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Embeddings into Orthomodular Lattices with Given Centers, State Spaces and Automorphism Groups

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Abstract. We prove that, given a nontrivial Boolean algebra B, a compact convex set S and a group G, there is an orthomodular lattice L with the center isomorphic to B, the automorphism group isomorphic to G, and the state space affinely homeomorphic to S. Moreover, given an orthomodular lattice J admitting at least one state, L can be chosen such that J is its subalgebra.

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1. Basic Definitions and Historical Overview of the Problem

Orthomodular lattices appear – besides their algebraical importance – as event structures of quantum mechanical systems. They allow to develop a generalized probability theory which admits the phenomenon of noncompatibility: Two events a, b that can be observed separately need not be jointly observable.

DEFINITION 1.1. An *orthomodular lattice* (abbr. *OML*) is a lattice L with a least and a greatest element (denoted by 0, 1, respectively) and with a unary operation $'$ (*orthocomplementation*) satisfying the following properties for all $a, b \in L$:

(1) $(a')' = a$,
(2) $a \leq b \implies b' \leq a'$,
(3) $a \vee a' = 1$,
(4) $a \leq b \implies b = a \vee (b \wedge a')$.

Here we always work with OMLs that are *nontrivial*, i.e., $0 \neq 1$. When necessary, we shall distinguish the bounds of OMLs by indices, e.g., $0_L, 1_L$. An element a of an OML L is called an *atom* if there is no $b \in L$ satisfying $0 < b < a$. We denote by $\mathcal{A}(L)$ the set of all atoms of L. We say that L is *atomistic* if each element of L can be expressed as a join of atoms. Two elements a, b of an OML are called *orthogonal* (in symbols $a \perp b$) if $a \leq b'$. If we require in Definition 1.1 the existence of joins only for orthogonal pairs of elements (and dually for meets), we obtain the definition of an *orthomodular poset* (abbr. *OMP*). Typical examples of OMLs are Boolean algebras and lattices of projections in Hilbert spaces. We refer to [6, 18] for basic facts on OMLs and OMPs.

Let L be an OML. A *Boolean subalgebra* of L is a subalgebra which – with the operations inherited from L – becomes a Boolean algebra. Two elements $a, b \in L$ are called *compatible* if they are contained in a Boolean subalgebra of L. The set $C(L) = \{a \in L : a$ is compatible to all $x \in L\}$ is a Boolean subalgebra called the *center* of L.

DEFINITION 1.2. Let L be an OML. A mapping $m : L \to [0, 1]$ is called a *state* on L if the following conditions are satisfied:

(1) $m(1) = 1$,
(2) $m(a \vee b) = m(a) + m(b)$ whenever $a \perp b$.

We denote by $\mathcal{S}(L)$ the *state space* (i.e., the set of all states) of an OML L. In contrast to Boolean algebras and lattices of projections in Hilbert spaces, the following situation is possible:

PROPOSITION 1.3 [4]. *There exists a finite OML which is* stateless, *i.e., it does not admit any state.*

In the physical interpretation, the center represents the classical part of the system. The state space represents all possible states of the system. The automorphism group is also of importance, because it determines the symmetries. It is natural to ask whether there is a kind of dependence between these attributes of the event structure of a quantum system (OML). As a motivation, in OMLs of projections in von Neumann algebras the state space determines the center uniquely [1]. We ask if a dependence of this type holds also for OMLs. Moreover, we pose our question under the additional requirement that a given subsystem is embedded. Our main result is to show there is no dependence between the above-mentioned attributes of OMLs.

Before stating the main theorem, let us make an overview of the long history of the problem. We use some of these preceding partial solutions, too.

The fact that an arbitrary Boolean algebra can occur as the center of an OML is obvious.

The state spaces were characterized by the following theorem due to Shultz (see [19] and also [8, 11] for simplified proofs):

THEOREM 1.4. *For every OML L, the state space $\mathcal{S}(L)$ is a compact convex subset of the space of real functionals on L with the product (= weak) topology. Conversely, every compact convex subset of a locally convex topological linear space is affinely homeomorphic to the state space of some OML.*

The fact that any group can be represented as the automorphism group of an OML was first mentioned by Kalmbach in [7]. A clear proof based on graph-theoretical methods was given by Kallus and Trnková [5]. Their construction also allows to embed an arbitrary atomistic OML as a subalgebra. This result was generalized to the nonatomistic case in [15]. (There is an alternative way of this generalization: Following [2], every OML can be embedded into an atomistic one in an automorphism-preserving manner. Then the construction from [5] can be applied.) For concrete (= set-representable) OMLs an analogous result was proved in [16]. OMLs with given group of automorphisms and given group of affine homeomorphisms of the state space are constructed in [20].

As the first result concerning the interplay of the attributes of OMLs, P. Pták [17] proved the existence of OMLs with given centers and state spaces. In [14], this result was extended by embedding an arbitrary OML as a subalgebra. The independence of the center, the state space, the automorphism group and a subalgebra was proved in [9], but in full generality only for orthomodular posets. The problem remained open for OMLs. In particular, it was not clear if OMLs with given centers (without any atom) and given nontrivial automorphism groups exist. The partial results of [9] for OMLs which we use here are formulated as follows:

THEOREM 1.5. *Let J be an OML admitting at least one state, B a nontrivial Boolean algebra, G a group and S a compact convex subset of a locally convex Hausdorff linear space. If G is the trivial group or B contains an atom, then there is an OML L such that*

(1) $J \leq L$,
(2) $C(L) \cong B$,
(3) $Aut(L) \cong G$,
(4) $\mathcal{S}(L)$ *is affinely homeomorphic to* S.

Here we give the answer for OMLs in the general case. Although we use the partial results of [5, 9] whenever possible, we needed completely new techniques at some steps.

To make our historical survey complete, it should be mentioned that analogous questions for σ-complete OMLs and σ-additive states are even more complicated. Partial results – the independence of the center, the state space and a subalgebra – were developed in [1, 12, 13, 18].

