Journal of Algebra **238**, 345–371 (2001)
doi:10.1006/jabr.2000.8622, available online at http://www.idealibrary.com on IDEAL®

Bounded Lattice Expansions

Mai Gehrke and John Harding

New Mexico State University, Las Cruces, New Mexico 88003
E-mail: mgehrke, jharding@nmsu.edu

Communicated by H. A. Priestley

Received May 24, 2000

The notion of a canonical extension of a lattice with additional operations is introduced. Both a concrete description and an abstract characterization of this extension are given. It is shown that this extension is functorial when applied to lattices whose additional operations are either order preserving or reversing, in each coordinate, and various results involving the preservation of identities under canonical extensions are established. © 2001 Academic Press

Key Words: algebras with a lattice reduct; canonical extensions; Galois connections; functoriality; preservation of identities.

1. INTRODUCTION

Many types of algebras, particularly ones arising from algebraic treatments of logics, consist of a Boolean algebra with additional operations. Often these operations preserve finite joins in each coordinate, in which case they are called operators. Examples of Boolean algebras with operators (BAOs) include relation algebras, modal algebras, cylindric algebras, and tense algebras. It was in the 1951 papers of Jónsson and Tarski [28, 29] that a systematic treatment of BAOs was begun. As a central ingredient of [28, 29], Jónsson and Tarski proved, starting from a recasting of a famous result of Stone, that every BAO can be extended (in an essentially unique manner) to a complete atomic BAO. This extension is called the canonical extension of the BAO. Canonical extensions of BAOs provide a representation theorem which can be of great use in algebraic investigations. Further, it has been recognized that canonical extensions play a fundamental role in completeness theorems for various extensions of classical logic such as modal logics. For these reasons, an extensive theory of canonical extensions of BAOs has been developed over the past 50 years [4, 19, 25–27, 34].

0021-8693/01 $35.00

There are also many types of algebras which consist of a distributive lattice with additional operations or, more generally, of a lattice with additional operations. Again, logic provides a rich source of examples. Roughly speaking, distributive lattices with additional operations, such as Heyting algebras, arise from algebraic studies of logics in which the classical negation has been weakened or eliminated [5, 6, 8, 9, 12, 18, 31–33, 35]. Lattices with additional operations arise from linear logics [1, 23, 30] and in studies of quantum logics [11]. The restriction to operators which has prevailed until recently is not a restriction motivated by applications but rather by the methods of proof. Even in the earliest application areas such a modal algebra or cylindric algebra, one is interested in considering both operators and order reversing operations, such as Boolean negation, or at least both operators and dual operators, such as $\Box$ and $\Diamond$ or existensial and universal quantifiers, simultaneously. In recent work [16, 17, 35] it has been possible to address questions concerning canonical extensions for broad classes of additional operations. Here we call a distributive lattice with additional operations a distributive lattice expansion (DLE) and we say a DLE is monotone if its additional operations are order preserving or inverting in each coordinate (note the Heyting implication $\rightarrow$ is order preserving in its first argument and inverting in its second). The term distributive lattice with operators (DLO) is reserved for a distributive lattice with additional operations which are additive in each argument. Similar definitions are used for lattice expansions (LEs), monotone lattice expansions, and lattices with operators (LOs).

In their 1994 paper [15], Gehrke and Jónsson introduced the notion of a canonical extension of a distributive lattice with operators and showed that all identities of a DLO are preserved by canonical extensions. This generalized the well known result [28] that all identities of a BAO not involving negation are preserved by canonical extensions. There are, however, identities in a BAO that are not preserved by canonical extensions [21] and the question of exactly which identities are preserved in a BAO is delicate [2, 19, 20, 26, 34]. Gehrke and Jónsson [16, 17] have addressed the more general question of determining which identities in a DLE are preserved by canonical extensions, obtaining strong results, some of which are new even when applied to the BAO situation.

It is the purpose of this paper to introduce the notion of canonical extensions of lattice expansions and to show that many of the results obtained for DLEs in [16] hold also for LEs. The proofs in the lattice setting are similar to those in the distributive case and are based on a fragment of an infinite distributive law shown to hold in the canonical extension of a lattice. The simplicity is perhaps surprising if one is familiar with Urquhart duality on which the canonical extension is based. Indeed, it is by providing an abstract characterization of the canonical extension of a lattice that such difficulties are avoided. The paper is organized as follows.

In Section 2 we define the canonical extension of a bounded lattice, show the existence of this extension, and show its uniqueness. In [22], Harding had obtained canonical extensions for lattices using the fact that lattices are exactly the images of Galois connections on Boolean algebras. Here we present a more direct construction of the canonical extension of an arbitrary (bounded) lattice and give a characterization that simplifies the ones given previously both in the distributive and non-distributive case. For a Boolean algebra, the canonical extension is isomorphic to the complete field of sets that the algebra is represented in via the topological dualities. Similarly for distributive lattices, the canonical extension is isomorphic to the complete ring of sets generated by the representation of the lattice obtained via the topological dualities. As is to be expected, the canonical extension of a lattice is isomorphic to the complete lattice it is represented in via the generalized topological dualities such as Urquhart duality [36] or Hartung's duality [24]. What is surprising is that describing the extension as well as working with it is no harder in the general case than it is in the distributive setting.

In Section 3 we develop some of the basic properties of canonical extensions of lattices. Even though their properties are not quite as strong in the general case as in the distributive case, they are remarkably strong and remarkably like those of canonical extensions of distributive lattices. For example, canonical extensions of distributive lattices are join generated by their completely join irreducibles. This is true in the non-distributive case as well. In the distributive case these completely join irreducibles are in one to one correspondence with the prime filters of the original lattice. In the non-distributive case the completely join irreducibles are in one to one correspondence with the maximal disjoint pairs of filters and ideals of the original lattice. Also, the canonical extensions of distributive lattices are completely distributive lattices. This of course cannot be true in the non-distributive case as it would force the underlying lattice that is being extended to be distributive. Nevertheless a very powerful restricted complete distributivity holds in canonical extensions of lattices. And it turns out this is sufficient to get many of the results on preservation of identities and the like previously obtained in the distributive case.

In Section 4 we give the definition of the extension of a map from one lattice to another and develop the basic properties of such extensions. In particular we consider extensions of order preserving maps and the interaction of extending and composing maps. The new definition for extending maps that we use here originates in [17]. It has the advantage of allowing extension of all maps whereas the original formula introduced in [28] only produces an extension of the underlying map when the latter is order preserving. In [17] several topologies on canonical extensions are explored in order to gain understanding and perspective on the properties of extensions of maps. We have decided not to explore these notions here and leave them for future work.

In Section 5 we define canonical extensions for arbitrary LEs and show that taking canonical extensions is functorial for monotone LEs. Furthermore, this functor preserves injectivity and surjectivity, which has as a consequence that the closure of a class under canonical extensions behaves well with respect to the formation of homomorphic images and subalgebras.

In Section 6 we give several preservation results. We show that the main result of [15] holds in the lattice setting in the sense that for any LE all identities involving only basic operations that are operators is preserved by canonical extensions. Notice this, however, does not imply that all identities are preserved for LEs in which all the additional operations are operators since the lattice meet is not itself an operator unless the lattice is distributive. We also show that a variety of monotone LEs is closed under canonical extensions if it is generated by a class K of LEs that is closed under canonical extensions and ultraproducts. In particular, all finitely generated varieties of monotone LEs are closed under canonical extensions.

2. CANONICAL EXTENSIONS OF LATTICES

Here, and in the remainder of the paper, all lattices will be assumed to have a least element 0 and a greatest element 1, and all homomorphisms preserve these bounds. All filters and ideals will be assumed non-empty. For a lattice L we use $\mathscr{F}_L$ and $\mathscr{I}_L$, or simply $\mathscr{F}$ and $\mathscr{I}$ when no confusion is likely, for the collections of all filters and ideals of L, respectively. For $p \in L$ we use $p\uparrow$ and $p\downarrow$ for the principal filter and ideal generated by p.

DEFINITION 2.1. A completion of a lattice L is a pair (e, c) where C is a complete lattice and $e\colon L \to C$ is a lattice embedding.

For a completion (e, C) of a lattice L, we say an element of C is open if it is a join of elements from the image of L and closed if it is a meet of elements from the image of L. The set of open elements of C will be denoted by O, and the set of closed elements will be denoted by K. It is customary to call a completion join dense if $O = C$ and meet dense if $K = C$. At the risk of clashing with existing terminology, we make the following definition.

DEFINITION 2.2. A completion (e, C) of a lattice L is called dense if every element of C can be expressed both as a join of meets and as a meet of joins of elements from the image of L.

Recall that a topological space is compact if for any family A of closed sets and any family B of open sets, $\bigcap A \subseteq \bigcup B$ iff there are finite subfamilies $A' \subseteq A$ and $B' \subseteq B$ with $\bigcap A' \subseteq \bigcup B'$. We make an analogous definition for completions.

DEFINITION 2.3. A completion (e, C) of a lattice L is called compact if for any set A of closed elements and any set of B of open elements, $\bigwedge A \leq \bigvee B$ iff there are finite subsets $A' \subseteq A$ and $B' \subseteq B$ with $\bigwedge A' \leq \bigvee B'$.

We emphasize that compactness is not a property of the lattice C alone but of the pair (e, C). It is worthwhile to note that to verify the compactness of (e, C), it suffices to consider subsets A, B of the image of the lattice L. Rephrased slightly, this is the content of the following lemma whose easy proof is left to the reader.

LEMMA 2.4. *For a completion (e, C) of a lattice L, the following are equivalent.*

(1) (e, C) *is compact.*

(2) $\bigwedge e[S] \leq \bigvee e[T]$ *iff* $\bigwedge S' \leq \bigvee T'$ *for some finite* $S' \subseteq S, T' \subseteq T$.

(3) $\bigwedge e[S] \leq \bigvee e[T]$ *iff* $F \cap I \neq \varnothing$ *for each* $S \subseteq F \in \mathscr{F}$ *and each* $T \subseteq I \in \mathscr{I}$.

DEFINITION 2.5. A canonical extension of a lattice L is a completion (e, C) of L which is dense and compact.

