Every lattice with 1 and 0 is embeddable in the lattice of topologies of some set by an embedding which preserves the 1 and 0

John Harding [*], Alex Pogel

New Mexico State University, Las Cruces, NM 88001-8003, USA

Abstract

We prove the result of the title, solving an open problem of Steven Watson (problem 172 in Open Problems in Topology). © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Lattice of topologies; Complemented lattice

AMS classification: 54A10; 06C15

From the axioms defining a topological space, it is trivial to show that the collection *Top X* of all topologies on a set *X* is closed under intersections, hence forms a complete lattice under set inclusion. The largest and smallest elements of this lattice (henceforth called the 1 and 0) are the discrete and indiscrete topologies respectively. While it has long been known that every lattice can be homomorphically embedded into the lattice *Top X* for some set *X* [4, Theorem 1.9], Watson asked whether the result could be strengthened as follows:

Problem. Can every lattice with 1 and 0 be homomorphically embedded in the lattice of topologies on some set?

This problem is listed as Problem 104 of Watson's article *Problems I wish I could solve* [6], which appeared in the book *Open Problems in Topology* (the problem is the 172nd problem in the book). After stating the problem he indicates that it is *"the most important problem in this section"*, and clearly explains that the required embedding must map the 1 and 0 of the given lattice to the 1 and 0 of the lattice of topologies.

---

[*] Corresponding author. E-mail: jharding@nmsu.edu.

Theorem. *Every lattice with* 1 *and* 0 *can be homomorphically embedded into the lattice of topologies of some set* X *by an embedding which maps the* 1 *and* 0 *of the given lattice to the* 1 *and* 0 *of Top* X.

Proof. Given any set X, the collection *Eq* X of all equivalence relations on X is closed under intersections, hence forms a complete lattice under set inclusion $\subseteq$. By the dual of *Eq* X we shall mean the lattice formed by taking $\supseteq$ as the partial ordering of the equivalence relations on X. Consider the map which associates to each equivalence relation θ on X the topology on X generated by taking the blocks of θ as basic open sets. It is well known [4, Theorem 1.9], and easy to verify, that this map homomorphically embeds the dual of *Eq* X into *Top* X and further maps the 1 and 0 of the dual of *Eq* X to the 1 and 0 of *Top* X, respectively.

By considering duals, it will suffice to show that each lattice has an appropriate homomorphic embedding into a lattice *Eq* X for some set X. If one does not care about preserving 1 and 0, as in [4, Theorem 1.9], this result is exactly Whitman's Theorem [7]. To find such an embedding that does preserve 1 and 0 we use the celebrated Grätzer–Schmidt Theorem [3] which states that every algebraic lattice is isomorphic to the congruence lattice of some general algebra.

Suppose L is a lattice with largest and least elements 1 and 0. Then L can be homomorphically embedded into the lattice of all non-empty ideals $\mathcal{I}(L)$ of L [1] by the map taking an element $x \in L$ to the principal ideal generated by x. Surely this map takes the 1 and 0 of L to the 1 and 0 of $\mathcal{I}(L)$. As $\mathcal{I}(L)$ is an algebraic lattice, the Grätzer–Schmidt Theorem provides that $\mathcal{I}(L)$ is isomorphic to the lattice of all congruences of some general algebra A. As the congruence lattice of A is a sublattice of *Eq* A which contains the 1 and 0 of *Eq* A, the result follows.   □

It is natural to ask whether the set X in the above theorem could be chosen finite if the given lattice is finite. Again, this question can be reduced to determining whether a given finite lattice has an appropriate embedding into *Eq* X for some finite set X. Neither Whitman's Theorem nor the Grätzer–Schmidt Theorem will help with this question, as they produce infinite sets (algebras) even for finite lattices. However, in another celebrated result, Pudlák and Tůma have proved [5] that every finite lattice can be homomorphically embedded into *Eq* X for some finite set X. This shows that every finite lattice can be homomorphically embedded into the lattice of topologies of some finite set, a observation already made in [2]. Pudlák and Tůma further state [5, p. 94], without proof, that their methods can be extended to provide an embedding which preserves 1 and 0. This statement implies that every finite lattice can be homomorphically embedded into the lattice of topologies of some finite set X by an embedding which maps the 1 and 0 of the given lattice to the 1 and 0 of *Top* X.

References

[1] P. Crawley, R.P. Dilworth, Algebraic Theory of Lattices, Prentice-Hall, 1973.

[2] C. Good, D.W. McIntyre, Finite intervals in the lattice of topologies, Preprint.

[3] G. Grätzer, E.T. Schmidt, Characterizations of congruence lattices of abstract algebras, Acta Sci. Math. (Szeged) 24 (1964) 34–59.

[4] R.E. Larson, S.J. Andima, The lattice of topologies, a survey, Rocky Mountain J. Math. 5 (1975) 177–198.

[5] P. Pudlák, J. Tůma, Every finite lattice can be embedded in a finite partition lattice, Algebra Universalis 10 (1980) 74–95.

[6] S. Watson, Problems I wish I could solve, in: J. van Mill, G.M. Reed (Eds.), Open Problems in Topology, North-Holland, Amsterdam, 1990, pp. 37–76.

[7] P. Whitman, Lattices, equivalence relations, and subgroups, Bull. Amer. Math. Soc. 52 (1946) 507–522.