2. Preliminaries

In this section we present some necessary notions and constructions used in the sequel.

DEFINITION 2.1. Let L be an OML. For $a, b \in L$, $a \leq b$, we define the *interval* $[a, b]_L = \{c \in L : a \leq c \leq b\}$.

We always consider an interval $[a, b]_L$ with the partial ordering inherited from L:

PROPOSITION 2.2. *Let L be an OML and let $e \in L \setminus \{0\}$. We endow the interval $K = [0, e]_L$ with the partial ordering $\leq_K$ inherited from L and with the orthocomplementation ${}'^K$ defined by $a'^K = a'^L \wedge_L e$. Then K is an OML.*

The latter proposition can be generalized to all (bounded) intervals, but we need it only for the intervals of the form $[0, e]_L$. These intervals are *principal ideals* in L.

DEFINITION 2.3. Let $\mathcal{F}$ be a family of OMLs. We take the Cartesian product $L = \prod_{K \in \mathcal{F}} K$ and we endow it with the partial ordering $\leq_L$ and orthocomplementation ${}'^L$ defined pointwise, i.e., for all $a, b \in L$, $a = (a_K)_{K \in \mathcal{F}}$, $b = (b_K)_{K \in \mathcal{F}}$, we define

$$a \leq_L b \iff \forall K \in \mathcal{F} : a_K \leq_K b_K,$$

$$a = b'^L \iff \forall K \in \mathcal{F} : a_K = (b_K)'^K.$$

Then L becomes an OML called the *product* of the family $\mathcal{F}$.

An OML is called *reducible* if it is isomorphic to a product of nontrivial OMLs; otherwise, it is called *irreducible*. A nontrivial OML is irreducible iff its center is the two-element Boolean algebra. An OML is called *simple* if it does not allow a nontrivial congruence. If an OML is simple, then it is irreducible. The reverse implication need not hold.

DEFINITION 2.4. Let $\mathcal{F}$ be a family of OMLs. We make a family $\mathcal{G}$ of copies of OMLs from $\mathcal{F}$ which are disjoint except that they have the same least element, 0, and the same greatest element, 1. Thus, for each $K, M \in \mathcal{G}$, $K \neq M$, we have $K \cap M = \{0, 1\}$. We take the union $L = \bigcup \mathcal{G}$ and we endow it with the partial ordering $\leq_L$ and orthocomplementation ${}'^L$ defined by

$$a \leq_L b \iff \exists K \in \mathcal{G} : (a, b \in K,\ a \leq_K b),$$

$$a = b'^L \iff \exists K \in \mathcal{G} : (a, b \in K,\ a = b'^K).$$

Then L becomes an OML called the *horizontal sum* of the family $\mathcal{F}$, denoted $\bigoplus_{K \in \mathcal{F}} K$, or in the case that $\mathcal{F}$ has two members K_1, K_2, as $K_1 \oplus K_2$.

A horizontal sum of a family $\mathcal{F}$ of OMLs is called *nontrivial* if $\mathcal{F}$ contains at least 2 OMLs with more than 2 elements. Then its center is trivial. We call an OML *totally irreducible* if it is irreducible and cannot be expressed as a nontrivial horizontal sum.

DEFINITION 2.5. Let K, L be OMLs. A mapping $\varphi: K \to L$ is called a *homomorphism* if it satisfies the following conditions:

(1) $\varphi(0) = 0$,
(2) $\varphi(a') = \varphi(a)'$ for all $a \in K$,
(3) $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$ for all $a, b \in K$.

If φ is bijective and both φ and φ^{-1} are homomorphisms, then φ is called an *isomorphism*. If $\varphi: K \to \varphi(K)$ is an isomorphism, then φ is called an *embedding*. If $K \subseteq L$ and the identity on K is an embedding into L, then K is a *subalgebra* of L; in symbols $K \leq L$.

We use the notation $\cong$ to denote isomorphisms of OMLs as well as group isomorphisms. Notice that isomorphisms of OMLs preserve centers and possible decompositions to products and horizontal sums.

For an OML L, we denote by $Aut(L)$ the group of all automorphisms $\varphi: L \to L$ and by id_L the identity on L. An OML is called *rigid* if its automorphism group is trivial, i.e., $Aut(L) = \{id_L\}$.

We shall use the following partial results from [5]:

THEOREM 2.6. *Let K be an OML, G be a group. There is a totally irreducible OML L such that $K \leq L$, $Aut(L) \cong G$.*

THEOREM 2.7. *There is a proper class of mutually non-isomorphic OMLs, each of which is totally irreducible, rigid, and of height three.*

3. An Example

In this section we present an explicit example of the simplest situation: a rigid OML with an empty state space and a trivial (= two-element) center. To describe it, we use hypergraphs called Greechie diagrams (see [4, 6]).

A *hypergraph* is a couple $H = (V, \mathcal{E})$ consisting of a nonempty set V (of *vertices*) and of a covering $\mathcal{E}$ of V by nonempty subsets (*edges*). A *loop of order* n in H ($n \geq 3$) is an n-tuple of edges $E_1, \ldots, E_n \in \mathcal{E}$ such that the intersections $E_1 \cap E_2, \ldots, E_{n-1} \cap E_n, E_n \cap E_1$ are nonempty and mutually disjoint.

Let L be a finite OML and let us denote by $\mathcal{A}(L)$ the set of all atoms of L. The *Greechie diagram* of L (see [4]) is a hypergraph $H = (V, \mathcal{E})$ such that $V = \mathcal{A}(L)$ and $\mathcal{E}$ consists of all maximal subsets of mutually orthogonal elements of $\mathcal{A}(L)$.

LOOP LEMMA (see [4, 6]). *Let $H = (V, \mathcal{E})$ be a hypergraph satisfying the following conditions*:

(1) $\forall E \in \mathcal{E}:\ \text{card}(E) \geq 3$,
(2) $\forall E, F \in \mathcal{E}, E \neq F:\ \text{card}(E \cap F) \leq 1$,
(3) *there is no loop of order less than* 5 *in* H.