PROPOSITION 2.6. *Every lattice has a canonical extension.*

Proof. Define $R \subseteq \mathscr{F} \times \mathscr{I}$ to be the set of all ordered pairs (F, I) with $F \cap I \neq \varnothing$. As with any binary relation, the polarities of R [3] provide a Galois connection ϕ: $\mathscr{P}\mathscr{F} \to \mathscr{P}\mathscr{I}$ and ψ: $\mathscr{P}\mathscr{I} \to \mathscr{P}\mathscr{F}$ between the power sets of $\mathscr{F}$ and $\mathscr{I}$. Specifically,

$$\phi(X) = \{I : I \cap F \neq \varnothing \text{ for each } F \in X\},$$
$$\psi(X) = \{F : F \cap I \neq \varnothing \text{ for each } I \in X\}.$$

In general, a pair of maps f: $L \to M$ and g: $M \to L$ between lattices L and M is called a Galois connection if both f and g are order inverting and $a \leq f(g(a))$ and $b \leq g(f(b))$ for all $a \in M$ and $b \in L$. This is clearly satisfied by the maps ϕ: $\mathscr{P}\mathscr{F} \to \mathscr{P}\mathscr{I}$ and ψ: $\mathscr{P}\mathscr{I} \to \mathscr{P}\mathscr{F}$. Setting $\mathscr{G}(\phi, \psi)$ to be the Galois closed elements of $\mathscr{P}\mathscr{F}$, that is, the elements of $\mathscr{P}\mathscr{F}$ that are in the image of ψ, and $\mathcal{G}(\psi, \phi)$ to be the Galois closed elements of $\mathcal{PI}$, that is, the elements of $\mathcal{PI}$ that are in the image of ϕ, it follows from general properties of Galois connections that $\mathcal{G}(\phi, \psi)$ and $\mathcal{G}(\psi, \phi)$ are complete lattices and ϕ, ψ restrict to mutually inverse dual order isomorphisms between these lattices.

Claim 1. For each $p \in L$

(1) $\psi\phi(\{p\uparrow\}) = \{F : p \in F\}$.

(2) $\phi\psi(\{p\downarrow\}) = \{I : p \in I\}$.

Thus, we may define maps $\alpha\colon L \to \mathcal{G}(\phi, \psi)$ and $\beta\colon L \to \mathcal{G}(\psi, \phi)$ by setting $\alpha(p) = \{F : p \in F\}$ and $\beta(p) = \{I : p \in I\}$. It is clear that α is an order embedding, β is a dual order embedding, $\phi \circ \alpha = \beta$, and $\psi \circ \beta = \alpha$.

Claim 2. For any $A \subseteq L$

(1) $\bigwedge \alpha[A] = \{F : A \subseteq F\}$.

(2) $\bigwedge \beta[A] = \{I : A \subseteq I\}$.

(3) $\bigvee \alpha[A] = \{F : F \cap I \neq \emptyset$ for all $A \subseteq I\}$.

(4) $\bigvee \beta[A] = \{I : I \cap F \neq \emptyset$ for all $A \subseteq F\}$.

The first assertion follows trivially as meets in $\mathcal{G}(\phi, \psi)$ are given by intersections. The second is similar. For the third, $\bigvee \alpha[A] = \psi\phi(\bigvee \alpha[A]) = \psi(\bigwedge \phi\alpha[A]) = \psi(\bigwedge \beta[A]) = \psi(\{I : A \subseteq I\})$, and the result follows from the definition of ψ. The fourth is similar.

Claim 3. For any $X \subseteq \mathcal{F}$ and $Y \subseteq \mathcal{I}$

(1) $\psi\phi(X) = \bigvee\{\bigwedge \alpha[F] : F \in X\}$.

(2) $\phi\psi(Y) = \bigvee\{\bigwedge \beta[I] : I \in Y\}$.

To begin, we verify the first assertion under the additional assumption that $X = \psi\phi(X)$. Then surely $X \subseteq \{G : G \supseteq F$ for some $F \in X\}$, but as X is in the image of ψ, we have equality. Rewrite this as $X = \bigcup\{\bigwedge \alpha[F] : F \in X\}$, and as $X = \psi\phi(X)$, this union is a join. Without the assumption that $X = \psi\phi(X)$, we have $\psi\phi(X) = \bigvee\{\bigwedge \alpha[F] : F \in \psi\phi(X)\}$, which clearly contains $\bigvee\{\bigwedge \alpha[F] : F \in X\}$. But this final set is closed and can easily be seen to contain X. The second assertion follows similarly.

We are now able to conclude the proof of the proposition. We know that the map $\alpha\colon L \to \mathcal{G}(\phi, \psi)$ is an order embedding of L into a complete lattice. From Claim 2 it follows that α is a lattice embedding and that this completion is compact. Part (1) of Claim 3 provides directly that every element in $\mathcal{G}(\phi, \psi)$ is a join of meets of elements from the image of α, and part (2) of Claim 3, in conjunction with the fact that ψ is a dual order isomorphism, provides that every element of $\mathcal{G}(\phi, \psi)$ is a meet of joins of elements from the image of α. ∎

PROPOSITION 2.7. *Any two canonical extensions of a lattice L are isomorphic by a unique isomorphism commuting with the embeddings of L.*

Proof. Let (e, C) be a canonical extension of L. The version of compactness given in part (3) of Lemma 2.4 obviously determines whether a meet of elements from the image $e[L]$ of L lies below a join of elements from $e[L]$. In particular, this determines when a meet of elements from $e[L]$ lies below a single element from $e[L]$, hence when a meet of elements from $e[L]$ lies below another meet of elements from $e[L]$. Thus, compactness completely determines the poset of closed elements and similarly the poset of open elements, and further, compactness determines whether a closed element lies below an open element. By density, elements of C correspond to subsets D of closed elements with the property that a closed element belongs to D iff it is a lower bound of every open element which is an upper bound of D. Thus, compactness and density completely determine the extension. ∎

Remark 2.8. The results of this section could have been presented in a much more general setting. Let P be any poset, let $\mathcal{F}$ be any collection of upsets of P which contains all principal upsets, and let $\mathcal{I}$ be any collection of downsets of P which contains all principal downsets. Without modification, the above results show the existence and uniqueness of a pair (e, C) consisting of a complete lattice C and an order embedding e: $P \to C$ with the following properties: (i) each element of C is a join of meets and a meet of joins of elements of the image of P, and (ii) for any $S, T \subseteq P$ we have $\bigwedge e[S] \leq \bigvee e[T]$ iff $F \cap I \neq \varnothing$ for any $S \subseteq F \in \mathcal{F}$ and any $T \subseteq I \in \mathcal{I}$. Should the members of $\mathcal{F}$ be closed under some specified set of meets, then the embedding e will preserve these meets, and should the members of $\mathcal{I}$ be closed under some specified set of joins, the embedding e will preserve these joins.

Many of the standard completions of a lattice L arise in this manner for suitable choices of $\mathcal{F}$ and $\mathcal{I}$. For the MacNeille completion take $\mathcal{F}$ to be the collection of all principal filters and take $\mathcal{I}$ to be the collection of all principal ideals. For the ideal lattice take $\mathcal{F}$ to be the collection of all principal filters and we take $\mathcal{I}$ to be the collection of all ideals. The situation for the lattice of complete ideals is similar.

Remark 2.9. One might naturally wish to develop a theory of canonical extensions of arbitrary (possibly not bounded) lattices. The easiest way to do this is to consider a functor F which adjoins bounds to each lattice L to produce a bounded lattice FL. There are several possibilities for F depending on whether one wishes to preserve any existing bounds. One then defines a canonical extension of L to be a canonical extension of the bounded lattice FL. The method described in the previous remark could be used to develop such canonical extensions of arbitrary lattices in a more direct manner. Given an arbitrary lattice L one might consider $\mathscr{F}$ to be the collection of all (possibly empty) filters of L and $\mathscr{I}$ to be the collection of all (possible empty) ideals of L if one wished an extension that did not preserve existing bounds. To preserve existing bounds, one could require all members of $\mathscr{F}$ to contain any existing upper bound and any members of $\mathscr{I}$ to contain any existing lower bound.

Remark 2.10. Canonical extensions were first considered by Jónsson and Tarski [28] as a means to give a purely lattice theoretic description of the natural embedding $e: B \to \mathscr{P}(Z)$ of a Boolean algebra into the power set of its Stone space. As the elements from the image of B are clopen, the compactness of the completion $(e, \mathscr{P}(Z))$ follows from the compactness of Z. As the image of B forms a basis of Z, the fact that Z is Hausdorff implies each singleton $\{z\}$ is the intersection of elements from the image of B; hence, as $\mathscr{P}(Z)$ is completely distributive, the extension is dense.

It seems that only much later [15] were canonical extensions of distributive lattices considered. In this setting, the canonical extension gives a purely lattice theoretic characterization of the natural embedding e: $D \to \mathscr{D}(Z)$ of a distributive lattice D into the collection of all downsets of its Priestley space. As above, compactness of the embedding follows from the compactness of Z, and density follows from the complete distributivity of $\mathscr{D}(Z)$ and the fact that Z is totally order disconnected.

For canonical extensions of general lattices Urquhart duality [36] plays a role analogous to that played by Stone and Priestley duality above. Although it is a non-trivial exercise, the reader familiar with [36] will be able to show that the natural embedding of a lattice L into the complete lattice of all stable subsets of its Urquhart dual space Z is a canonical extension. As the points of Z are maximally disjoint pairs of filters and ideals of L, this shows that a canonical extension of L can be constructed solely from the maximally disjoint filters and ideals of L, rather from the collections $\mathscr{F}$ of all filters and $\mathscr{I}$ of all ideals of L as in Proposition 2.6. Lemma 3.4 provides an explanation for this phenomenon. Finally, we note that the Galois connection used in the proof of Proposition 2.6 was considered by Hartung [24] in his reformulation (and generalization) of Urquhart duality in terms of contexts and their concept lattices.

While canonical extensions of lattices naturally arise from Urquhart duality and from canonical extensions of Boolean algebras with operators (see remark 5.5), the results here, and in [22], seem to be the first to abstractly characterize canonical extensions. Next, we develop the basic properties of such extensions.

3. PROPERTIES OF CANONICAL EXTENSIONS

A fundamental notion for lattices and ordered sets in general is that of order duals. The order dual (or just dual as this is mostly the kind of dual we will talk about here) of a poset $(P, \leq)$ is the ordered set $(P, \geq)$ which we will denote by P^d. For $f: L \to M$, let $f^d: L^d \to M^d$ be the same set mapping as f, but with the dual orders on the domain and codomain. Given an order property, the dual property is the one obtained by considering the original property applied to the dual order. Canonical extensions behave quite nicely with respect to duality and for this reason many properties come in dual pairs. We will state such properties in pairs, labelling one as the dual of the other and only proving one of them.

PROPOSITION 3.1. *Let (e, C) be a canonical extension of L. Then (e^d, C^d) is a canonical extension of L^d.*

Proof. Compactness, denseness, and being a completion are self-dual properties. ∎

Here, and in the remainder of the paper, we shall use the following notation. For a family X of non-empty sets, we let $\Phi(X)$ denote the family of choice functions for X. Specifically $\Phi(X) = \{\alpha: X \to \bigcup X : \alpha(A) \in A$ for each $A \in X\}$. The following lemma will be of key importance in developing the theory of canonical extensions.

