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Epimorphisms in Certain Varieties of Algebras

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Abstract. We prove a lemma which, under restrictive conditions, shows that epimorphisms in V are surjective if this is true for epimorphisms from irreducible members of V. This lemma is applied to varieties of orthomodular lattices which are generated by orthomodular lattices of bounded height, and to varieties of orthomodular lattices which are generated by orthomodular lattices which are the horizontal sum of their blocks. The lemma can also be applied to obtain known results for discriminator varieties.

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1. Introduction

Let $V \subseteq W$ be varieties of algebras and $A, B \in V$. A homomorphism $f: A \to B$ is a W-epimorphism iff for every $C \in W$ and all homomorphisms $g, h: B \to C$, $g \circ f = h \circ f$ implies $g = h$. The concept applies to arbitrary categories, but we are only concerned here with the case of varieties. It is obvious that if f is surjective then it is a W-epimorphism. The question thus arises in any concrete case of structured sets whether epimorphisms are surjective. The question is generally not easy to answer and in many important cases the answer is not known. An excellent survey article concerning epimorphisms and other fundamental concepts in universal algebra can be found in [12].

The Main Lemma (Section 3) of this paper answers the question whether epimorphisms are surjective in several cases. Loosely speaking it says that, under many restrictive conditions, W-epimorphisms in V are surjective if this is true for W-epimorphisms from (directly) irreducible members of V.

The first application of our Main Lemma, Theorem 1 of Section 4, deals with discriminator varieties. The result is known, see [1]. We include the result here because it is also an easy consequence of our Main Lemma. The remaining applications deal with varieties of orthomodular lattices, abbreviated OMLs. The first application, Theorem 2 of Section 5, is concerned with varieties of OMLs which are generated by OMLs of bounded height. The last application, Theorem 3 of Section 6, deals with varieties of OMLs which are generated by OMLs which are the horizontal sums of their blocks.

The proof of our Main Lemma is divided into several smaller lemmata. In an earlier version of this paper, Lemma 1 was part of the assumptions of the Main Lemma. We are grateful to R. Willard for pointing out to us that Lemma 1 is a consequence of the remaining assumptions of our Main Lemma. The proof of Lemma 1 is his.

2. Preliminaries

A concept closely related to that of a W-epimorphism is that of an epic subalgebra. A subalgebra S of an algebra A is epic in a variety W containing A, in symbols $S \leq A$ epic in W, iff the inclusion map of S into A is a W-epimorphism, in other words, if maps g, h from A to some $C \in W$ which coincide on S are equal. It is an easy exercise to see that W-epimorphisms in V are surjective iff no $A \in V$ has a proper subalgebra which is epic in W.

A concept which has been used extensively before and will also play an important role in this paper is that of a factor pair and a factor congruence. A factor pair in an algebra A is a pair (θ, ψ) of congruence relations in A satisfying $\theta \cap \psi = \Delta_A$ and $\theta \circ \psi = \nabla_A$, where Δ_A is the diagonal of A^2 and $\nabla_A = A^2$. It is well known and easily seen that a pair (θ, ψ) of congruences in A is a factor pair iff the map f from A into the product $(A/\theta) \times (A/\psi)$ defined by $f(a) = (a/\theta, a/\psi)$ is an isomorphism. A factor congruence in A is a congruence θ for which there exists ψ such that (θ, ψ) is a factor pair. For an algebra A let $Con(A)$ (or $Con\ A$) be the lattice of all congruence relations in A and let $C(A)$ be the set of all factor congruences in A.

Some more notation. If θ is a congruence of L then κ_θ is the canonical map onto the quotient, if L is an algebra then $[L]$ is the variety generated by L, and if p is a term and L an algebra then p_L is the polynomial in L induced by p.

3. The Main Lemma

Let $U \subseteq V \subseteq W$ be varieties and let p, q be n-ary terms such that

(1) *V is congruence permutable and for every $L \in V$ the set $C(L)$ of all factor congruences of L is a distributive (and hence Boolean) sublattice of $Con(L)$.*
(2) *Directly irreducibles in V are simple.*
(3) *No $L \in U$ has a proper epic subalgebra in W.*
(4) *No irreducible $L \in V$ has a proper epic subalgebra in W.*
(5) *$U = \{L \in V : p = q$ holds in $L\}$.*
(6) *If $L \in V$, $a \in L^n$ then the pair $(p_L(a), q_L(a))$ generates a factor congruence in L.*

(7) *If* $L \in V$ *then there exists* $a \in L^n$ *such that* $p_L(a) = q_L(a)$.

Under these assumptions no $L \in V$ *has a proper epic subalgebra in* W.

We divide the proof in a sequence of lemmata.

