety of ortholattices, a Boolean algebra is weakly projective iff it has two, four, or eight elements.*

**Proof.** Let $B$ be a Boolean algebra. If $B$ has one, two, or four elements the result is trivial. Kearnes established the result for an eight element Boolean algebra. If $B$ has more than eight elements then $B$ does not satisfy Whitman's condition. By (6.10) every free ortholattice satisfies Whitman's condition. Therefore $B$ is not a subalgebra of a free ortholattice, hence is not weakly projective. ∎

**Theorem 9.15** *In the variety of OMLs, a Boolean algebra is weakly projective iff it has more than one element and is at most countable.*

A proof of this result is found in [15, 16]. There are a number of miscellaneous results that may provide some feel for the topic. First, in the variety of OLs benzene is weakly projective. More generally, one obtains weakly projectives in OLs by replacing the intervals on the sides of benzene with a weakly projective lattice and its dual. Second, $MO_2 \times 2$ is weakly projective in the variety of OMLs. Third, $MO_3 \times 2$ is not weakly projective in the variety of MOLs. The first two results are (slight modifications of) well known and easily proved results. The third is much more difficult, requiring in part, the delicate construction of an infinite MOL with rather particular properties [14]. Also established in [15] is the following.

**Proposition 9.16** *Let $V$ be a variety of OMLs generated by a class of OMLs with a finite upper bound on the lengths of their chains. If $A \in V$ is finite, then $2 \times A$ is weakly projective in $V$.*

No doubt the reader is aware there are many open questions in this area.

**Problems.** Characterize the weakly projectives in the variety of OLs, OMLs, and MOLs.

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