LEMMA 3.2. *Let (e, C) be a canonical extension of L. Suppose X and Y are collections of subsets of C such that each set in X is a downwardly directed set of closed elements and each set in Y is an upwardly directed set of open elements. Then*

(1) $\bigvee\{\bigwedge A : A \in X\} = \bigwedge\{\bigvee \mathrm{Im}(\alpha) : \alpha \in \Phi(X)\}$.

$(1)^d$ $\bigwedge\{\bigvee B : B \in Y\} = \bigvee\{\bigwedge \mathrm{Im}(\alpha) : \alpha \in \Phi(Y)\}$.

These will be called the restricted distributive laws.

Proof. Suppose $A \in X$ and $\alpha \in \Phi(X)$. As $\alpha(A) \in A$, it follows that $\bigwedge A \leq \alpha(A) \leq \bigvee \mathrm{Im}(\alpha)$; hence the left side of the required equation is less than or equal to the right. As the completion is dense, each element can be expressed as a meet of open elements. Thus, it suffices to show that any open element y which is greater than or equal to the left side is also greater than or equal to the right. For such y we have $\bigwedge A \leq y$ for each $A \in X$; hence, using compactness and the fact that each $A \in X$ is downwardly directed, for each $A \in X$ there is $a \in A$ with $a \leq y$. Thus, there is $\alpha \in \Phi(X)$ with $\bigvee \mathrm{Im}(\alpha) \leq y$, showing the right side is less than or equal y. ∎

LEMMA 3.3. *Let (e, C) be a canonical extension of a lattice L. Then*

(1) *The open elements O are a sublattice of C which is isomorphic to $\mathscr{I}_L$.*

$(1)^d$ *The closed elements K are a sublattice of C which is isomorphic to $\mathscr{F}_L$.*

Proof. Consider the map $\varphi: \mathscr{I}_L \to O$ defined by setting $\varphi(I) = \bigvee e[I]$. By general principles this map is order preserving, by compactness it is an order embedding, and by the definition of O it is a mapping onto O. Thus $\mathscr{I}_L$ is isomorphic to O, and it remains only to show that O is a sublattice of C. Surely the join of two open elements is open, so the task is to show that the meet of two open elements x, y is open. For such open x, y there are ideals I, J of L with $x = \bigvee e[I]$ and $y = \bigvee e[J]$. Setting $Y = \{e[I], e[J]\}$, we have $x \wedge y = \bigwedge\{\bigvee B : B \in Y\}$; hence, by restricted distributivity, $x \wedge y = \bigvee\{\bigwedge \mathrm{Im}(\alpha) : \alpha \in \Phi(Y)\}$. Therefore, $x \wedge y = \bigvee\{e(a \wedge b) : a \in I, b \in J\}$ is a join of elements in the image of L, hence open. ∎

An element j of a complete lattice C is called completely join irreducible if $J = \bigvee A$ implies $j \in A$. The set of completely join irreducibles in C will be denoted $J(C)$, and dually, the set of completely meet irreducibles in C will be denoted $M(C)$. For a lattice L we call a pair (F, I) consisting of a filter and an ideal of L a maximal pair if F is maximal among all filters disjoint from I and I is maximal among all ideals disjoint from F.

LEMMA 3.4. *Let (e, C) be a canonical extension of L.*

(1) $x \in J(C)$ *iff* $x = \bigwedge e[F]$ *for some maximal pair* (F, I) *of* L.

$(1)^d$ $x \in M(C)$ *iff* $x = \bigvee e[I]$ *for some maximal pair* (F, I) *of* L.

Further, each element of C is a join of completely join irreducibles and a meet of completely meet irreducibles.

Proof. We will show for any maximal pair (F, I) that $\bigwedge e[F]$ is completely join irreducible and that each closed element is a join of completely join irreducibles arising from such maximal pairs. As every element of C is a join of closed elements, it follows that every element of C is a join of completely join irreducibles arising from such maximal pairs, and therefore each completely join irreducible must arise from such a maximal pair.

For (F, I) a maximal pair in L, let $x = \bigwedge e[F]$, and suppose $x = \bigvee M$. We must show $x \in M$. As every element of C is a join of closed elements, we may assume without loss of generality that M is a set of closed elements. Suppose that $m < x$ for each $m \in M$. Then for each $m \in M$ the set $F_m = \{a \in L : m \leq e(a)\}$ is a filter in L which properly contains F and hence intersects I non-trivially. We may therefore choose an element

$b_m \in F_m \cap I$ for each $m \in M$. It follows that $\bigwedge e[F] = x = \bigvee M \leq \bigvee\{e(b_m) : m \in M\} \leq \bigvee e[I]$. Using the compactness condition of Lemma 2.4 we have $F \cap I \neq \emptyset$, contrary to (F, I) being a maximal pair.

We next show that each closed element k is a join of completely join irreducibles arising from such maximal pairs. As every element of C is a meet of open elements, it is sufficient to show that $k \nleq y$, for y open, implies there is a completely join irreducible x arising from a maximal pair with $x \leq k$ and $x \nleq y$. As k is closed there is a filter G of L, namely $G = k\uparrow \cap L$, with $k = \bigwedge e[G]$, and as y is open, there is an ideal J of L, namely $J = y\downarrow \cap L$, with $y = \bigvee e[J]$. As $k \nleq y$, we have $G \cap J \neq \emptyset$, so by a standard application of Zorn's lemma, there is a maximal pair (F, I) which extends (G, J). Set $x = \bigwedge e[F]$. Surely $x \leq k$. As F and I are disjoint, compactness provides $\bigwedge e[F] \nleq \bigvee e[I]$; hence $x \nleq y$. ∎

In the special case of canonical extensions of distributive lattices, a good deal more can be said.

PROPOSITION 3.5. *Let D be a bounded distributive lattice and let (e, c) be a canonical extension of D.*

(1) *C is completely distributive.*

(2) *Both C and its dual are algebraic lattices.*

(3) *$J(C)$ and $M(C)$ are isomorphic posets which are directly complete.*

Proof. The natural embedding $e: D \to \mathscr{D}(Z)$ of a distributive lattice D into the collection of all downsets of the Priestley space Z of D yields a canonical extension of D (see Remark 2.10). Therefore all three of the above statements follow directly from well known results. The first two can be found in [10]; the third follows as the poset of join irreducibles of $\mathscr{D}(Z)$ corresponds to the poset of prime ideals of D. ∎

Remark 3.6. It is natural to form many conjectures about canonical extensions based on the above result. Unfortunately, none of the statements in the above proposition hold for canonical extensions of general lattices. Obviously, a canonical extension of a non-distributive lattice will not be distributive, let alone completely distributive. However, one might reasonably hope that canonical extensions do preserve lattice identities. This is not the case; an example in [22] shows a canonical extension of a modular lattice need not be modular. Also in [22] is an example of a canonical extension which is not meet continuous, hence not algebraic (although Lemma 3.2 shows that meets do distribute over up-directed joins when all elements involved lie in the image of L). Finally, even for finite lattices there is in general no isomorphism between the posets of join and meet irreducibles, and examples can be constructed (necessarily infinite) of canonical completions where the posets of join and meet irreducibles are not directly complete.

For the sequel, it will be convenient to introduce some terminology.

DEFINITION 3.7. For each lattice L, let L^σ be the[1] (unique up to isomorphism) lattice containing L so that the identical embedding $L \to L^\sigma$ is a canonical extension. Note, by the definition of canonical extensions, that we have (up to isomorphism) $(L^d)^\sigma = (L^\sigma)^d$ and $(L_1 \times \cdots \times L_n)^\sigma = L_1^\sigma \times \cdots \times L_n^\sigma$.

4. EXTENSIONS OF MAPS

In this section, we introduce methods to extend a unary map $f\colon L \to M$ between lattices to a map between the canonical extensions L^σ and M^σ. The basic properties of the extensions introduced will also be investigated. The results of this, and the following, section are closely based on results obtained in the distributive case [16].

DEFINITION 4.1. Let L and M be lattices and let $f\colon L \to M$ be any map. Define maps $f^\sigma, f^\pi\colon L^\sigma \to M^\sigma$ by setting

$$f^\sigma(x) = \bigvee\{\bigwedge\{f(a) : a \in L \text{ and } p \le a \le q\} : p \in K, q \in O \text{ and } p \le x \le q\}$$

$$f^\pi(x) = \bigwedge\{\bigvee\{f(a) : a \in L \text{ and } p \le a \le q\} : p \in K, q \in O \text{ and } p \le x \le q\}$$

If $f^\sigma = f^\pi$ we shall say f is smooth.

LEMMA 4.2. *Let L and M be lattices and let $f\colon L \to M$ be any set map.*

(1) *f^σ and f^π both extend f.*

(2) *$f^\sigma \le f^\pi$ under the pointwise order.*

Proof. (1) Let $x \in L$. If $p \in K$, $q \in O$, and $p \le x \le q$, then we surely have $\bigwedge\{f(a) : a \in L$ and $p \le a \le q\} \le f(x)$. Hence $f^\sigma(x) \le f(x)$. Conversely, as $x \in L$ we have $x \in K$, $x \in O$; hence $f^\sigma(x) \ge \bigwedge\{f(a) : a \in L$ and $x \le a \le x\} = f(x)$. That f^π extends f follows similarly. (2) If $p_1, p_2 \in K$, $q_1, q_2 \in O$, and $p_1, p_2 \le x \le q_1, q_2$, we must show $\bigwedge\{f(a) : a \in L$ and $p_1 \le a \le q_1\} \le \bigvee\{f(b) : b \in L$ and $p_2 \le b \le q_2\}$. Set $p = p_1 \vee p_2$ and $q = q_1 \wedge q_2$. It follows from Lemma 3.3 that $p \in K$, $q \in O$, and surely $p \le x \le q$. Hence, by compactness, there is some $c \in L$ with

[1] Alternately, one can use the axiom of foundation to prove the existence of a function $\sigma\colon \mathscr{L} \to \mathscr{L}$ from the class of all lattices to itself such that (i) L is a sublattice of L^σ, (ii) the identical embedding $L \to L^\sigma$ is a canonical extension, (iii) $(L^d)^\sigma = (L^\sigma)^d$, and (iv) $(L_1 \times \cdots \times L_n)^\sigma = L_1^\sigma \times \cdots \times L_n^\sigma$ (the equalities in parts (iii) and (iv) are to be taken literally).

$p \leq c \leq q$. Then, the left side of the above inequality lies below $f(c)$ and the right side of the inequality lies above $f(c)$. ∎

Naturally, extensions of order preserving maps will play a prominent role. For an order preserving map $f\colon L \to M$ between Boolean algebras L and M, the definition provided above for the extension f^σ reduces to the one considered by Jónsson and Tarski in their original 1951 paper [28] on Boolean algebras with operators. We next collect a few simple, but useful, observations on extensions of order preserving maps.