Then there is a unique OML L such that H is the Greechie diagram of L.

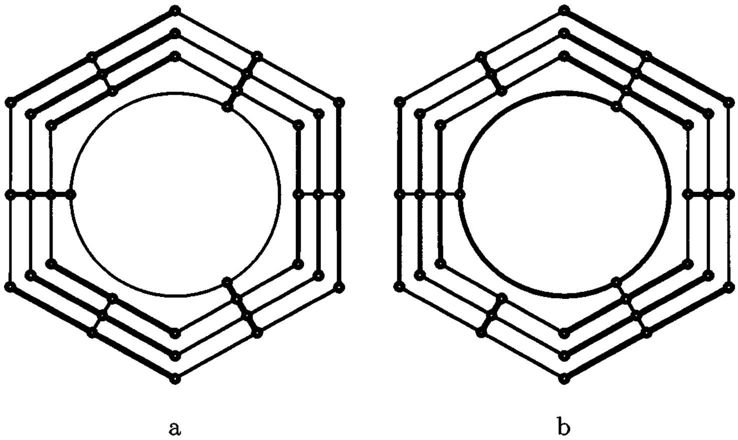

Figure 1. An OML admitting no states and many automorphisms.

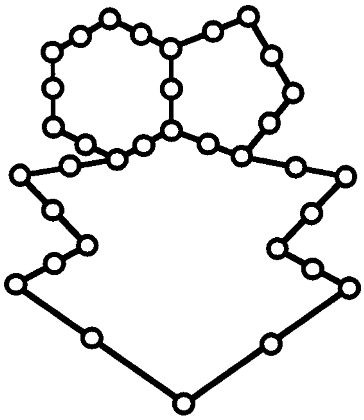

Figure 2. A rigid OML admitting many states.

In figures, vertices of Greechie diagrams are denoted by small circles and edges by line segments or smooth curves. States on a finite OML are in a one-to-one correspondence with nonnegative evaluations of vertices of its Greechie diagram such that for each edge the sum of evaluations of its vertices is 1.

The OML described by its Greechie diagram in Figure 1a, resp. b – "the web" – admits no states. Indeed, all its vertices can be disjointly covered by 12, resp. 13 edges, so each state s has to satisfy

$$12 = \sum_{a \in \mathcal{A}(L)} s(a) = 13,$$

a contradiction. This OML admits many automorphisms.

The OML corresponding to the Greechie diagram in Figure 2 is rigid because the three loops are of different order.

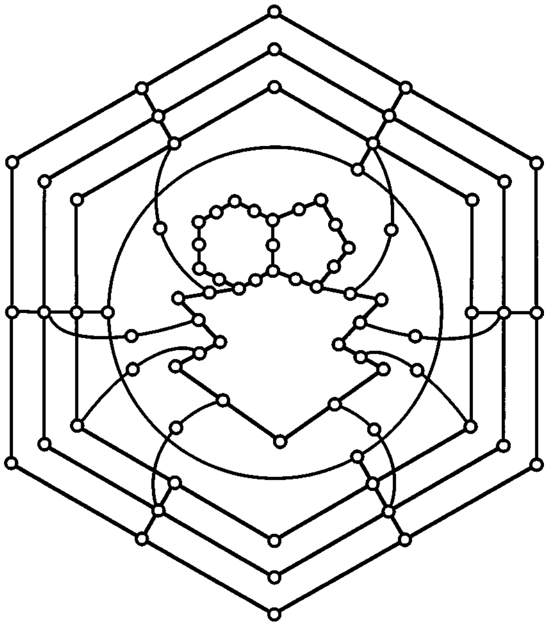

Figure 3. A rigid OML admitting no states.

We connect the latter two examples by additional edges and obtain the "spider in web" in Figure 3. It is the Greechie diagram of an OML admitting no states because it contains the OML from Figure 1 as a subalgebra. As the "body of the spider" (Figure 2) admits no nontrivial automorphisms and it fixes the automorphisms of the whole structure, the resulting OML is rigid. No element different from $0, 1$ is central, so the center is trivial, too.

We obtained an example demonstrating a solution of the simplest possible case of our task. Before proving the general result, we shall formulate and prove two lemmas.

4. The First Lemma

LEMMA 4.1. *Suppose K, Q are nontrivial OMLs, and $\varphi\colon Q \to K$ is a homomorphism with*

(1) *Q irreducible,*
(2) *$\varphi \circ \mu = \varphi$ for all $\mu \in Aut(Q)$,*
(3) *K is rigid.*

Then, for any nontrivial Boolean algebra B, there is an OML L with $Q \leq L$, $Aut(L) \cong Aut(Q)$, and $C(L) \cong B$. Moreover, each interval beneath a central element of L contains a homomorphic image of Q as a subalgebra.

Proof. Let X be the Stone space of B and fix $y \in X$. According to Theorem 2.7, there is a proper class of nonisomorphic rigid OMLs which are not decomposable to a horizontal sum. It follows that there is a family $\{M_U \mid U \subseteq X$ is clopen and $y \notin U\}$ of such nonisomorphic OMLs, none of which is isomorphic to a horizontal summand of K. For each $x \in X$ with $x \neq y$ let T_x be the horizontal sum of $\{M_U \mid x \in U$ and $y \notin U\}$ and K, and set $T_y = Q$.

CLAIM 1. *For each $x \neq y$, T_x is rigid.*

Proof. Suppose $\lambda: T_x \to T_x$ is an automorphism. Then λ takes horizontal summands of T_x to other horizontal summands of T_x. As no summand of the form M_U is isomorphic to any summand of the form M_V with $V \neq U$, nor isomorphic to a summand of K, λ fixes each summand of T_x. As all the summands are rigid (see condition (3)), λ must equal the identity.

CLAIM 2. *For each $x, z \in X$, $T_x \cong T_z$ implies $x = z$.*

Proof. If $x \neq z$, then there is a clopen U not containing y and containing exactly one of x, z. Assume U contains x. So M_U is a horizontal summand of T_x, but not of T_z.