LEMMA 1. *If* (θ, θ^*) *is a factor pair in* $L \in V$ *and if* ϕ *is a completely meet-irreducible congruence in* L, *then* $\theta \subseteq \phi$ *or* $\theta^* \subseteq \phi$.

Proof (Willard). By assumption V is congruence permutable and for each $A \in V$ the set $C(A)$ of all factor congruences of A is a Boolean sublattice of $Con(A)$. It follows that the five element modular, non-distributive lattice M_3 is not a bounded sublattice of $Con(A)$, hence, by exercise 5.2, p. 55 of [7], V does not contain non-trivial Abelian algebras. Therefore by Theorem 5.8, p. 85 of [7], for each $A_1, A_2 \in V$ and each $\psi \in Con(A_1 \times A_2)$, $\psi = (\ker \pi_1 \vee \psi) \wedge (\ker \pi_2 \vee \psi)$. As (θ, θ^*) is a factor pair of L, $L \cong L/\theta \times L/\theta^*$, hence $\phi = (\theta \vee \phi) \wedge (\theta^* \vee \phi)$. As ϕ is completely meet irreducible, the result follows. □

LEMMA 2. *If* $L \in V$, $S \leq L$ *epic in* $[L]$ *then the only congruence* θ *in* L *satisfying* $\theta \mid S = \nabla_S$ *is* ∇_L.

Proof. Assume $\theta \in Con\, L$ and $\theta \mid S = \nabla_S$. Then it follows from one of the isomorphism theorems that $\kappa_\theta(S)$ is a one element subalgebra $\{a\}$ of L/θ. Thus the map $f\colon L \to \{a\}$ is a homomorphism and $f \mid S = \kappa_\theta(S)$. Since $S \leq L$ epic in $[L]$ it follows that $f = \kappa_\theta$ and hence that $\theta = \nabla_L$. □

If $S \leq L$ and if $\theta \in Con\, S$ let $\overline{\theta}$ be the congruence in L generated by θ.

LEMMA 3. *If* $L \in V$, $S \leq L$ *epic in* W *and if* (θ, θ^*) *is a factor pair in* S *then* $(\overline{\theta}, \overline{\theta^*})$ *is a factor pair in* L.

Proof. Let ϕ be a completely meet-irreducible congruence in L and $\psi = \phi \mid S$. Then $\kappa_\phi(S) \leq L/\phi$ is epic in W, hence, since L/ϕ us irreducible, $\kappa_\phi(S) = L/\phi$ and $S/\psi \cong L/\phi$. Thus S/ψ is s.i. and it follows from Lemma 1 that $\theta \subseteq \psi$ or $\theta^* \subseteq \psi$, hence $\overline{\theta} \subseteq \phi$ or $\overline{\theta^*} \subseteq \phi$, hence $\overline{\theta} \cap \overline{\theta^*} \subseteq \phi$. Since the intersection of all completely meet-irreducible congruences is Δ_L, it follows that $\overline{\theta} \cap \overline{\theta^*} = \Delta_L$. But $\nabla_S = \theta \vee \theta^* \subseteq (\overline{\theta} \vee \overline{\theta^*}) \mid S$, thus, by Lemma 2, $\overline{\theta} \vee \overline{\theta^*} = \nabla_L$, which implies the claim since we are in a congruence permutable variety. □

LEMMA 4. *If* $L \in V$, $S \leq L$ *epic in* W *then the map* $\theta \rightsquigarrow \overline{\theta}$ *is a Boolean algebra embedding of* $C(S)$ *into* $C(L)$.

Proof. By Lemma 3 the map $\theta \rightsquigarrow \overline{\theta}$ maps $C(S)$ into $C(L)$ and preserves complements. To show that it preserves meets assume that $\phi, \psi \in C(S)$. Then $\phi \cap \psi \subseteq \overline{\phi} \cap \overline{\psi}$ and hence $\overline{\phi \cap \psi} \subseteq \overline{\phi} \cap \overline{\psi}$. To show equality it is enough to show that $\overline{(\phi \cap \psi)} \vee (\overline{\phi} \cap \overline{\psi})^* = \nabla_L$. But $\overline{(\phi \cap \psi)} \vee (\overline{\phi} \cap \overline{\psi})^* = \overline{(\phi \cap \psi)} \vee \overline{\phi}^* \vee \overline{\psi}^* = \overline{(\phi \cap \psi)} \vee \overline{\phi^*} \vee \overline{\psi^*} \supseteq (\phi \cap \psi) \vee \phi^* \vee \psi^* = (\phi \cap \psi) \vee (\phi \cap \psi)^* = \nabla_S$. Equality thus holds by Lemma 2 and hence our map is a Boolean algebra homomorphism. But $\overline{\theta} = \Delta_L$ clearly implies $\theta = \Delta_S$, proving our map is an embedding. □