LEMMA 4.3. *Let L and M be lattices and let $f\colon L \to M$ be an order preserving map.*

(1) $f^\sigma(p) = \bigwedge\{f(a) : a \in L \text{ and } p \leq a\}$ *for all* $p \in K$.

$(1)^d$ $f^\pi(q) = \bigvee\{f(a) : a \in L \text{ and } a \leq q\}$ *for all* $q \in O$.

(2) $f^\sigma(x) = \bigvee\{f^\sigma(p) : p \in K \text{ and } p \leq x\}$ *for all* $x \in L^\sigma$.

$(2)^d$ $f^\pi(x) = \bigwedge\{f^\pi(q) : q \in O \text{ and } x \leq q\}$ *for all* $x \in L^\sigma$.

(3) f^σ *and* f^π *agree on* $K \cup O$.

Thus, $f^\sigma | K$ is the largest order preserving extension of f to K, and f^σ is the least order preserving extension of $f^\sigma | K$ to L^σ. Similarly, $f^\pi | O$ is the least order preserving extension of f to O, and f^π is the largest order preserving extension of $f^\pi | O$.

Proof. The first four statements are trivial consequences of the definition. For the fifth statement, we show f^σ and f^π agree on K; that they also agree on O follows similarly. Let $p \in K$. By Lemma 4.2 we know $f^\sigma(p) \leq f^\pi(p)$. Suppose then that $a \in L$ and $p \leq a$. As $a \in O$, it follows from (4) that $f^\pi(p) \leq f^\pi(a) = f(a)$. Thus, by part (1), $f^\pi(p) \leq f^\sigma(p)$. ∎

Here, and in the remainder of the paper, we shall have occasion to consider extensions of maps which preserve some specified set of joins or meets. We note first that the join of the empty set is 0 and the meet of the empty set is 1. As f^σ is an extension of f, we trivially have that f preserving the empty join (meet) implies that f^σ also preserves the empty join (meet). Much more interesting results will follow when we consider maps f that preserve all finite non-empty joins (meets) or, equivalently, maps which preserve binary joins (meets). We will show that extensions of such maps preserve all non-empty joins (meets). As a final comment, we note that, by definition, directed sets are non-empty.

LEMMA 4.4. *Let f: $L \to M$.*

(1) *If f^σ preserves non-empty joins, then f is smooth.*

$(1)^d$ *If f^π preserves non-empty meets, then f is smooth.*

(2) *If f^π preserves upwardly directed joins, then f is smooth.*

$(2)^d$ *If f^σ preserves downwardly directed meets, then f is smooth.*

Proof. (1) By Lemma 4.2 $f^\sigma \leq f^\pi$. Let $x \in L^\sigma$. Set X to be the collection of all filters F of L with $\bigwedge F \leq x$ and set X' to be the collection of all $f[F]$, where $F \in X$. By Lemma 4.3 $f^\sigma(x) = \bigvee\{\bigwedge G : G \in X'\}$; hence, by restricted distributivity, $f^\sigma(x) = \bigwedge\{\bigvee \mathrm{Im}(\alpha) : \alpha \in \Phi(X')\}$. For $\alpha \in \Phi(X')$ there is $\beta \in \Phi(X)$ with $\mathrm{Im}(\alpha) = f[\mathrm{Im}(\beta)]$; hence, as f^σ preserves non-empty joins, $\bigvee \mathrm{Im}(\alpha) = \bigvee f[\mathrm{Im}(\beta)] = f^\sigma(\bigvee \mathrm{Im}(\beta))$. But $\bigvee \mathrm{Im}(\beta)$ is open and $x \leq \bigvee \mathrm{Im}(\beta)$. Thus $f^\sigma(x) \geq \bigwedge\{f^\sigma(q) : q \in O$ and $x \leq q\}$, so by Lemma 4.3, $f^\sigma(x) \geq f^\pi(x)$.

(2) By Lemma 4.2 $f^\sigma \leq f^\pi$. Let $x \in L^\sigma$. Then $x = \bigvee\{p : p \in K$ and $p \leq x\}$ and by Lemma 3.3 this join is directed. So $f^\pi(x) = \bigvee\{f^\pi(p) : p \in K$ and $p \leq x\}$. By Lemma 4.3 f^σ and f^π agree on K; hence $f^\pi(x) = \bigvee\{f^\sigma(p) : p \in K$ and $p \leq x\} = f^\sigma(x)$. ∎

We now address extensions of composite mappings, a topic that will lie at the heart of determining the equational properties preserved by extensions. The second and third parts of the following lemma have their origins in the early work of Jónsson and Tarski [28]; the analogue of the fourth and fifth parts seem to only have been discovered much more recently [16].

LEMMA 4.5. *Let f: $L \to M$ and g: $M \to N$ be order preserving.*

(1) $(gf)^\sigma \leq g^\sigma f^\sigma \leq g^\sigma f^\pi, g^\pi f^\sigma \leq g^\pi f^\pi \leq (gf)^\pi$ *with equality on* $K \cup O$.

(2) *If g^σ preserves upwardly directed joins, then $(gf)^\sigma = g^\sigma f^\sigma$.*

$(2)^d$ *If g^π preserves downwardly directed meets, then $(gf)^\pi = g^\pi f^\pi$.*

(3) *If f^σ preserves non-empty meets, then $(gf)^\sigma = g^\sigma f^\sigma$.*

$(3)^d$ *If f^π preserves non-empty joins, then $(gf)^\pi = g^\pi f^\pi$.*

Proof. (1) We will show $(gf)^\sigma \leq g^\sigma f^\sigma$. $g^\pi f^\pi \leq (gf)^\pi$ is dual. The remaining inequalities follow from Lemma 4.2, and Lemma 4.3 provides $(gf)^\sigma$ and $(gf)^\pi$ agree on $K \cup O$. To show $(gf)^\sigma \leq g^\sigma f^\sigma$, it is enough to show this inequality holds on K, as $(gf)^\sigma$ is the least order preserving extension of $(gf)^\sigma | K$ to L^σ (see Lemma 4.3). Let $p \in K$. By Lemma 4.3, $f^\sigma(p)$ is closed, so $g^\sigma f^\sigma(p) = \bigwedge\{g(a) : a \in M$ and $f^\sigma(p) \leq a\}$. Suppose then that $a \in M$ and $f^\sigma(p) \leq a$. As $f^\sigma(p) = \bigwedge\{f(b) : b \in L$ and $p \leq b\}$ is a downwardly directed meet, compactness provides some $b \in L$ with $p \leq b$ and $f(b) \leq a$. Then $(gf)^\sigma(p) \leq (gf)(b) \leq g(a)$; hence $(gf)^\sigma(p) \leq g^\sigma f^\sigma(p)$.

(2) By Lemma 4.3, $(gf)^\sigma(x) = \bigvee\{(gf)^\sigma(p) : p \in K \text{ and } p \le x\}$. In (1) we have shown $(gf)^\sigma$ and $g^\sigma f^\sigma$ agree on K, so $(gf)^\sigma(x) = \bigvee\{g^\sigma f^\sigma(p) : p \in K \text{ and } p \le x\}$. Setting $D = \{f^\sigma(p) : p \in K \text{ and } p \le x\}$, we have $(gf)^\sigma(x) = \bigvee\{g^\sigma(d) : d \in D\}$. As K is a sublattice of L^σ (see Lemma 3.3) D is directed. Thus, $(gf)^\sigma(x) = g^\sigma(\bigvee D)$. But, by Lemma 4.3, $\bigvee D = f^\sigma(x)$; hence $(gf)^\sigma(x) = g^\sigma f^\sigma(x)$.

(3) By (1) we know that $(gf)^\sigma \le g^\sigma f^\sigma$ with equality on closed elements and on open elements. To show the reverse inequality, let $x \in L^\sigma$. We first establish the following claim: if $p \in M$ is closed and $p \le f^\sigma(x)$, then there is $p' \in L$ closed with $p' \le x$ and $p \le f^\sigma(p')$. Indeed, as f^σ preserves non-empty meets, we have by Lemma 4.4 that $f^\sigma = f^\pi$; hence $p \le f^\pi(x) = \bigwedge\{f^\pi(q) : q \in O \text{ and } x \le q\}$. Then for each $q \in O$ with $x \le q$, we have $p \le f^\pi(q) = \bigvee\{f(a) : a \in L \text{ and } a \le q\}$, where this join is upwardly directed. By compactness, for each $q \in O$ with $x \le q$ there is $a_q \in L$ with $a_q \le q$ and $p \le f(a_q)$. Set $p' = \bigwedge\{a_q : q \in O \text{ and } x \le q\}$. Then p' is closed, $p' \le x$, and as f^σ preserves non-empty meets, $p \le f^\sigma(p')$.

By Lemma 4.3, $g^\sigma f^\sigma(x) = \bigvee\{g^\sigma(p) : p \in K \text{ and } p \le f^\sigma(x)\}$, so by the above claim $g^\sigma f^\sigma(x) \le \bigvee\{g^\sigma f^\sigma(r) : r \in K \text{ and } r \le x\}$. As $(gf)^\sigma$ and $g^\sigma f^\sigma$ agree on closed elements, we have that $g^\sigma f^\sigma(x) \le \bigvee\{(gf)^\sigma(r) : r \in K \text{ and } r \le x\} = (gf)^\sigma(x)$. ∎

Having seen several lemmas demonstrating the utility of knowing that f^σ preserves non-empty, or upwardly directed, joins, we now show that such maps exist in relative abundance. We say that a map $f\colon L^n \to M$ preserves binary joins in the ith coordinate if each of the induced unary maps formed by fixing the arguments in all but the ith coordinate preserves binary joins. Recall that $(L^n)^\sigma = (L^\sigma)^n$; hence f^σ is also a map from $(L^\sigma)^n \to M^\sigma$.

LEMMA 4.6. *Let $f\colon L^n \to L$ be an order preserving map. If f preserves binary joins in the ith coordinate, then f^σ preserves non-empty joins in the ith coordinate. Dually, if f preserves binary meets in the ith coordinate, then f^π preserves non-empty meets in the ith coordinate.*

Proof. We show the statement involving joins. For convenience, assume $f\colon L^2 \to L$ and that f preserves binary joins in the second coordinate. We first show for $p \in L^\sigma$ closed and $S \subseteq L^\sigma$ a set of closed elements that $f^\sigma(p, \bigvee S) \le \bigvee\{f^\sigma(p, s) : s \in S\}$, hence equality. Let X be the collection of all filters F of L such that $\bigwedge F \in S$ and let Y be the collection of all subsets of L of the form $G_F = \{f(a, b) : p \le a \text{ and } b \in F\}$ for some $F \in X$. Next, we note that $\bigvee\{f^\sigma(p, s) : s \in S\} = \bigvee\{\bigwedge G : G \in Y\} = \bigwedge\{\bigvee \mathrm{Im}(\alpha) : \alpha \in \Phi(Y)\}$, and $f^\sigma(p, \bigvee S) = \bigvee\{f^\sigma(p, t) : t$ is closed and

$t \leq \bigvee S\}$. So we must show that t closed, $t \leq \bigvee S$, and $\alpha_0 \in \Phi(Y)$ imply $f^\sigma(p,t) \leq \bigvee \mathrm{Im}(\alpha_0)$.