Define a map $\Phi: \prod_{x \in X} T_x \to K \times \prod_{x \in X \setminus \{y\}} T_x$ by setting

$$(\Phi f)(x) = \begin{cases} f(x) & \text{if } x \neq y, \\ \varphi(f(y)) & \text{if } x = y. \end{cases}$$

Note that Φ is a homomorphism. We now define

$$L = \left\{ f \in \prod_{x \in X} T_x \mid \text{for each } x \in X,\ \Phi f \text{ is constant on a neighborhood of } x \right\}.$$

CLAIM 3. *For $f, g \in L$ and $x \in X$, if $f(x) = g(x)$, then f and g agree on a neighborhood of x.*

Proof. As $f, g \in L$, there is a neighborhood N of x on which Φf and Φg are constant, hence Φf and Φg agree on N. First consider the case $x \neq y$. We may assume $y \notin N$ and hence Φf agrees with f and Φg agrees with g on N. If $x = y$, then Φf and Φg take the common value k on N, where k is some element of K. Then f and g have a common value k on $N \setminus \{y\}$ and by assumption, $f(y) = g(y)$.

CLAIM 4. *$L \leq \prod_{x \in X} T_x$, so operations in L are coordinatewise.*

Proof. Let $f, g \in L$ and $x \in X$. Let N be a neighborhood of x on which both $\Phi f, \Phi g$ are constant. Then $\Phi(f \wedge g) = \Phi f \wedge \Phi g$ (as Φ is a homomorphism) is also constant on N. So L is closed under binary meets, and similarly is closed under orthocomplementation, hence is a subalgebra.

CLAIM 5. *Q is isomorphic to a subalgebra of L.*

Proof. Let $q \in Q$ and define $\Gamma: Q \to \prod_{x \in X} T_x$ by setting

$$\Gamma(q)(x) = \begin{cases} \varphi(q) & \text{if } x \neq y, \\ q & \text{if } x = y. \end{cases}$$

Note $(\Phi\Gamma(q))(x) = \varphi(q)$ for each $x \in X$, so $\Phi\Gamma(q)$ is constant on X, hence $\Gamma(q) \in L$. As φ is a homomorphism, $\Gamma(q \wedge r)(x) = \Gamma(q)(x) \wedge \Gamma(r)(x)$ and $\Gamma(q')(x) = (\Gamma(q)(x))'$ for each $x \in X$, hence Γ is a homomorphism. But $q \neq r$ implies $\Gamma(q)(y) \neq \Gamma(r)(y)$, hence Γ is an embedding.

CLAIM 6. *For any $x \in X$ and any $t \in T_x$, there is $f \in L$ with $f(x) = t$.*

Proof. If $x = y$ we take the element $\Gamma(t)$ from above. If $x \neq y$ we consider two cases. First, if $t \in M_U$ for some clopen set U containing x but not y, let f be the function with constant value t on U and 0 elsewhere. Finally, if $x \neq y$ and $t \in K$, choose a clopen neighborhood V of x which does not contain y and set f to be the function with constant value t on V and 0 elsewhere.

CLAIM 7. *$f \in C(L)$ iff $f = \chi_U$ for some clopen $U \subseteq X$ (where χ_U denotes the characteristic function of U).*

Proof. Surely each χ_U is central and is in L. Conversely, suppose $f \in L$ and $f(x) \neq 0, 1$ for some $x \in X$. Note that $C(T_x)$ is trivial for each $x \in X$. (By condition (1), $T_y = Q$ is assumed irreducible. For $x \neq y$, T_x is a horizontal sum. If X is a two element Boolean space and $K = 2$ we have only one horizontal summand M_U of T_x, but this can be chosen irreducible.) So, there is some $t \in T_x$ which does not commute with $f(x)$. Use the above claim to produce a function g with $g(x) = t$. Then f and g do not commute. Hence, the range of f is contained in $\{0, 1\}$. A standard compactness argument shows $f = \chi_U$ for some clopen set U.

Any automorphism $\alpha: L \to L$ clearly restricts to an automorphism of $C(L)$. By Stone duality, associated to each such automorphism α is a homeomorphism $\beta: X \to X$ with $\alpha(\chi_U) = \chi_{\beta(U)}$.

CLAIM 8. *If $f(x) = g(x)$, then $(\alpha f)(\beta x) = (\alpha g)(\beta x)$.*

Proof. By Claim 3, if $f(x) = g(x)$, then for some clopen neighborhood U of x we have $f \wedge \chi_U = g \wedge \chi_U$. Hence $\alpha(f \wedge \chi_U) = \alpha(g \wedge \chi_U)$, giving $\alpha f \wedge \chi_{\beta(U)} = \alpha g \wedge \chi_{\beta(U)}$, and in particular that $(\alpha f)(\beta x) = (\alpha g)(\beta x)$.

CLAIM 9. *The map $\alpha_x: T_x \to T_{\beta x}$ defined by $\alpha_x(f(x)) = (\alpha f)(\beta x)$ is an isomorphism.*

Proof. Using Claims 6 and 8 the map α_x is well defined. As $\alpha_x(f(x) \wedge g(x)) = \alpha_x((f \wedge g)(x)) = \alpha(f \wedge g)(\beta x) = (\alpha f)(\beta x) \wedge (\alpha g)(\beta x) = \alpha_x(f(x)) \wedge \alpha_x(g(x))$ the map α_x preserves binary meets. Similarly, it preserves orthocomplementation, hence is a homomorphism. We obtain its inverse if we repeat the argument with α^{-1} in place of α (as the homeomorphism associated with α^{-1} is β^{-1}).