LEMMA 5. *If $L \in V$, $S \leq L$ epic in W and $\theta \in C(S)$ then $\overline{\theta} \mid S = \theta$.*

Proof. Let (θ, θ^*) be a factor pair in S. Then, by Lemma 3, $(\overline{\theta}, \overline{\theta^*})$ is a factor pair in L, in particular $\overline{\theta} \cap \overline{\theta^*} = \Delta_L$. Thus $(\overline{\theta} \mid S) \cap \theta^* \subseteq (\overline{\theta} \cap \overline{\theta^*}) \mid S = \Delta_S$. It follows that θ and $\overline{\theta} \mid S$ are complements of θ^* in $Con\, S$. It follows from congruence modularity that they are equal. □

NOTATION. If $S \leq L$ and if I is a prime ideal in $C(S)$ define $\overline{I} = \{\overline{\theta} : \theta \in I\}$.

LEMMA 6. *If $L \in V$, $S \leq L$ epic in W and if I is a prime ideal in $C(S)$ then $\bigcup I$ is a congruence in S, $\bigcup \overline{I}$ is a congruence in L and $S/\bigcup I \cong S/\bigcup \overline{I}$. (Here $S/\bigcup \overline{I}$ is defined to be $\{s/\bigcup \overline{I} : s \in S\}$.)*

Proof. Since $I, \overline{I}$ are up-directed their unions are trivially congruences. The map $s \rightsquigarrow s/\bigcup \overline{I}$ is clearly a homomorphism of S onto $S/\bigcup \overline{I}$ and the kernel of it is $\bigcup I$ by Lemma 5, proving the last claim. □

LEMMA 7. *If $L \in V$ and if I is a prime ideal in $C(L)$ then $L/\bigcup I$ is irreducible or belongs to U.*

Proof. Assume $A = L/\bigcup I \notin U$. We have to show that A is irreducible, or equivalently, has no non-trivial factor pair. But $Con(L/\bigcup I)$ is canonically isomorphic with the interval $[\bigcup I, \nabla_L]$ of $Con\, L$ and under this isomorphism permuting congruences correspond to permuting congruences. Thus to every factor pair in A there exist congruences θ_1, θ_2 in L such that $\theta_1 \cap \theta_2 = \bigcup I$ and $\theta_1 \circ \theta_2 = \nabla_L$. As $A \cong (L/\theta_1) \times (L/\theta_2) \notin U$ we may assume w.l.o.g. that $L/\theta_1 \notin U$. Then, by assumption (5), there exist $a_1, \dots, a_n \in L$ such that $p_L(a_1, \dots, a_n)/\theta_1 \neq q_L(a_1, \dots, a_n)/\theta_1$ and by assumption (7) there exist $b_1, \dots, b_n \in L$ such that $p_L(b_1, \dots, b_n) = q_L(b_1, \dots, b_n)$. Since $(a_i, b_i) \in \nabla_L = \theta_1 \circ \theta_2$ there exist $c_1 \in L$ such that $a_i \theta_1 c_i \theta_2 b_i$. We thus have

$$\begin{aligned} p_L(c_1, \dots, c_n)/\theta_1 &= p_L(a_1, \dots, a_n)/\theta_1 \\ &\neq q_L(a_1, \dots, a_n)/\theta_1 \\ &= q_L(c_1, \dots, c_n)/\theta_1, \\ p_L(c_1, \dots, c_n)/\theta_2 &= p_L(b_1, \dots, b_n)/\theta_2 \\ &= q_L(b_1, \dots, b_n)/\theta_2 \\ &= q_L(c_1, \dots, c_n)/\theta_2. \end{aligned}$$

ϕ be the congruence generated by $(p_L(c_1, \dots, c_n), q_L(c_1, \dots, c_n))$. By assumption (6), ϕ is a factor congruence. Clearly $\phi \not\subseteq \theta_1$, hence $\phi \notin I$, hence $\phi^* \in I$, hence $\phi^* \subseteq \theta_2$. But by construction $\phi \subseteq \theta_2$. Thus $\theta_2 = \nabla_L$. □

LEMMA 8. *If $L \in V$, $S \leq L$ epic in W and if I is a prime ideal in $C(L)$ then $S/\bigcup I = L/\bigcup I$.*

Proof. We have $S/\bigcup I \leq L/\bigcup I$ epic in W and by Lemma 7 $L/\bigcup I$ is either irreducible or belongs to U. In the first case the claim follows from assumption (4), in the second case from assumption (3). □