For each $F \in X$, let (a_F, b_F) be some element with $p \leq a_F$, $b_F \in F$, and $f(a_F, b_F) = \alpha_0(G_F)$. Define a map $\beta_0 \in \Phi(X)$ by setting $\beta_0(F) = b_F$. As $t \leq \bigvee S = \bigvee\{\bigwedge F : F \in X\} = \bigwedge\{\bigvee \mathrm{Im}(\beta) : \beta \in \Phi(X)\}$, we have $t \leq \bigvee \mathrm{Im}(\beta_0)$. As t is closed, compactness yields $t \leq b_{F_1} \vee \cdots \vee b_{F_n}$ for some $F_1, \ldots, F_n \in X$. Set $a = a_{F_1} \wedge \cdots \wedge a_{F_n}$. Then $f^\sigma(p,t) \leq f^\sigma(a, b_{F_1} \vee \cdots \vee b_{F_n})$; hence $f^\sigma(p,t) \leq f^\sigma(a, b_{F_1}) \vee \cdots \vee f^\sigma(a, b_{F_n}) \leq f(a_{F_1}, b_{F_1}) \vee \cdots \vee f(a_{F_n}, b_{F_n})$. This yields $f^\sigma(p,t) \leq \bigvee \mathrm{Im}(\alpha_0)$, as required.

Suppose $x \in L^\sigma$ and $Z \subseteq L^\sigma$. To show $f^\sigma(x, \bigvee Z) \leq \bigvee\{f^\sigma(x,z) : z \in Z\}$, hence equality, it suffices to show $f^\sigma(x, \bigvee Z) \leq \bigvee\{f^\sigma(x,z) : z \in Z'\}$ where Z' is the set of all closed elements lying beneath some element of Z. Note first that $f^\sigma(x, \bigvee Z) = \bigvee\{f^\sigma(p,t) : (p,t)$ is closed and $(p,t) \leq (x, \bigvee Z)\}$. For such (p,t) we have $f^\sigma(p,t) \leq f^\sigma(p, \bigvee Z')$, which, by the above remarks, yields $f^\sigma(p,t) \leq \bigvee\{f^\sigma(p,z) : z \in Z'\}$. ∎

COROLLARY 4.7. *Let f: $L \to M$.*

(1) *If f preserves binary joins, then f^σ preserves non-empty joins.*

$(1)^d$ *If f preserves binary meets, then f^π preserves non-empty meets.*

(2) *If f preserves binary meets, then f^σ preserves non-empty meets.*

$(2)^d$ *If f preserves binary joins, then f^π preserves non-empty joins.*

Proof. (1) This is a special case of Lemma 4.6.

(2) That f^π preserves non-empty meets is also a special case of Lemma 4.6. But Lemma 4.4 provides $f^\sigma = f^\pi$. ∎

A map f: $L^n \to M$ is called an operator if it preserves binary joins in each coordinate (note that this does not imply that f preserves binary joins), and f is called a complete operator if it preserves non-empty joins in each coordinate. The dual notions are named dual operator and complete dual operator, respectively. The following is also evident from Lemma 4.6.

COROLLARY 4.8. *If f: $L^n \to M$ is an operator, then f^σ is a complete operator. Dually, if f: $L^n \to M$ is a dual operator, then f^π is a complete dual operator.*

We collect one final application of these results.

LEMMA 4.9. *Let f: $L \to M$ be a lattice homomorphism.*

(1) *f is one to one iff f^σ is one to one.*

$(1)^d$ *f is one to one iff f^π is one to one.*

(2) *f is onto iff f^σ is onto.*

$(2)^d$ *f is onto iff f^π is onto.*

Proof. (1) Assume f is one to one and $x, y \in L^\sigma$ with $x \neq y$. Assume, without loss of generality, that $y \nleq x$. As every element of L^σ is a join of closed elements and a meet of open elements, there are $p \in K$ and $q \in O$ with $p \leq y$, $x \leq q$, and $p \nleq q$. If $f^\sigma(x) = f^\sigma(y)$, then $f^\sigma(p) \leq f^\sigma(q)$. By Lemma 4.3 it follows that $f^\sigma(p) = \bigwedge\{f(a) : a \in L, p \leq a\} \leq \bigvee\{f(b) : b \in L, b \leq q\} = f^\sigma(q)$. As this is a downwardly directed meet below an upwardly directed join, compactness provides some $a, b \in L$ with $p \leq a, b \leq q$ and $f(a) \leq f(b)$. As f is an embedding, $a \leq b$, providing $p \leq a \leq b \leq q$, a contradiction. Thus, if f is one to one, then f^σ is one to one. The converse is trivial.

(2) Note first that by Corollary 4.7 f^σ preserves non-empty joins and meets. If f is a mapping onto M, then as every element of M^σ is a join of meets of elements of M, it follows that f^σ is a mapping onto M^σ. Conversely, suppose f^σ is a mapping onto M^σ. Then for $m \in M$ there is $x \in L^\sigma$ with $f^\sigma(x) = m$. Let P be the set of closed elements of L^σ with $p \leq x$. Then $m = \bigvee\{f^\sigma(p) : p \in P\}$, and in particular $f^\sigma(p) \leq m$ for each $p \in P$. As $f^\sigma(p) = \bigwedge\{f(a) : a \in L, p \leq a\}$, compactness provides, for each $p \in P$, some $a_p \in L$ with $p \leq a_p$ and $f(a_p) \leq m$. Thus, $m = \bigvee\{f(a_p) : p \in P\}$, and by compactness there is some finite $P' \subseteq P$ with $m = \bigvee\{f(a_p) : p \in P'\}$. As f is a homomorphism, $m = f(\bigvee P')$, showing f maps L onto M. ∎

5. EXTENSIONS OF LATTICES WITH OPERATIONS

An n-ary operation on a lattice L is a map $f: L^n \to L$. As L^n is a lattice, we obtain an extension $f^\sigma: (L^n)^\sigma \to L^\sigma$. But $(L^n)^\sigma = (L^\sigma)^n$. Thus, the results of the previous section extend an n-ary operation f on a lattice L to an n-ary operation f^σ on the canonical extension L^σ.

LEMMA 5.1. *For L a lattice, $\wedge^\sigma, \wedge^\pi: L^\sigma \times L^\sigma \to L^\sigma$ are both the meet in L^σ. Dually $\vee^\sigma, \vee^\pi: L^\sigma \times L^\sigma \to L^\sigma$ are both the join in L^σ.*

Proof. Let $f: L \times L \to L$ be meet in the lattice L. Note (p_1, p_2) is closed in the completion $L \times L \to L^\sigma \times L^\sigma$ iff p_1 and p_2 are closed in $L \to L^\sigma$. Thus $p_1 \wedge p_2 = \bigwedge\{a_1 : a_1 \in L \text{ and } p_1 \leq a_1\} \wedge \bigwedge\{a_2 : a_2 \in L \text{ and } p_2 \leq a_2\}$. As each of these meets is non-empty, $p_1 \wedge p_2 = \bigwedge\{a_1 \wedge a_2 : (a_1, a_2) \in L \times L \text{ and } (p_1, p_2) \leq (a_1, a_2)\} = f^\sigma(p_1, p_2)$. For $(x_1, x_2) \in L^\sigma \times L^\sigma$, $x_1 \wedge x_2 = \bigvee\{p : p \in L^\sigma \text{ is closed and } p \leq x_1 \wedge x_2\}$; hence $x_1 \wedge x_2 = \bigvee\{p_1 \wedge p_2 : p_1, p_2 \in L^\sigma \text{ are closed and } (p_1, p_2) \leq (x_1, x_2)\}$. As $p_1 \wedge p_2 = f^\sigma(p_1, p_2)$, it follows from 4.3 that $x_1 \wedge x_2 = f^\sigma(x_1, x_2)$. Finally, as $\wedge: L^2 \to L$ preserves binary meets, by Lemma 4.4 and Corollary 4.7, it is smooth. ∎

DEFINITION 5.2. Let $L = (L, \wedge, \vee, (f_i)_{i \in I})$ be a lattice expansion (abbreviated: LE). Define the canonical extension L^σ to be $(L^\sigma, \wedge^\sigma, \vee^\sigma, (f_i^\sigma)_{i \in I})$ and define the dual canonical extension L^π to be $(L^\sigma, \wedge^\pi, \vee^\pi, (f_i^\pi)_{i \in I})$. Both L^σ and L^π are LEs of the same type as L and the underlying lattices for these extension are the same.

Note that the extension of a map $f: L^n \to M$ depends on the ordering of the lattices L^n and M. To apply the results of the previous section to maps which are order preserving in some coordinates and order inverting in others, we introduce some notation. Let $2 = \{0, 1\}$ and $\alpha \in 2^n$. We define $L^\alpha = L^{\alpha_1} \times \cdots \times L^{\alpha_n}$ where L^0 is used to denote the lattice L and L^1 denotes the dual lattice L^d. For $f: L^n \to M$ we then define $f^\alpha: L^\alpha \to M$ to be the same set mapping as f, but with the appropriate modifications to the order on the domain. Finally, for $f_i: L_i \to M_i$ $(i = 1, \ldots, n)$ let $\langle f_1, \ldots, f_n \rangle: L_1 \times \cdots \times L_n \to M_1 \times \cdots \times M_n$ be the natural product map.

LEMMA 5.3. (1) *For* $f: L^n \to M$ *and* $\alpha \in 2^n, (f^\alpha)^\sigma = (f^\sigma)^\alpha$.

$(1)^d$ *For* $f: L^n \to M$ *and* $\alpha \in 2^n, (f^\alpha)^\pi = (f^\pi)^\alpha$.

(2) *For* $f: L \to M, (f^d)^\sigma = (f^\pi)^d$.

$(2)^d$ *For* $f: L \to M, (f^d)^\pi = (f^\sigma)^d$.

(3) *For* $f_i: L_i \to M_i$ $(i = 1, \ldots, n), \langle f_1, \ldots, f_n \rangle^\sigma = \langle f_1^\sigma, \ldots, f_n^\sigma \rangle$.