CLAIM 10. *$\Lambda: Aut(L) \to Aut(Q)$ defined by $\Lambda(\alpha) = \alpha_y$ is a group isomorphism.*

Proof. Suppose $\alpha \in Aut(L)$ with $\beta: X \to X$ the associated homeomorphism. Then $\alpha_x: T_x \to T_{\beta x}$ is an isomorphism for each $x \in X$, hence by Claim 2, $\beta = id_X$. The definition of $\alpha_x: T_x \to T_x$ reduces to $\alpha_x(f(x)) = (\alpha f)(x)$. Thus, for α, δ automorphisms of L, $(\alpha_y \circ \delta_y)(f(y)) = \alpha_y((\delta f)(y)) = (\alpha\delta f)(y) = (\alpha\circ\delta)_y(f(y))$, showing Λ is a group homomorphism. Suppose $\alpha \neq id_L$. Then $(\alpha f)(x) \neq f(x)$ for some $f \in L$ and $x \in X$, hence $\alpha_x \neq id_{T_x}$. But, by Claim 1, T_x is rigid for each $x \neq y$, so $x = y$. This shows Λ is an embedding. Finally, let $\mu: Q \to Q$ be an automorphism. Define $\alpha: L \to \prod_{x\in T} T_x$ by setting

$$(\alpha f)(x) = \begin{cases} f(x) & \text{if } x \neq y, \\ \mu(f(y)) & \text{if } x = y. \end{cases}$$

By condition (2), $\Phi(\alpha f) = \Phi f$, hence $\alpha: L \to L$. Clearly α is an automorphism, and $\alpha_y = \mu$. Hence Λ is a group isomorphism.

Claim 10 establishes $Aut(L) \cong Aut(Q)$, Claim 5 establishes that Q is isomorphic to a subalgebra of L, and Claim 7 establishes $C(L) \cong B$. This concludes the proof of the lemma. □

5. The Second Lemma

LEMMA 5.1. *For any nontrivial OML M there exist nontrivial OMLs Q, K and a surjective homomorphism $\varphi: Q \to K$ such that*

(1) *Q is irreducible,*
(2) *K is rigid,*
(3) *$M \leq Q$,*
(4) *$Aut(Q) \cong Aut(M)$,*
(5) *$\varphi \circ \mu = \varphi$ for all $\mu \in Aut(Q)$.*

Proof. Using Theorem 2.6, we can find totally irreducible OMLs H and K such that (i) $M \leq H \leq K$, (ii) H is of height at least four, (iii) $Aut(H) \cong Aut(M)$, and (iv) K is rigid. Using Theorem 2.7, we can find a totally irreducible, rigid OML R of height 3 such that R is not isomorphic to a subalgebra of an interval of K.

Recursively define a sequence of OMLs P_n for $n \in \mathbb{N}$ by setting

$$P_1 = R \oplus H \quad \text{and} \quad P_{n+1} = R \oplus (K \times P_n).$$

For $n \in \mathbb{N}$ put $Q_n = K \times P_n$, so $Q_{n+1} = K \times (R \oplus Q_n)$, and let $\varphi_n: Q_n \to K$ be the natural projection onto the first coordinate.

Next, for each $n \leq m$ we will define a map $\lambda_{nm}: Q_n \to Q_m$. For each $n \in \mathbb{N}$ let λ_{nn} be the identity map on Q_n, and define $\lambda_{n,n+1}: Q_n \to K \times Q_n \subseteq Q_{n+1}$ by setting

$$\lambda_{n,n+1}(k, p) = (k, (k, p)).$$

Then for $n < m$ set $\lambda_{nm} = \lambda_{m-1,m} \circ \cdots \circ \lambda_{n,n+1}$. Note that each $\lambda_{n,n+1}$, hence all $\lambda_{n,m}$ for $n \leq m$ are embeddings.

As $\lambda_{mk} \circ \lambda_{nm} = \lambda_{nk}$ for each $n \leq m \leq k$, the OMLs Q_n, together with the maps λ_{nm}, form a directed family [3, Chapter 3, Sec. 21] of OMLs. We form the direct limit (also called the inductive limit) of this family (see also [15]), obtaining an OML Q and maps $\lambda_{n\infty}: Q_n \to Q$ for each $n \in \mathbb{N}$. The following claims are standard properties of direct limits.

CLAIM 1. *For all $n \leq m$, $\lambda_{m\infty} \circ \lambda_{nm} = \lambda_{n\infty}$. Further, as each λ_{nm} is an embedding, the $\lambda_{n\infty}$ are also embeddings.*

CLAIM 2. *For each $q \in Q$ there is $n \in \mathbb{N}$ with $q \in \lambda_{m\infty}(Q_m)$ for all $m \geq n$.*

CLAIM 3. *As $\varphi_m \circ \lambda_{nm} = \varphi_n$ for all $n \leq m$, there exists a unique homomorphism $\varphi: Q \to K$ with $\varphi \circ \lambda_{n\infty} = \varphi_n$ for each $n \in \mathbb{N}$.*

It is the OML Q and the map $\varphi: Q \to K$ that will be used to establish the lemma. We proceed to verify their properties.

CLAIM 4. *Q is irreducible.*

Proof. Suppose $q \in C(Q)$. Then $q = \lambda_{n\infty}(k, p)$ for some $n \in \mathbb{N}$ and some $(k, p) \in Q_n$. As $\lambda_{n\infty} = \lambda_{n+1,\infty} \circ \lambda_{n,n+1}$ and $\lambda_{n+1,\infty}$ is an embedding, we must have $\lambda_{n,n+1}(k, p) = (k, (k, p))$ is central in $Q_{n+1} = K \times (R \oplus Q_n)$. As both K and $R \oplus Q_n$ are irreducible, there are only four central elements in Q_{n+1}, of these only the bounds 0, 1 are in the image of $\lambda_{n,n+1}$. Hence $C(Q) = \{0, 1\}$.

This verifies the first condition required by the lemma. The second, that K is rigid, is satisfied by our choice of K. The third is given by

CLAIM 5. *M is isomorphic to a subalgebra of Q.*

Proof. In our choices of H and K, we required that $M \leq H \leq K$. Thus M is isomorphic to a subalgebra of $K \times (R \oplus H) = Q_1$. The result follows as $\lambda_{1\infty}: Q_1 \to Q$ is an embedding.