LEMMA 9. *If $L \in V$, $S \leq L$ epic in W and $S \in U$ then $S = L$.*

Proof. By assumption (3) it is enough to show that $L \in U$. Assume not. Then, by assumption (5), there would exist $a \in L^n$ such that $p_L(a) \neq q_L(a)$. Let θ be the congruence in L generated by $(p_L(a), q_L(a))$. By assumption (6) θ is a factor congruence. Clearly $\theta \neq \Delta_L$ and hence $\theta^* \neq \nabla_L$. Thus there exists a prime ideal I in $C(L)$ with $\theta^* \in I$. But $(p_L(a), q_L(a)) \in \bigcup I$ would imply the existence of $\phi \in I$ such that $(p_L(a), q_L(a)) \in \phi$, hence $\theta \subseteq \phi$ and we would obtain $\theta, \theta^* \in I$, a contradiction. Thus $(p_L(a), q_L(a)) \notin \bigcup I$ and hence the equation $p = q$ is not valid in $L/\bigcup I$. But by Lemma 8, $S/\bigcup I = L/\bigcup I$. Thus the equation would not be valid in $S/\bigcup I$ and hence not in S, contradicting the assumption (5). Thus $L \in U$. □

LEMMA 10. *If $L \in V$, $S \leq L$ epic in W and if S is irreducible then $S = L$.*

Proof. By assumption (4) it is enough to show that L is irreducible. To do this let θ, ϕ be completely meet-irreducible congruences in L. Then $\kappa_\theta(S) \leq L/\theta$ epic in W and since L/θ is irreducible we obtain from assumption (4) that $\kappa_\theta(S) = L/\theta$. Thus $\kappa_\theta \mid S$ is a map of S onto L/θ. But S is irreducible and hence, by assumption (2), simple. It follows that $\kappa_\theta \mid S$ is an isomorphism between S and L/θ. Thus $\kappa_\phi \circ (\kappa_\theta \mid S)^{-1} \circ \kappa_\theta$ and κ_ϕ are homomorphisms of L into L/ϕ and they obviously coincide on S. Since $S \leq L$ epic in W they are equal. Assume now that $(a, b) \in \theta$. Then

$$\kappa_\phi(a) = (\kappa_\phi \circ (\kappa_\theta \mid S)^{-1} \circ \kappa_\theta)(a) = (\kappa_\phi \circ (\kappa_\theta \mid S)^{-1} \circ \kappa_\theta)(b) = \kappa_\phi(b)$$

and hence $(a, b) \in \phi$. Thus $\theta \subseteq \phi$ and, by symmetry, $\theta = \phi$. Thus there is only one completely meet-irreducible congruence in L. Since the meet of all completely meet-irreducible congruences is Δ_L it follows that Δ_L is completely meet irreducible, hence that L is s.i. and hence irreducible. □

LEMMA 11. *If $L \in V$, $S \leq L$ epic in W and if I is a prime ideal in $C(S)$ then $S/\bigcup \overline{I} = L/\bigcup \overline{I}$.*

Proof. Using Lemma 6 we have $S/\bigcup I \cong S/\bigcup \overline{I} \leq L/\bigcup \overline{I}$ epic in W. By Lemma 7 $S/\bigcup I$ is then irreducible or belongs to U and hence the same is true for $S/\bigcup \overline{I}$. The claim then follows from Lemmata 9 and 10. □

We are now in a position to prove the Main Lemma. Assume $L \in V$, $S \leq L$ epic in W and $a \in L$. We have to show that $a \in S$. Let Σ be the set of all prime ideals in $C(S)$ and for $\theta \in C(S)$ define $\beta(\theta) = \{I : \theta \in I \in \Sigma\}$. Define furthermore

$$\Omega = \{\theta \in C(S) : \text{ there exists } s \in S \text{ with } (a, s) \in \overline{\theta}\}.$$

We show first

($\star$) If $\theta_1, \theta_2 \in \Omega$ then $\theta_1 \cap \theta_2 \in \Omega$.

If $\theta_1, \theta_2 \in \Omega$ there exist $s_1, s_2 \in S$ such that $(a, s_i) \in \overline{\theta_i}$, hence $(s_1, s_2) \in \overline{\theta_1} \vee \overline{\theta_2} = \overline{\theta_1 \vee \theta_2}$ and, by Lemma 5, $(s_1, s_2) \in \theta_1 \vee \theta_2 = \theta_1 \circ \theta_2$. Thus, there exists $s \in S$ such that $(s_i, s) \in \theta_i$. But this clearly implies that $(a, s) \in \overline{\theta_1} \cap \overline{\theta_2} = \overline{\theta_1 \cap \theta_2}$ and hence $\theta_1 \cap \theta_2 \in \Omega$, proving ($\star$).