$(3)^d$ *For* $f_i: L_i \to M_i$ $(i = 1, \ldots, n), \langle f_1, \ldots, f_n \rangle^\pi = \langle f_1^\pi, \ldots, f_n^\pi \rangle$.

Proof. (1) We illustrate a typical instance. Suppose that $f: L^2 \to M$ and $\alpha_1 = 0$, $\alpha_2 = 1$. Then $f^\alpha: L \times L^d \to M$. By Definition 4.1 we have $(f^\alpha)^\sigma(x_1, x_2)$ is the join of all $\bigwedge\{f(a_1, a_2) : (p_1, p_2) \leq_\alpha (a_1, a_2) \leq_\alpha (q_1, q_2)\}$ where (p_1, p_2) is closed in $L \times L^d$, (q_1, q_2) is open in $L \times L^d$, and $(p_1, p_2) \leq_\alpha (x_1, x_2) \leq_\alpha (q_1, q_2)$. Therefore $(f^\alpha)^\sigma$ is the join of all $\bigwedge\{f(a_1, a_2) : p_1 \leq a_1 \leq q_1$ and $q_2 \leq a_2 \leq p_2\}$, where p_1, q_2 are closed in L^σ and q_1, p_2 are open in L^σ. Hence, $(f^\alpha)^\sigma = f^\sigma$.

(2) This follows directly from Definition 4.1 as joins in $(M^d)^\sigma = (M^\sigma)^d$ correspond to meets in M^σ, etc.

(3) This follows as joins and meets in $M^\sigma \times M^\sigma$ are computed componentwise. ∎

We say an n-ary operation $f: L^n \to L$ is monotone if there is $\alpha \in 2^n$ with $f^\alpha: L^\alpha \to L$ order preserving. In other words, f is monotone if it is either order preserving or order reversing in each coordinate. For a given type τ, we let $\mathscr{C}_\tau$ be the category whose objects are all LEs of type τ with monotone operations and whose morphisms are all homomorphisms.

THEOREM 5.4. *For any type* $\tau, \sigma: \mathscr{C}_\tau \to \mathscr{C}_\tau$ *is a functor. Further,* σ *preserves and reflects one to one and onto mappings. Dually,* $\pi: \mathscr{C}_\tau \to \mathscr{C}_\tau$ *is a functor that preserves and reflects one to one and onto mappings.*

Proof. Suppose L is an object in $\mathscr{C}_\tau$. Then for $f\colon L^n \to L$ a basic operation of L, there is some $\alpha \in 2^n$ with $f^\alpha\colon L^\alpha \to L$ order preserving. Therefore $(f^\alpha)^\sigma\colon (L^\alpha)^\sigma \to L^\sigma$ is order preserving, and as $(f^\alpha)^\sigma = (f^\sigma)^\alpha$, we have f^σ is monotone. Thus L^σ is also an object in $\mathscr{C}_\tau$.

Suppose L and M are objects in $\mathscr{C}_\tau$ and $h\colon L \to M$ is a homomorphism, hence a morphism of $\mathscr{C}_\tau$. We must show $h^\sigma\colon L^\sigma \to M^\sigma$ is also a homomorphism, hence a morphism of $\mathscr{C}_\tau$. Let f be an n-ary operation of L, and let g be the corresponding n-ary operation of M. Knowing $h \circ f = g \circ \langle h, \dots, h \rangle$, we must show $h^\sigma \circ f^\sigma = g^\sigma \circ \langle h^\sigma, \dots, h^\sigma \rangle$.

As f, g are monotone there are $\alpha, \beta \in 2^n$ such that f^α and g^β are order preserving. Then as h is a homomorphism, Corollary 4.7 shows h^σ preserves non-empty joins, so Lemma 4.5 yields $(h \circ f^\alpha)^\sigma = h^\sigma \circ (f^\alpha)^\sigma$. Thus $((h \circ f)^\sigma)^\alpha = ((h \circ f)^\alpha)^\sigma = (h \circ f^\alpha)^\sigma = h^\sigma \circ (f^\alpha)^\sigma = h^\sigma \circ (f^\sigma)^\alpha = (h^\sigma \circ f^\sigma)^\alpha$; hence $(h \circ f)^\sigma = h^\sigma \circ f^\sigma$.

For $i \le n$ let $h_i = h$ if $\beta_i = 0$ and let $h_i = h^d$ if $\beta_i = 1$. As h is a homomorphism it is smooth, so $h_i^\sigma = h^\sigma$ if $\beta_i = 0$ and $h_i^\sigma = (h^d)^\sigma = (h^\pi)^d = (h^\sigma)^d$ if $\beta_i = 1$. Therefore $g^\beta \circ \langle h_1, \dots, h_n \rangle = (g \circ \langle h, \dots, h \rangle)^\beta$ and $(g^\sigma)^\beta \circ \langle h_1^\sigma, \dots, h_n^\sigma \rangle = (g^\sigma \circ \langle h^\sigma, \dots, h^\sigma \rangle)^\beta$. As g^β is order preserving and $\langle h_1, \dots, h_n \rangle$ is a homomorphism, hence preserves binary meets, Lemma 4.5 provides $(g^\beta \circ \langle h_1, \dots, h_n \rangle)^\sigma = (g^\beta)^\sigma \circ \langle h_1, \dots, h_n \rangle^\sigma$. Piecing this together, we have $((g \circ \langle h, \dots, h \rangle)^\sigma)^\beta = ((g \circ \langle h, \dots, h \rangle)^\beta)^\sigma = g^\beta \circ \langle h_1, \dots, h_n \rangle)^\sigma = (g^\beta)^\sigma \circ \langle h_1, \dots, h_n \rangle^\sigma = (g^\sigma)^\beta \circ \langle h_1^\sigma, \dots, h_n^\sigma \rangle = (g^\sigma \circ h^\sigma, \dots, h^\sigma \rangle)^\beta$; hence $(g \circ \langle h, \dots, h \rangle)^\sigma = g^\sigma \circ \langle h^\sigma, \dots, h^\sigma \rangle$. Therefore $h^\sigma \circ f^\sigma = (h \circ f)^\sigma = (g \circ \langle h, \dots, h \rangle)^\sigma = g^\sigma \circ \langle h^\sigma, \dots, h^\sigma \rangle$, showing h^σ is a homomorphism and hence a morphism of $\mathscr{C}_\tau$.

Clearly $(\mathrm{id}_L)^\sigma = \mathrm{id}_{L^\sigma}$, and by Lemma 4.5 and Corollary 4.7 if $h\colon L \to M$ and $e\colon M \to N$ are homomorphisms, then $(e \circ h)^\sigma = e^\sigma \circ h^\sigma$. Thus $\sigma\colon \mathscr{C}_\tau \to \mathscr{C}_\tau$ is a functor. The further comments follow from Lemma 4.9. ∎

Before concluding this section, we make a few comments about another possible route to developing canonical extensions of lattice expansions.

Remark 5.5. For a lattice M, let B_1 be the Boolean subalgebra of the power set of M generated by the principal ideals of M, and let B_2 be the Boolean subalgebra of the power set of M generated by the principal filters of M. Define $U\colon B_1 \to B_2$ by setting $U(X) = \{a : x \le a \text{ for all } x \in X\}$ and define $L\colon B_2 \to B_1$ by setting $L(Y) = \{a : a \le y \text{ for all } y \in Y\}$. One can show that L, U are a Galois connection between B_1 and B_2 and, further, that the Galois closed elements of B_1 are all of the form $a\!\downarrow$ for some $a \in M$; hence they form a lattice isomorphism to M. One can then show that the extensions $U^\sigma\colon B_1^\sigma \to B_2^\sigma$ and $L^\sigma\colon B_2^\sigma \to B_1^\sigma$ are a Galois connection between B_1^σ and B_2^σ and, further, that the Galois closed elements of B_1^σ yield a canonical extension of the lattice M. The relationship between this realization of a canonical extension of M and the one produced in Proposition 2.6 can be made clear. One can show that B_1^σ is isomorphic to the power set of the collection $\mathscr{I}$ of all ideals of M, that B_2^σ is isomorphic to the power set of the collection $\mathscr{F}$ of all filters of M, and that the maps ψ and ϕ given in Proposition 2.6 correspond, via these isomorphisms, with the maps U^σ and L^σ.

These results can be reformulated in a somewhat different way. Let $B = B_1 \times B_2$, and define maps $f, g\colon B \to B$ by setting $f(X,Y) = (M, UX)$ and $g(X,Y) = (LY, M)$. Then, the maps f, g are a Galois connection on the Boolean algebra B, and the Galois closed elements form a lattice isomorphic to M (see [22] for complete details). If we call a triple (B, f, g) consisting of a Boolean algebra with a Galois connection a conjugated algebra, we then have that every lattice is isomorphic to the closed elements of some conjugated algebra. Further, if M is isomorphic to the closed elements of the conjugated algebra (B, f, g), then $(B, f, g)^\sigma$ is also a conjugated algebra, and its closed elements yield a canonical extension of M.

Suppose $(M, (h_i)_I)$ is a monotone LE and (B, f, g) is a conjugated algebra whose closed elements are equal to M. Note that, as the closure operator $g \circ f$ is a map from B to M, we have $(g \circ f)^n\colon B^n \to M^n$. Consider the algebra $(B, (\overline{h_i})_I)$ where, for an n-ary operation h_i, we define $\overline{h_i} = h_i \circ (g \circ f)^n$. One can show that $(\overline{h_i})^\sigma\colon (B^\sigma)^n \to (M^\sigma)$ and that the restriction of $(\overline{h_i})^\sigma$ to $(M^\sigma)^n$ is the canonical extension of h_i in the sense of Definition 4.1. Thus, the theory of canonical extensions of monotone LEs could be developed within the theory of Boolean algebra expansions.

6. PRESERVATION OF IDENTITIES

Historically, a good deal of attention has been paid to identities preserved by canonical extensions. In their original 1951 paper, Jónsson and Tarski [28] showed that canonical extensions, when applied to Boolean algebras, preserve all identities built from basic operations which are operators. A similar result for canonical extensions of distributive lattices was recently given by Gehrke and Jónsson [15]. Our first task here is to present a similar result for lattices.

Definition 6.1. Let $s(x_1, \ldots, x_n)$ be a term whose variables are among $x_1, \ldots, x_n$. For a LE L of the appropriate type, we use $s^L(x_1, \ldots, x_n)$ for the n-ary operation on L induced by s.

Lemma 6.2. *Let L be a LE and let $s(x_1, \ldots, x_n)$ be a term of the appropriate type. If all the basic operations comprising s interpret to operators of L, then $(s^L(x_1, \ldots, x_n))^\sigma$ is equal to $s^{L^\sigma}(x_1, \ldots, x_n)$. Dually, if all the basic* *operations comprising s interpret to dual operators of L, then* $(s^L(x_1, \ldots, x_n))^\pi$ *is equal to* $s^{L^\pi}(x_1, \ldots, x_n)$.