It remains to establish the final two conditions of the lemma involving automorphisms. For each $\alpha \in Aut(H)$, define recursively maps $\alpha_n: Q_n \to Q_n$ by setting

$$\alpha_1(k, p) = \begin{cases} (k, \alpha(p)) & \text{if } p \in H, \\ (k, p) & \text{if } p \in R \end{cases}$$

and

$$\alpha_{n+1}(k, p) = \begin{cases} (k, \alpha_n(p)) & \text{if } p \in Q_n, \\ (k, p) & \text{if } p \in R. \end{cases}$$

Note, each α_n is an automorphism of Q_n and $\alpha_{n+1} \circ \lambda_{n,n+1} = \lambda_{n,n+1} \circ \alpha_n$ for each $n \in \mathbb{N}$. Then by a standard argument for direct limits we have

CLAIM 6. *For each $\alpha \in Aut(H)$ there is a unique automorphism $\Lambda(\alpha)\colon Q \to Q$ with $\Lambda(\alpha) \circ \lambda_{n\infty} = \lambda_{n\infty} \circ \alpha_n$ for each $n \in \mathbb{N}$.*

CLAIM 7. *$\Lambda\colon Aut(H) \to Aut(Q)$ is a group embedding.*

Proof. Let $\alpha, \beta \in Aut(H)$. By an obvious induction $\alpha_n \circ \beta_n = (\alpha \circ \beta)_n$ for each $n \in \mathbb{N}$. Therefore $\Lambda(\alpha) \circ \Lambda(\beta) \circ \lambda_{n\infty} = \Lambda(\alpha) \circ \lambda_{n\infty} \circ \beta_n = \lambda_{n\infty} \circ \alpha_n \circ \beta_n = \lambda_{n\infty} \circ (\alpha \circ \beta)_n$ for each $n \in \mathbb{N}$. By the previous claim $\Lambda(\alpha) \circ \Lambda(\beta) = \Lambda(\alpha \circ \beta)$, showing Λ is a group homomorphism. If $\alpha \neq id_H$, then $\Lambda(\alpha)$ is not the identity map of Q, hence Λ is an embedding.

Noting that $\varphi \circ \Lambda(\alpha) \circ \lambda_{n\infty} = \varphi \circ \lambda_{n\infty} \circ \alpha_n = \varphi_n \circ \alpha_n = \varphi_n$ we easily obtain

CLAIM 8. *For each $\alpha \in Aut(H)$, $\varphi \circ \Lambda(\alpha) = \varphi$.*

Thus, if we show that Λ maps $Aut(H)$ onto $Aut(Q)$, the remaining two conditions of the lemma, that $Aut(M) \cong Aut(Q)$ and that $\varphi \circ \mu = \varphi$ for all $\mu \in Aut(Q)$ will be satisfied (recall H was chosen with $Aut(H) \cong Aut(M)$). Showing that Λ is onto will require some effort. For convenience, we let q_n denote the element $(0_K, 1_{P_n})$ of $Q_n = K \times P_n$ for $n \in \mathbb{N}$.

CLAIM 9. *For each $n \in \mathbb{N}$, $\lambda_{n\infty}$ restricts to an isomorphism between $[0, q_n]_{Q_n}$ and $[0, \lambda_{n\infty}(q_n)]_Q$.*

Proof. For $k \in \mathbb{N}$, $\lambda_{k,k+1}(q_k) = (0, (0, 1))$. Hence, if $x \in Q_{k+1} = K \times (R \oplus Q_k)$ with $x \leq \lambda_{k,k+1}(q_k)$, then x is in the range of $\lambda_{k,k+1}$. Thus, for $z \leq q_k$, we have $\lambda_{k,k+1}[0, z]_{Q_k} = [0, \lambda_{k,k+1}(z)]_{Q_{k+1}}$, and by an obvious induction, $\lambda_{kj}[0, z]_{Q_k} = [0, \lambda_{kj}(z)]_{Q_j}$ for each $k \leq j$.

To prove the claim, note it is enough to show for $n \leq m$ and $x \in Q_m$ that $\lambda_{m\infty}(x) \leq \lambda_{n\infty}(q_n)$ implies $\lambda_{m\infty}(x) \in \lambda_{n\infty}[0, q_n]_{Q_n}$. But $\lambda_{m\infty}(x) \leq \lambda_{n\infty}(q_n) = \lambda_{m\infty}\lambda_{nm}(q_n)$ implies $x \leq \lambda_{nm}(q_n)$, hence $x \in \lambda_{nm}[0, q_n]_{Q_n}$.

CLAIM 10. *If $\mu \in Aut(Q)$ and $q \in Q$, then $\varphi(q) = 0$ implies $\varphi \circ \mu(q) = 0$.*

Proof. Choose n so that $q, \mu(q) \in \lambda_{n\infty}(Q_n)$. Say $q = \lambda_{n\infty}(x)$. As $\varphi \circ \lambda_{n\infty} = \varphi_n$, we have $\varphi_n(x) = 0$, so $x \leq q_n = (0_K, 1_{P_n})$. Consider the map $f = \varphi \circ \lambda_{n\infty}$: $Q_n \to K$. We claim that $f(q_n) = 0$, hence $f(x) = 0$. Indeed, as the principal ideal of Q_n generated by q_n is isomorphic to P_n, the principal ideal of K generated by $f(q_n)$ contains a homomorphic image of P_n as a subalgebra. But P_n is a non-trivial horizontal sum, hence simple, so any homomorphic image of P_n is either isomorphic to P_n or is a one-element OML. As R is a subalgebra of P_n, but not isomorphic to a subalgebra of any interval of K, we must have that the principal ideal of K generated by $f(q_n)$ is trivial, hence $f(q_n) = 0$ as required.