Now assume $I \in \Sigma$. By Lemma 11 there exists $s \in S$ such that $(a, s) \in \bigcup \overline{I}$ and hence there exists $\theta \in I$ such that $(a, s) \in \overline{\theta}$. Thus $I \in \beta(\theta)$ for some $\theta \in \Omega$. It follows that the sets $\beta(\theta)$ with $\theta \in \Omega$ form an open cover of the Stone space of $C(S)$. By compactness there exist $\theta_1, \dots, \theta_n$ such that $\Sigma = \beta(\theta_1) \cup \cdots \cup \beta(\theta_n) = \beta(\theta_1 \cap \cdots \cap \theta_n)$. By ($\star$) we have $\theta = \theta_1 \cap \cdots \cap \theta_n \in \Omega$, hence $\Sigma = \beta(\theta)$, which obviously implies that $\theta = \Delta_S$. But by definition of Ω there exists $s \in S$ with $(a, s) \in \overline{\theta} = \Delta_L$, hence $a = s \in S$, proving the Main lemma.

4. Discriminator Varieties

As was pointed out in the introduction the following Theorem 1 is known. It is, in slightly different formulation, contained in part (ii) of Corollary 3 of [1]. We show here that, modulo well known results concerning discriminator varieties, it is also a consequence of our Main Lemma.

THEOREM 1. *If V is a discriminator variety and if no irreducible member of V has a proper epic subalgebra in V then no member of V has a proper epic subalgebra in V.*

Proof. We apply our lemma with $W = V$ and U the trivial variety. Assumption (1) is a consequence of the well known result fact that discriminator varieties are arithmetical, see [5], Theorem 9.4, p. 165 or [10], Theorem 2.4, p. 388. Assumption (2) is also part of Theorem 9.4 of [5]. Assumption (3) is trivial and assumption (4) of the Main Lemma is explicitly assumed in the theorem. Define $p(x, y) = x, q(x, y) = y$. Assumptions (5) and (7) are obvious and assumption (6) is Theorem 5.6 of [10]. □

Comer [6] has given an example of a discriminator variety V in which epimorphisms are not onto. H. Werner [14] points out in his perceptive review of Comer's paper that in Comer's example epimorphisms from simple algebras are not onto. Theorem 1 shows that there are no other examples.

5. Varieties Generated by OMLs of Bounded Height

THEOREM 2. *Let $V \subseteq W$ be a varieties. If V is generated by OMLs of height at most n and no irreducible member of V has a proper epic subalgebra in W, then no member of V has a proper epic subalgebra in W.*

The proof is divided in a sequence of lemmata based on notions developed in [8]. We briefly recall a few facts from [8]. A partial matrix in an OML L is a rectangular matrix $M = (m_{ij})$ whose entries are elements of L. It is not required that each cell of M has an entry. The size $N(M)$ of the matrix is the finite sequence $\langle n_1, \dots, n_r \rangle$

of natural numbers where r is the number of rows of M and n_i is the number of entries in the i^{th} row of M. A partial matrix is called admissible if it satisfies a certain technical condition described on p. 557 of [8]. Upon inspection of these conditions it is clear that every OML has at least one admissible partial matrix. The following is contained in Lemma 1 of [8].

LEMMA 12. *If K is a non-empty class of irreducible OMLs of height at most n then $\{N(R) : R$ is admissible in some $L \in K\}$ is a finite chain under the lexicographical ordering on sequences of natural numbers.*

Denote the maximum of this chain by $S(K)$, or by $S(L)$ if $K = \{L\}$.

LEMMA 13. *Suppose $\alpha = \langle n_1, \ldots, n_r\rangle$ is the maximal size of an admissible partial matrix in some irreducible OML of height at most n. There is an $(n_1 + \cdots + n_r + 1)$-ary ortholattice term $h_\alpha(\vec{x}, y)$ with the following property. If L is irreducible of height at most n with $S(L) \leq \alpha$, then for M a partial matrix in L of size α and $z \in L$*

$$h_\alpha(M, z) = \begin{cases} 1 & \textit{if } M \textit{ is admissible and } z \neq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Set $h_\alpha(\vec{x}, y) = p_\alpha(\vec{x}, y) \wedge t_\alpha(\vec{x})$ where p_α is the term given by Lemma 2 of [8] and t_α is the term given by Lemma 6 of [8]. Note that these terms are constructed solely from α, n and the results proved in [8] apply equally to any class K of irreducible OMLs of height at most n with $S(K) = \alpha$. In particular they apply to the class $K = \{L, Q\}$ where Q is some irreducible OML of height at most n with $S(Q) = \alpha$. □

Let V be a variety generated by OMLs of height at most n and let K be the s.i. members of V. By Jónsson's lemma [9] K is a class of irreducible OMLs of height at most n. Consider the set

$$\{N(R) : R \text{ is admissible of maximal size in some } L \in K\}.$$

By Lemma 12 this is a finite chain, say $\alpha_1 < \cdots < \alpha_k$. For $0 \leq i \leq k$ define

$$K_i = \{L \in K : h_{\alpha_j} = 0 \text{ holds in } L \text{ for each } i \leq j \leq k\}$$

and let V_i be the variety generated by K_i.