Proof. We introduce some notation. For $f_1, \ldots, f_m$ a family of maps from $L^n \to L$, let $[f_1, \ldots, f_m]$ be the induced map from $L^n \to L^m$. We leave to the reader the routine verification that $[f_1, \ldots, f_m]^\sigma = [f_1^\sigma, \ldots, f_m^\sigma]$.

We now prove our claim by induction on the complexity of the term s. If s has no basic operations, then s must be one of the coordinate projections, and the verification in this case is left to the reader. Suppose $s = f(s_1, \ldots, s_m)$. Then $s^L = f^L \circ [s_1^L, \ldots, s_m^L]$. As f^L is an operator, by Corollary 4.8 we have that $(f^L)^\sigma$ is a complete operator and therefore preserves upwardly directed joins. It then follows from Lemma 4.5 that $(s^L)^\sigma = (f^L)^\sigma \circ [s_1^L, \ldots, s_m^L]^\sigma$. From the above remarks $(s^L)^\sigma = f^{L^\sigma} \circ [(s_1^L)^\sigma, \ldots, (s_m^L)^\sigma]$. The inductive hypothesis then yields $(s^L)^\sigma = f^{L^\sigma} \circ [s_1^{L^\sigma}, \ldots, s_m^{L^\sigma}]$, which is equal to s^{L^σ}. ∎

THEOREM 6.3. *Let L be a LE and let* $s \simeq t$ *be an identity holding in L. If all of the basic operations comprising s, t interpret to operators of L, then* $s \simeq t$ *holds in* L^σ. *Dually, if all of the basic operations comprising s, t interpret to dual operators of L, then* $s \simeq t$ *holds in* L^π.

Proof. Assume the variables of s and t are among $x_1, \ldots, x_n$. Then as $s \simeq t$ holds in L, we have that the n-ary maps s^L and t^L are equal. Thus their extensions $(s^L)^\sigma$ and $(t^L)^\sigma$ are equal, and by the previous lemma $s^{L^\sigma} = t^{L^\sigma}$. Therefore, the identity $s \simeq t$ holds in L^σ. ∎

Remark 6.4. This result sheds light on the reason distributivity is preserved by canonical extensions while other lattice identities, such as modularity, are not preserved. The binary operation of lattice meet is an operator iff the lattice is distributive. It should be noted that even though this result comes in a dual pair, dealing both with operators and dual operators, it does not allow mixing these. In a non-distributive lattice the join is certainly an operator as it is even join preserving and the meet is a dual operator being meet preserving, but join is not a dual operator nor is meet an operator, so for such lattices identities involving both the meet and the join, such as modularity, are not covered by the theorem.

While it is not our intent here to give a thorough analysis of which identities are preserved by canonical extensions, we present two particular cases of interest. The first was obtained in [22].

PROPOSITION 6.5. *The canonical extension of an ortholattice is an ortholattice.*

Proof. Suppose $f\colon L \to L$ is an orthocomplementation on the lattice L, that is to say, that f is an anti-isomorphism of period two and satisfies

$a \vee f(a) = 1$. Let $g\colon L^d \to L$ be the same set mapping as f, but with the indicated alteration in the ordering of the domain (g is the map f^α for $\alpha = (1)$). Note, by Lemma 5.3, that the maps g^σ, $(g^d)^\sigma$, and f^σ agree as set mappings.

As g is a homomorphism, Corollary 4.7 yields that g^σ preserves non-empty joins and meets. So, by Lemma 4.5, $\mathrm{id}_{L^\sigma} = (\mathrm{id}_L)^\sigma = (g \circ g^d)^\sigma = g^\sigma \circ (g^d)^\sigma$. This shows that f^σ is a period two dual homomorphism and hence an anti-isomorphism.

We next show that $y \in L^\sigma$ and $y \geq f^\sigma y$ implies $y = 1$. Suppose $q \in L^\sigma$ is open and $q \geq y$. Then $q \geq y \geq f^\sigma(y) = f^\sigma(\bigwedge\{r : r \text{ is open and } y \leq r\})$. As f^σ takes meets to joins, $q \geq \bigvee\{f^\sigma(r) : r \text{ is open and } y \leq r\}$. Thus $q \geq f^\sigma(q)$. Therefore, $\bigvee\{a : a \in L \text{ and } a \leq q\} = q \geq f^\sigma(q) = \bigwedge\{f(b) : b \in L \text{ and } b \leq q\}$. As this join and meet are directed, compactness yields some $a, b \in L$ with $a, b \leq q$ and $a \geq f(b)$. Then setting $c = a \vee b$, we have $c \leq q$ and $c \geq f(c)$. This implies $c = 1$ and hence $q = 1$, and as q is an arbitrary open element above y, that $y = 1$, as required. Suppose that $x \in L^\sigma$. Then $x \vee f^\sigma(x) \geq f^\sigma(x) \wedge x = f^\sigma(x \vee f^\sigma(x))$; hence $x \vee f^\sigma(x) = 1$. ∎

The following result about Galois connections was first demonstrated in the case of Boolean algebras in [28] under the guise of conjugated operators.

PROPOSITION 6.6. *If f, g are a Galois connection on a lattice L, then f^σ, g^σ are a Galois connection on L^σ.*

Proof. Surely f^σ and g^σ are order inverting. Let $\tilde{f}\colon L \to L^d$ and $\tilde{g}\colon L^d \to L$ be the same set maps as f and g, but with the indicated changes in the order on the domain or range. Note that $\tilde{f}^\sigma = f^\pi$ and $\tilde{g}^\sigma = g^\sigma$, and both $\tilde{f}$ and $\tilde{g}$ are order preserving. Then as $\mathrm{id}_L \leq g \circ f$, we have $\mathrm{id}_L \leq \tilde{g} \circ \tilde{f}$; hence $(\mathrm{id}_L)^\sigma \leq (\tilde{g} \circ \tilde{f})^\sigma$. So, by Lemma 4.5, $(\mathrm{id}_L)^\sigma \leq \tilde{g}^\sigma \circ \tilde{f}^\pi$, yielding $\mathrm{id}_{L^\sigma} \leq g^\sigma \circ f^\sigma$. Similarly, $\mathrm{id}_{L^\sigma} \leq f^\sigma \circ g^\sigma$. This shows f^σ, g^σ are a Galois connection on L^σ. ∎

We shall present one more preservation theorem originally proved in the distributive case in [16]. The proof is based on the following technical lemma with its origins in [14]. For background on Boolean products, see [7].

LEMMA 6.7. *Let $(L_x)_{x \in X}$ be a family of LEs of the same type. If $L \leq \prod_X L_x$ is a Boolean product, then $L^\sigma = \prod_X L_x^\sigma$ and dually $L^\pi = \prod_X L_x^\pi$.*

Proof. We first show that the identical embedding of L into $\prod_X L_x^\sigma$ is a canonical extension. As each L_x^σ is complete, the product $\prod_X L_x^\sigma$ is a complete lattice. Suppose $x \in X$ and $p \in L_x^\sigma$ is closed. Define $u_{x,p} \in$

$\prod_X L_X^\sigma$ by setting $u_{x,p}(x) = p$ and $u_{x,p}(y) = 0$ for $y \neq x$. We first show that $u_{x,p}$ is a meet in $\prod_X L_x^\sigma$ of elements from L. It then follows that every element of $\prod_X L_x^\sigma$ is a join of meets of elements of L and, by a similar argument, a meet of joins of elements of L.

To show that $u_{x,p}$ is a meet of elements of L, note first that p is a meet in L_x^σ of a family S of elements of L_x. As $L \leq \prod_X L_x$ is subdirect, for each $s \in S$ there is some $u_s \in L$ with $u_s(x) = s$. Using the patchwork property, for each clopen neighborhood N of x, and each $s \in S$, we have $u_s|N \cup 0|N^c$ is an element of L. Then, the meet of $\{u_s|N \cup 0|N^c : s \in S, x \in N \text{ clopen}\}$ is equal to $u_{x,p}$. This shows that the identical embedding of L into $\prod_X L_x^\sigma$ is dense.

Next, we show that the identical embedding of L into $\prod_X L_x^\sigma$ is compact. Suppose that S is a filter of L, T is an ideal of L, and $\bigwedge S \leq \bigvee T$. For each $x \in X$ let $S_x = \{u(x) : u \in S\}$ and let $T_x = \{v(x) : v \in T\}$. Then $\bigwedge S_x \leq \bigvee T_x$ for each $x \in X$. As L_x^σ is a canonical extension of L_x, $S_x \cap T_x \neq \emptyset$; hence there are $u_x \in S$ and $v_x \in T$ with $u_x(x) = v_x(x)$. As equalizers in a Boolean product are clopen, u_x and v_x agree on some clopen neighborhood N_x of x. Then, as X is compact, and $\{N_x : x \in X\}$ is an open cover of X, there is a finite family $x_1, \dots, x_n$ with $N_{x_1}, \dots, N_{x_n}$ a cover of X. We assume, without loss of generality, that $N_{x_1}, \dots, N_{x_n}$ are pairwise disjoint. Let w be the function which agrees with u_{x_i}, hence also with v_{x_i}, on N_{x_i} for $i = 1, \dots, n$. By the patchwork property, w is an element of L. Also, w is the join of the n functions agreeing with u_{x_i} on N_{x_i} and being 0 elsewhere; hence w is in the ideal S. Similarly w is the meet of the n functions agreeing with v_{x_i} on N_{x_i} and being 1 elsewhere; hence w is in the filter T. Thus, $S \cap T \neq \emptyset$. This shows that the identical embedding of L into $\prod_X L_x^\sigma$ is compact and hence a canonical extension.

It remains to show that the operations of $\prod_X L_x^\sigma$ are the canonical extensions of the operations of L. Suppose $x \in X$; let K_x and O_x be the closed and open elements of L_x^σ, respectively.

Claim. Let $u \in L^\sigma = \prod_X L_x^\sigma$.

(1) $\{p(x) : p \in K, p \leq u\} = \{c : c \in K_x, c \leq u(x)\}$.

(2) $\{q(x) : q \in O, u \leq q\} = \{e : e \in O_x, u(x) \leq e\}$.

(3) $\{a(x) : a \in L, p \leq a \leq q\} = \{d : d \in L_x, p(x) \leq d \leq q(x)\}$ any $p \in K, q \in O$.