CLAIM 11. *If $P_n \cong P_m$, then $n = m$.*

Proof. Assume $n \leq m$. Proceed by induction on n. If $n = 1$, then P_1 is a horizontal sum of two irreducible summands, and for any $m > 1$, $P_m = R \oplus (K \times P_{m-1})$ is the horizontal sum of one irreducible and one reducible summand. Assume $n > 1$, hence $m > 1$. Then $P_n = K \times (R \oplus P_{n-1})$ and $P_m = K \times (R \oplus P_{m-1})$. Therefore $P_{n-1} \cong P_{m-1}$, and the result follows from the inductive hypothesis.

CLAIM 12. *If $n, m \in \mathbb{N}$ and P_n is isomorphic to the principal ideal of Q_m generated by $x \in Q_m$, then $n \leq m$ and $x = \lambda_{nm}(q_n)$.*

Proof. By induction on m. Assume $m = 1$. If P_n is isomorphic to the principal ideal generated by $(k, p) \in Q_1 = K \times (R \oplus H)$, then as P_1 is irreducible either $k = 0$ or $p = 0$. Then, as R is a subalgebra of P_n, but not isomorphic to a subalgebra of a principal ideal of either H or K, nor isomorphic to a subalgebra of a proper principal ideal of R (as R is of height three), we have $(k, p) = q_1$. Hence $P_n \cong P_1$, which implies $n = 1$.

Assume $m > 1$ and that the principal ideal of $Q_m = K \times (R \oplus Q_{m-1})$ generated by x is isomorphic to P_n. By the above reasoning, it follows that $x = (0, p)$ for some $p \in Q_{m-1}$. If $p \neq 1_{Q_{m-1}}$, then the principal ideal of Q_{m-1} generated by p is isomorphic to the principal ideal of Q_m generated by $(0, p)$, hence to P_n. Thus, we may apply the inductive hypothesis (with p in place of x) to obtain $n \leq m - 1$ and $p = \lambda_{n,m-1}(q_n)$, hence $(0, p) = \lambda_{nm}(q_n)$. If $p = 1_{Q_{m-1}}$, then $x = q_m$ and $P_n \cong P_m$, giving $n = m$.

CLAIM 13. *If $\mu \in Aut(Q)$, then μ restricts to an automorphism of $[0, \lambda_{n\infty}(q_n)]_Q$ for each $n \in \mathbb{N}$.*

Proof. Let $q = \lambda_{n\infty}(q_n)$ and suppose $\mu(q) = \lambda_{m\infty}(x)$ for some $m \in \mathbb{N}$ and $x \in Q_m$. Then as $\varphi(q) = \varphi_n(q_n) = 0$, Claim 10 gives $\varphi\mu(q) = \varphi_m(x) = 0$, hence $x \leq q_m$. Two applications of Claim 9 then give $[0, x]_{Q_m} \cong [0, \mu(q)]_Q \cong [0, q]_Q \cong [0, q_n]_{Q_n} \cong P_n$. Hence, by Claim 12, $x = \lambda_{nm}(q_n)$ and $\mu(q) = q$.

CLAIM 14. *If $\mu \in Aut(Q)$ restricts to the identity on $[0, \lambda_{1\infty}(q_1)]_Q$, then μ restricts to the identity on $[0, \lambda_{n\infty}(q_n)]_Q$ for each $n \in \mathbb{N}$.*

Proof. By Claims 9 and 13, for each $n \in \mathbb{N}$ the map $\mu_n = \lambda_{n\infty}^{-1} \circ \mu \circ \lambda_{n\infty}$ is an automorphism of $[0, q_n]_{Q_n}$. One easily checks that $\lambda_{n,n+1} \circ \mu_n = \mu_{n+1} \circ \lambda_{n,n+1}$ for each $n \in \mathbb{N}$. In view of Claim 9 it is enough to show $\mu_n = id$ for each $n \in \mathbb{N}$. We proceed by induction on n.

The claim assumes $\mu_1 = id$. Suppose $n > 1$. As $Q_n = K \times (R \oplus Q_{n-1})$, the interval $[0, q_n]_{Q_n}$ is equal to $\{0\} \times (R \oplus Q_{n-1})$. As R is rigid and $R \not\cong Q_{n-1}$, it follows that μ_n restricts to the identity on $\{0\} \times R$. It remains to show that μ_n restricts to the identity on $\{0\} \times Q_{n-1} = \{0\} \times (K \times P_{n-1})$. For any $p \in P_{n-1}$ we have $\mu_n(0, (0, p)) = \mu_n \lambda_{n-1,n}(0, p) = \lambda_{n-1,n}\mu_{n-1}(0, p)$, so the inductive hypothesis gives $\mu_n(0, (0, p)) = (0, (0, p))$. As the automorphism μ_n respects orthocomplements in $[0, q_n]_{Q_n}$, it follows that μ_n also fixes $(0, (1, 0))$, hence as K

is rigid, $\mu_n(0, (k, 0)) = (0, (k, 0))$ for each $k \in K$. Then for any $(k, p) \in Q_{n-1}$ we have $\mu_n(0, (k, p)) = \mu_n(0, (k, 0)) \vee \mu_n(0, (0, p)) = (0, (k, p))$.