LEMMA 14. $V_0 \subseteq \cdots \subseteq V_k$. *Further V_0 is the trivial variety and $V_k = V$.*

Proof. It is enough to show $\emptyset = K_0 \subseteq \cdots \subseteq K_k = K$. Let $L \in K$. As every OML has an admissible partial matrix, choose an admissible partial matrix M in L of maximal size. Then $N(M) = \alpha_j$ for some j. By Lemma 13 L does not satisfy $h_{\alpha_j} = 0$. Hence $K_0 = \emptyset$. Trivially $K_i \subseteq K_{i+1}$ and $K_k = K$. □

LEMMA 15. $V_i = \{L \in V : h_{\alpha_j} = 0$ *holds in L for each* $i \leq j \leq k\}$.

Proof. As V_i is generated by a subset of the right side, we have containment from left to right. But the right side is a variety whose s.i. members are contained in the left side. □

LEMMA 16. *If* $1 \le i$ *and* $L \in V_i$ *is s.i. then* $S(L) \le \alpha_i$.

Proof. Otherwise $S(L) = \alpha_j$ for some $i < j$. Then for M an admissible partial matrix in L of size α_j, Lemma 13 gives $h_{\alpha_j}(M, 1) = 1$, contrary to Lemma 15. □

LEMMA 17. *For* $1 \le i$, *each* $L \in V_i$ *satisfies* $\gamma(h_{\alpha_i}(\vec{x}, y), z) = 0$.

Proof. It is enough to show each s.i. $L \in V_i$ satisfies this equation and this is a trivial consequence of Lemma 13 and Lemma 16. □

We are now able to prove Theorem 2. By induction on i we show no algebra in V_i has a proper epic subalgebra in W. For $i = 0$ this is obvious as V_0 is the trivial variety. Assume no algebra in V_i has a proper epic subalgebra in W, we show the same is true of V_{i+1}. Apply the Main Lemma to $V_i \subseteq V_{i+1} \subseteq W$, with the term $h_{\alpha_{i+1}}$ and the constant term 0. Seven assumptions must be verified.

Assumption (1) is true of any variety of OMLs. Assumption (2) follows from Theorem 1 of [8] as V_{i+1} is generated by a class of OMLs of height at most n. Assumption (3) is the inductive hypothesis and assumption (4) is an assumption of the theorem being proved. Assumption (5) follows from Lemma 15. If $L \in V_{i+1}$ then, by Lemma 17, the range of $h_{\alpha_{i+1}}$ is contained in the centre of L, and assumption (6) follows. If $L \in V_{i+1}$ is s.i. then by Lemma 16 $S(L) \le \alpha_{i+1}$, hence, by Lemma 13, L satisfies $h_{\alpha_{i+1}}(\vec{x}, 0) = 0$. Assumption (7) follows.

COROLLARY 1. *If* V *is a variety generated by a finite number of finite OMLs there is an effective procedure to determine if any member of* V *has a proper epic subalgebra in* V.

Proof. By Theorem 2 it is enough to see if an irreducible member of V can have a proper epic subalgebra in V. By Theorem 1 of [8] each irreducible member of V is simple, so by Jónsson's lemma there are only finitely many irreducible members of V. Suppose $L \in V$ is irreducible. If $S \le L$ is a proper epic subalgebra in V then there is some $M \in V$ and maps $f, g: L \to M$ which agree on S but are not equal. But M is a subdirect product of s.i. members of V, so there is a s.i. M' and maps $f, g: L \to M'$ which agree on S but are not equal. Thus it suffices to determine whether one of the finitely many irreducible members in V has a proper epic subalgebra in the finite set of irreducible members of V. □

One can easily check that the variety generated by MO2 has no proper epic subalgebras, and that the variety generated by the orthomodular house (the five loop) has proper epic subalgebras. However, the orthomodular house clearly has no proper epic subalgebras in the variety of all OMLs. Using Theorem 2 this last observation can be considerably strengthened.

COROLLARY 2. *Let V be the variety generated by the class of all OMLs of height at most three. Then no member of V has a proper epic subalgebra in V.*

Proof. Let $L \in V$ be irreducible. We must show L has no proper epic subalgebra in V. As each irreducible member of V is simple, see Theorem 1 of [8], it follows from Jónsson's lemma that L has height at most three. Assume $S \leq L$ is a proper subalgebra. Let L_1, L_2 be two copies of L with $L_1 \cap L_2 = S$, let $M = L_1 \cup L_2$ and let $\leq$ be the union of the partial orderings on L_1, L_2.