For the first, $p \in K$ and $p \leq u$ clearly implies $p(x) \in K_x$ and $p(x) \leq u(x)$. Suppose $c \in K_x$ and $c \leq u(x)$. The map $u_{x,c}$ described above belongs to K, lies below u, and satisfies $u_{x,c}(x) = c$. The second follows similarly. For the third, note that $a \in L$ with $p \leq a \leq q$ clearly implies $a(x) \in L_x$ and $p(x) \leq a(x) \leq q(x)$. For the converse, assume $d \in L_x$ with $p(x) \leq d \leq$

$q(x)$. As $L \leq \prod_X L_x$ is a subdirect product, there is $h \in L$ with $h(x) = d$. As $p(x) = \bigwedge\{j(x) : j \in L, p \leq j\} \leq d$, the compactness of L_x^σ provides some $j \in L$ with $p \leq j$ and $j(x) \leq d$. Similarly we get $k \in L$ with $k \leq q$ and $d \leq k(x)$. Finally, using the compactness of L^σ and $p \leq q$ we get some $m \in L$ with $p \leq m \leq q$. Set N to be the intersection of the equalizers of $j \leq h$ and $h \leq k$. Using the patchwork property and the fact that equalizers are clopen, $a = h|N \cup m|N^c$ is an element of L with $p \leq a \leq q$ and $a(x) = d$.

We can now show the operations of $\prod_X L_x^\sigma$ are the canonical extensions of the operations of L. Let f be a unary operation symbol in our language. We must show that for any $x \in X$ and any $u \in \prod_X L_x^\sigma$ that $(f^L)^\sigma(u)(x) = f^{L_x^\sigma}(u(x))$. From 4.1, the first quantity is $\bigvee\{\bigwedge\{f^{L_x}(a(x)) : p \leq a \leq q\} : p \in K, q \in O, p \leq u \leq q\}$ while the second is $\bigvee\{\bigwedge\{f^{L_x}(d) : c \leq d \leq e\} : c \in K_x, e \in O_x, c \leq u(x) \leq e\}$. Equality then follows easily from the above claim. The argument for n-ary operations is obviously similar. ▮

To state the following theorem, it is convenient to introduce some terminology. Let $\mathscr{K}$ be a class of LEs of the same type, each of which is monotone. We say $\mathscr{K}$ is a monotone class if for each $L, M \in \mathscr{K}$ and each basic operation symbol f, if f^L is order preserving (reversing) in the ith coordinate, then f^M is also order preserving (reversing) in the ith coordinate. Note that a product of members of a monotone class will necessarily be monotone.

THEOREM 6.8. *If $\mathscr{K}$ is a monotone class which is closed under (dual) canonical extensions and ultraproducts, then the variety generated by $\mathscr{K}$ is closed under (dual) canonical extensions.*

Proof. Let L be in the variety generated by $\mathscr{K}$. Then there is a family $(M_i)_I$ in $\mathscr{K}$, a subalgebra $S \leq \prod_I M_i$, and an onto homomorphism φ: $S \to L$. As $\mathscr{K}$ is a monotone class, $\prod_I M_i$ is monotone; hence S is monotone, and therefore L is monotone. So $\prod_I M_i$, S, and L are objects of the category $\mathscr{C}_\tau$, where τ is the type of the algebra in question. By Theorem 5.4, $S^\sigma \leq (\prod_I M_i)^\sigma$, and φ^σ: $S^\sigma \to L^\sigma$ is an onto homomorphism. Thus, it suffices to show $(\prod_I M_i)^\sigma$ is in the variety generated by $\mathscr{K}$.

As with any product of a family of algebras, we can represent $M = \prod_I M_i$ as a Boolean product $M \leq \prod_X L_x$ where the stalks L_x are all ultraproducts of the family $(M_i)_I$. As $\mathscr{K}$ is closed under canonical extensions and ultraproducts, it follows that L_x^σ belongs to $\mathscr{K}$ for each $x \in X$. But, by Lemma 6.7, $M^\sigma = \prod_X L_X^\sigma$; hence it is a product of members of $\mathscr{K}$ and therefore in the variety generated by $\mathscr{K}$. ▮

As the canonical extension of a finite lattice is just the lattice itself, we immediately obtain the following.

COROLLARY 6.9. *If L is a monotone LE whose underlying set is finite, then the variety generated by L is closed under canonical extensions and dual canonical extensions*.

Note that this corollary also provides an explanation why the varieties of distributive lattices, and Boolean algebras, are closed under canonical extensions.

7. CONCLUSIONS

While this work has developed the basics of canonical extensions of lattice expansions, there remains much to be done. The preservation results obtained in Section 6 are likely only the tip of the iceberg. Certainly a more complete account of various types of identities preserved by canonical extensions could be developed, and perhaps results from BAOs, such as Sahlqvist's theorem [26, 34], could be adapted to LEs.

Another matter deserving attention is the relationship between canonical extensions of LEs and the current interest in developing a type of Kripke semantics for various non-classical logics such as linear logic [1] and its BCK fragment [30]. Certainly the relationship between canonical extensions of BAOs and Kripke semantics for modal logics, etc., has been thoroughly investigated and put to very good use. Essentially, canonical extensions provide a purely algebraic route to completeness theorems. While we have not explored the matter, the situation for extensions of LEs is likely analogous. In this case, the advantages provided by the algebraic methods may be even more apparent due to the difficulty in working with Urquhart duality.

REFERENCES

1. G. Allwein and J. M. Dunn, Kripke models for linear logic, *J. Symbolic Logic* **58** (1993), 514–545.
2. J. A. F. K. van Benthem, Some kinds of modal completeness, *Studia Logica* **39** (1980), 125–141.
3. G. Birkhoff, "Lattice Theory," 3rd ed., American Mathematical Society Colloquium Publications, Vol. 25, Amer. Math. Soc., Providence, 1967.
4. W. J. Blok, The lattice of modal logics: an algebraic investigation, *J. Symbolic Logic* **45** (1980), 221–236.
5. T. S. Blyth and J.C. Varlet, "Ockham Algebras," Oxford Univ. Press, London, 1994.

6. V. Boicescu, A. Filipoiu, G. Georgescu, and S. Rudeanu, "Lukasiewicz–Moisil Algebras," Annals of Discrete Mathematics, Vol. 49, North-Holland, Amsterdam, 1991.
7. S. Burris and H. P. Sankappanavar, "A Course in Universal Algebra," Springer-Verlag, Berlin/New York, 1981.
8. S. Celani and R. Jansana, A new semantic for positive modal logic, *Notre Dame J. Formal Logic* **38** (1997), 1–18.
9. R. Cignoli, Quantifiers on distributive lattices, *Discrete Math.* **96** (1991), 183–197.
10. P. Crawley and R. P. Dilworth, "Algebraic Theory of Lattices," Prentice Hall, New York, 1973.
11. M. L. Dalla Chiara, Quantum logic, *in* "Handbook of Philosophical Logic," Vol. III, pp. 427–469, Reidel, Dordrecht, 1986.
12. J. M. Dunn, Positive modal logic, *Studia Logica* **55**, No. 2 (1995), 301–317.
13. J. M. Dunn and C. Hartonas, Stone duality for lattices, *Algebra Universalis* **37**, No. 3 (1997), 391–401.
14. M. Gehrke, The order structure of Stone spaces and the T_D-separation axiom, *Z. Math. Logik Grundlag. Math.* **37**, No. 1 (1991), 5–15.
15. M. Gehrke and B. Jónsson, Bounded distributive lattices with operators, *Math. Japon.* **40**, No. 2 (1994), 207–215.
16. M. Gehrke and B. Jónsson, Monotone bounded distributive lattice expansions, *Math. Japon.* **52**, No. 2 (2000), 197–213.
17. M. Gehrke and B. Jónsson, Bounded distributive lattice expansions, Preprint.
18. S. Ghilardi and G. Meloni, Constructive canonicity in non-classical logics, *Ann. Pure Appl. Logic* **86** (1997), 1–32.
19. R. Goldblatt, Varieties of complex algebras, *Ann. Pure Appl. Logic* **44** (1989), 173–242.
20. R. Goldblatt, On closure under canonical embedding algebras, *in* "Algebraic Logic, Proc. Conf. Budapest 1988," Colloq. Math. Soc. J. Bolyai, Vol. 54, pp. 217–231, North-Holland, Amsterdam, 1991.
21. R. Goldblatt, The McKinsey axiom is not canonical, *J. Symbolic Logic* **56** (1991), 554–562.
22. J. Harding, Canonical completions of lattices and ortholattices, *Tatra Mt. Math. Publ.* **15** (1998), 85–96.
23. C. Hartonas, Duality for lattice-ordered algebras and normal algebraizable logics, *Studia Logica* **58** (1997), 403–450.
24. G. Hartung, A topological representation of lattices, *Algebra Universalis* **29** (1992), 273–299.
25. L. Henkin, Extending Boolean operations, *Pacific J. Math.* **32** (1970), 723–752.
26. B. Jónsson, On the canonicity of Sahlqvist identities, *Studia Logica* **53**, No. 4 (1994), 473–491.
27. B. Jónsson, A survey of Boolean algebras with operators, *in* "Algebras and Orders (Montreal, PQ, 1991)," NATO Adv. Sci. Inst. Ser. C. Math. Phys. Sci., Vol. 389, pp. 239–286, Kluwer Academic, Dordrecht, 1993.
28. B. Jónsson and A. Tarski, Boolean algebras with operators, I, *Amer. J. Math.* **73** (1951), 891–993.
29. B. Jónsson and A. Tarski, Boolean algebras with operators, II, *Amer. J. Math.* **74** (1952), 127–162.
30. W. MacCaull, Kripke semantics for logics with BCK implication, *Bull. Sect. Logic Univ. Łódź* **25**, No. 1 (1996), 41–51.
31. H. A. Priestley, Natural dualities for varieties of distributive lattices with a quantifier, *in* "Algebraic Methods in Logic and Computer Science (Warsaw, 1991)," Banach Center Publ., Vol. 28, pp. 291–310, Polish Acad. Sci., Warsaw, 1993.

32. H. A. Priestley, Varieties of distributive lattices with unary operations, I, *J. Austral. Math. Soc. Ser. A* **63**, No. 2 (1997), 165–207.
33. H. A. Priestley and R. Santos, Varieties of distributive lattices with unary operations, II, *Portugal. Math.* **55**, No. 2 (1998), 135–166.
34. H. Sahlqvist, Completeness and correspondence in the first and second order semantics for modal logic, *in* "Proceedings of the Third Scandinavian Logic Symposium (Univ. Uppsala, Uppsala, 1973)," Stud. Logic Found. Math., Vol. 82, pp. 110–143, North-Holland, Amsterdam, 1975.
35. V. Sofronie-Stokkermans, Duality and canonical extensions of bounded distributive lattices with operators, and applications to the semantics of non-classical logics, I, II, *Studia Logica* **64** (2000), 93–132 and 151–172.
36. A. Urquhart, A topological representation theory for lattices, *Algebra Universalis* **8** (1978), 45–58.