CLAIM 15. *If $\mu \in Aut(Q)$ restricts to the identity on $[0, \lambda_{1\infty}(q_1)]_Q$, then $\mu = id_Q$.*

Proof. We first note that $q \in Q$ and $\varphi(q) = 0$ imply there is some $n \in \mathbb{N}$ with $q \in [0, \lambda_{n\infty}(q_n)]_Q$, so by the previous claim $\mu(q) = q$. For the general case, consider any element $q \in Q$ and choose n so that q and $\mu(q)$ are in $\lambda_{n\infty}(Q_n)$. As $Q_n = K \times P_n$, there are $k, \hat{k} \in K$ and $p, \hat{p} \in P_n$ with $q = \lambda_{n\infty}(k, p)$ and $\mu(q) = \lambda_{n\infty}(\hat{k}, \hat{p})$. Then $q = \lambda_{n+1,\infty}(k, (k, p))$ and $\mu(q) = \lambda_{n+1,\infty}(\hat{k}, (\hat{k}, \hat{p}))$. We obtain $\mu(q) = \mu\lambda_{n+1,\infty}(k, (k, p)) = \mu\lambda_{n+1,\infty}(k, (0, 0)) \vee \mu\lambda_{n+1,\infty}(0, (k, p)) = \mu\lambda_{n+1,\infty}(k, (0, 0)) \vee \lambda_{n+1,\infty}(0, (k, p))$, where the final equality follows as $\varphi\lambda_{n+1,\infty}(0, (k, p)) = 0$. Comparing the result with $\mu(q) = \lambda_{n+1,\infty}(\hat{k}, (\hat{k}, \hat{p}))$ we obtain the inequality $(0, (k, p)) \leq (\hat{k}, (\hat{k}, \hat{p}))$ giving $(k, p) \leq (\hat{k}, \hat{p})$. Thus $q \leq \mu q$ for all $q \in Q$, and as μ is an automorphism of an OML, $\mu = id_Q$.

We are now in a position to prove the final result required to establish the lemma.

CLAIM 16. *$\Lambda: Aut(H) \to Aut(Q)$ is onto.*

Proof. Let $\mu \in Aut(Q)$. By Claims 9 and 13, $\mu_1 = \lambda_{1\infty}^{-1} \circ \mu \circ \lambda_{1\infty}$ is an automorphism of $[0, q_1]_{Q_1} = \{0\} \times (R \oplus H)$. Thus, there is an automorphism $\alpha \in Aut(H)$ with $\mu_1(0, h) = (0, \alpha(h))$ for all $h \in H$. Then $\Lambda(\alpha)$ and μ agree on $[0, \lambda_{1\infty}(q_1)]_Q$, hence $\Lambda(\alpha) \circ \mu_1^{-1}$ restricts to the identity on this interval. By the previous claim $\Lambda(\alpha) \circ \mu^{-1} = id_Q$, hence $\Lambda(\alpha) = \mu$. □

6. Conclusion

We may combine the two lemmas to the following

THEOREM 6.1. *Let J be a nontrivial OML, B a nontrivial Boolean algebra and G a group. Then there is an OML L such that*

(1) $J \leq L$,
(2) $C(L) \cong B$,
(3) $Aut(L) \cong G$.

Proof. By [15] there is an OML M with $J \leq M$ and $Aut(M) \cong G$. By the second lemma, there are nontrivial OMLs Q, K and a homomorphism $\varphi: Q \to K$ satisfying (1) Q is irreducible, (2) K is rigid, (3) $J \leq M \leq Q$, (4) $Aut(Q) \cong Aut(M) \cong G$, and (5) $\varphi \circ \mu = \varphi$ for all $\mu \in Aut(Q)$. Then applying the first lemma, there is an OML L with (1) $J \leq Q \leq L$, (2) $C(L) \cong B$, and (3) $Aut(L) \cong Aut(Q) \cong G$. □

Taking into account the state spaces, we prove the following

THEOREM 6.2. *Let J be an OML admitting at least one state, B a nontrivial Boolean algebra, G a group and S a compact convex subset of a locally convex Hausdorff linear space. Then there is an OML L such that*

(1) $J \leq L$,
(2) $C(L) \cong B$,
(3) $Aut(L) \cong G$,
(4) $\mathcal{S}(L)$ *is affinely homeomorphic to* S.

Proof. According to Theorem 1.5, we may restrict attention to $B \ncong 2$ and consider it as a product $B = B_1 \times B_2$ of two nontrivial Boolean algebras. Applying Theorem 1.5 in the case of the trivial automorphism group, we construct an OML L_1 such that $J \leq L_1$, $C(L_1) \cong B_1$, L_1 is rigid, and $\mathcal{S}(L_1)$ is affinely homeomorphic to S.

Take a stateless OML I from Proposition 1.3, and a simple OML T which is not isomorphic to a subalgebra of an interval of L_1, and form the horizontal sum $J_2 = J \oplus I \oplus T$. We apply the previous theorem to J_2, B_2 and G and obtain an OML L_2 such that $J_2 \leq L_2$, $C(L_2) \cong B_2$, and $Aut(L_2) \cong G$. Note further that $\mathcal{S}(L_2) = \emptyset$ as L_2 contains a stateless subalgebra I. Further, the proof of the previous theorem shows that for each nonzero central element $c \in C(L_2)$, the interval $[0, c]_{L_2}$ contains a subalgebra isomorphic to a quotient of J_2, hence contains a subalgebra isomorphic to T.

Take the product $L = L_1 \times L_2$. We will verify conditions (1)–(4) of the theorem for L. As J is a subalgebra of both L_1 and L_2, we have $J \leq L$, hence condition (1). Also, $C(L) = C(L_1 \times L_2) = C(L_1) \times C(L_2) \cong B_1 \times B_2 = B$, giving condition (2). As L_2 is stateless, it follows that each state s on $L = L_1 \times L_2$ satisfies $s(x, y) = s(x, 0)$, hence there is an affine homeomorphism between $\mathcal{S}(L)$ and $\mathcal{S}(L_1)$, yielding condition (4). It remains only to verify condition (3).

We claim that $\alpha(1, 0) = (1, 0)$ for each $\alpha \in Aut(L)$, showing that each $\alpha \in Aut(L)$ is of the form $\alpha = \alpha_1 \times \alpha_2$ for some $\alpha_1 \in Aut(L_1)$ and some $\alpha_2 \in Aut(L_2)$, and hence providing condition (3). Considering inverses, it suffices to show $\alpha(1, 0) \leq (1, 0)$ for each $\alpha \in Aut(L)$. But T is not isomorphic to a subalgebra of an interval of $[(0, 0), (1, 0)]_L$ and is isomorphic to a subalgebra of an interval of $[(0, 0, (c, d)]_L$ for each central (c, d) with $d \neq 0$. □

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