If $x \leq_1 y \leq_2 z$, then $y \in S$, hence, as L has height at most three, either $x \in S$ or $z \in S$. It follows that $\leq$ is transitive, hence a partial ordering. If $x, y \in L_i$ then $x \vee_i y$ is the join of x, y in M, and if $x \in L_1 \setminus L_2$, $y \in L_2 \setminus L_1$ then the join of x, y in M is the least member of S above each. Therefore M is a lattice. As $x < y$ in M implies $x, y \in L_i$ for some $i = 1, 2$, it follows that $y \wedge x' \neq 0$, hence M is orthomodular. The obvious isomorphisms $\varphi_i \colon L \to L_i$ for $i = 1, 2$ are maps from L into M which agree on S. As $S \leq L$ is proper, these maps are not equal. □

The following observation shows no direct comparison can be made between Theorems 1 and 2.

PROPOSITION 1. *The only varieties of OMLs that are discriminator varieties are the trivial variety and the variety of all Boolean algebras.*

Proof. It is well known that any variety of OMLs other than the trivial variety and the variety of all Boolean algebras contains MO2. As MO2 is simple and has a subalgebra which is not simple no variety which contains MO2 can be a discriminator variety, see Theorem 9.4 [5]. □

6. Varieties Generated by OMLs which Are the Horizontal Sum of their Blocks

PROPOSITION 2. *If an OML L is the horizontal sum of its blocks then L has no proper epic subalgebras in the variety it generates.*

Proof. If S is a proper subalgebra of L then there exists a block B of L such that $S \cap B$ is a proper subalgebra of B. Assume first that B is finite. Then there exists an atom a of $S \cap B$ which is not an atom of B and hence contains at least two atoms of B. Let f be an isomorphism of L which permutes the atoms of B underneath a and maps the rest of L identically. Clearly f and the identity map of L coincide on S but are not equal. Assume next that B is infinite. Then, since Boolean algebras have the strong amalgamation property, there exist a Boolean algebra C and Boolean embeddings α, β of B into C which satisfy $\alpha(x) = \beta(x)$ exactly for the elements $x \in S \cap B$. Let M be the OML obtained from L by replacing the block B by C. Since L and M have (up to isomorphism) the same finitely generated (and hence finite) subalgebras, L and M generate the same variety. Let f, g be maps of L into M defined by

$$f(x) = \begin{cases} \alpha(x) & \text{if } x \in B, \\ x & \text{otherwise,} \end{cases} \qquad g(x) = \begin{cases} \beta(x) & \text{if } x \in B, \\ x & \text{otherwise.} \end{cases}$$

Clearly f and g coincide on S but are not equal. Thus S is not an epic subalgebra of L in [L]. □

THEOREM 3. *Let V be a variety generated by OMLs which are the horizontal sum of their blocks. Then no member of V has a proper epic subalgebra in V.*

Proof. We apply our Main Lemma with U the variety of all Boolean algebras, $W = V$, $p(x, y) = \gamma(x, y) = (x \vee y) \wedge (x \vee y') \wedge (x' \vee y) \wedge (x' \vee y')$, the commutator of x and y, and $q(x, y) = 0$. We show that all assumptions of the Main Lemma are satisfied. Assumption (1) is well known for OMLs in general, see [11], p. 83. The variety V satisfies $\gamma(x, \gamma(y, z)) = 0$, see Theorem 2 of [4]. Then for all $a, b \in L \in V$, $\gamma(a, b)$ is central in L. It follows that if L is irreducible then $\gamma(a, b) \in \{0, 1\}$, which implies that L is the horizontal sum of its blocks. This is well known to imply that L is simple, proving assumption (2) and, in view of Proposition 2, assumption (4). Assumption (3) is a consequence of the well known fact that no Boolean algebra has a proper epic subalgebra in the variety of all Boolean algebras. Assumption (5) is the well known fact that the equation $\gamma(x, y) = 0$ characterizes Boolean algebras among OMLs. Assumption (6) follows as $\gamma(a, b)$ is central for all $a, b \in L \in V$. Finally, assumption (7) is satisfied since $\gamma(a, a) = 0$ in any OML. This proves Theorem 3. □

7. Epic Subalgebras in the Variety of All OMLs

While we are unable to determine whether an OML can have a proper epic subalgebra in the variety of all OMLs, we can demonstrate the following partial result.

THEOREM 4. *Every OML can be embedded into an OML that has no proper epic subalgebras in the variety it generates.*

One step in the proof requires a result that, thus far, does not appear in the literature [13]. A proof of this result, which should be credited to Bruns, Greechie, Kalmbach, and Schröder, is provided through a sequence of lemmata.

LEMMA 18. *Given an OML L and $x \in L$ there is an OML $L(x)$ such that*

(i) $L \leq L(x)$,
(ii) *each atom of L is an atom of $L(x)$, and*
(iii) *x is a join of two or fewer atoms in $L(x)$.*

Proof. If x is either 0 or an atom of L simply set $L(x) = L$. Otherwise use Greechie's paste job to paste L and $([0, x'] \cup [x, 1) \times 2$ along the sections $[0, x'] \cup [x, 1]$ and $([0, x'] \times \{0\}) \cup ([x, 1] \times \{1\})$. □

The construction in the above proof is Kalmbach's coatom extension [11, p. 310], its simplified form is due to Roddy.

LEMMA 19. *Given an OML L there is an OML L^* such that*

(i) $L \leq L^*$,
(ii) *each atom of L is an atom of L^*, and*
(iii) *each element of L is a join of two or fewer atoms of L^*.*

Proof. Let $(x_\alpha)_\kappa$ be an indexing over a cardinal κ of L. Define recursively $L_0 = L$, $L_{\alpha+1} = L_\alpha(x_{\alpha+1})$, and $L_\alpha = (\bigcup_{\beta<\alpha} L_\alpha)(x_{\alpha+1})$ for α a limit ordinal. Set $L^* = L_\kappa$. □

LEMMA 20. *Given an OML L there is an OML $\hat{L}$ such that*

(i) $L \leq \hat{L}$, *and*
(ii) *each element of $\hat{L}$ is the join of two or fewer atoms of $\hat{L}$.*

Proof. Define recursively $L^0 = L$, $L^{n+1} = (L^n)^*$. Set $\hat{L} = \bigcup_n L^n$. □

We are now situated to prove Theorem 4. Assume L is an OML and $(a_\alpha)_\kappa$ is an indexing of the atoms of $\hat{L}$. For each $\alpha \in \kappa$ let M_α be a copy of $MO2 \times 2$ with c_α the central atom of M_α. Let P be the OML formed by pasting each M_α to $\hat{L}$ along the atoms a_α and c_α. We claim P has no proper epic subalgebra in the variety it generates. Assume $S \leq P$ is a proper subalgebra. As each element of $\hat{L}$ is a join of two or fewer atoms, $S \leq P$ proper implies $M_\alpha \not\subseteq S$ for some $\alpha \in \kappa$. As any proper subalgebra of M_α is contained in one of the two blocks of M_α, there is a non-trivial automorphism φ of M_α which fixes c_α and $M_\alpha \cap S$. Let φ' be the automorphism of P which agrees with φ on M_α and with the identity map elsewhere. Then φ' and the identity map on P agree on S, hence S is not an epic subalgebra in the variety generated by P.

References

1. Bergman, C. (1987) Saturated algebras in filtral varieties, *Algebra Universalis* **24**, 101–110.
2. Bruns, G., Greechie, R., Harding, J. and Roddy, M. (1990) Completions of orthomodular lattices, *Order* **7**, 67–76.
3. Bruns, G. and Harding, J. (1998) Amalgamation of ortholattices, *Order* **14**, 193–209.
4. Bruns, G. and Kalmbach, G. (1971) Varieties of orthomodular lattices, *Canad. J. Math.* **23**, 802–810.
5. Burris, S. and Sankappanavar, H. P. (1981) *A Course in Universal Algebra*, Springer-Verlag, New York.
6. Comer, S. D. (1983) Epimorphisms in discriminator varieties, *Lectures in Universal Agebra*, Szeged, 41–48.
7. Freese, R. and McKenzie, R. (1987) *Commutator Theory in Congruence Modular Varieties*, Cambridge University Press.
8. Harding, J. (1992) Irreducible orthomodular lattices which are simple, *Algebra Universalis* **29**, 556–563.
9. Jónsson, B. (1967) Algebras whose congruence lattices are distributive, *Math. Scand.* **21**, 110–121.

10. Jónsson, B. (1995) Congruence distributive varieties, *Math. Japon.* **42**, 353–401.
11. Kalmbach, G. (1983) *Orthomodular Lattices*, Academic Press.
12. Kiss, E. W., Márki, L., Pröhle, P. and Tholen, W. (1983) Categorical algebraic properties. Compendium on amalgamation, congruence extension, epimorphisms, residual smallness, and injectivity, *Studia Sci. Math. Hungar.* **18**, 79–141.
13. Schröder, B. Embedding orthomodular lattices into atomistic orthomodular lattices, Manuscript.
14. Werner, H. Mathematical Reviews MR 87m:08012.