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The free orthomodular lattice on countably many generators is a subalgebra of the free orthomodular lattice on three generators

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Abstract. We show every at most countable orthomodular lattice is a subalgebra of one generated by three elements. As a corollary we obtain that the free orthomodular lattice on countably many generators is a subalgebra of the free orthomodular lattice on three generators. This answers a question raised by Bruns in 1976 [2] and listed as Problem 15 in Kalmbach's book on orthomodular lattices [6].

1. Introduction

In comparison to the situation for free lattices, little is known about the structure of free orthomodular lattices (abbreviated: OMLs). There is a complete description of the free OML on two generators (it is finite with 96 elements) and the free OML on three generators is known to be infinite. However, there is no useful description of the free OML on three or more generators, no solution to the free word problem on three or more generators, and no description of the subalgebras of free OMLs on three or more generators.

It is our purpose here to provide one small additional piece of information — that the free OML on countably many generators is a subalgebra of the free OML on three generators. This result was anticipated by Bruns in his 1976 paper on free ortholattices [2] but he was unable to find a proof. The corresponding result for lattices has long been known, having been established by Whitman in his classic 1940's papers [8, 9].

Our method of proof is to show every at most countable OML is a subalgebra of a three-generated OML. The result then follows from the projectivity of free algebras. A similar result on three-generation was established by Greechie in 1977 [5] where he showed any complete atomic OML with at most countably many atoms is a subalgebra of one completely generated (generated using the operations of infinite join and meet) by three elements. Our result on three-generation is proved in a

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similar way to Greechie's, using many of the same ideas. We are able to prove the stronger result largely because of the (surprisingly recent) discovery that every OML can be embedded into an atomic OML in which every element is a join of two or fewer atoms.

This paper is organized as follows. In the second section we examine some properties of the OML P of subspaces of a three dimensional vector space over the rationals. In the third section we construct from P a three-generated OML called the scaffold. In the fourth section we consider a countable OML M such that each element of M is a join of finitely many atoms. We give a method to combine M with the scaffold to produce a three-generated OML with M (but not the scaffold) as a subalgebra. In the fifth section we describe the result that every OML can be embedded into an OML in which every element is a join of two or fewer atoms. In the final section we prove our result on three-generation, Greechie's result on three-generation, the result that the free OML on countably many generators is a subalgebra of the free OML on three generators, and give an explicit free generating set of the free OML on countably many generators in terms of the generators of the free OML on three generators. We also remark that all our results can be obtained even if we enforce a comparability $x \leq y$ between two of the three generators.

Throughout we assume the reader is familiar with orthomodular lattices. Consult [6] for background on OMLs, [4] for background on free algebras.

2. The rational projective plane

Let P be the OML of all subspaces of a three dimensional vector space over the rationals with the orthocomplement of a subspace A being its orthogonal subspace $A^{\perp}$. It is well known that P is modular and that every maximal chain in P has four elements. Thus every element in P is either a bound, an atom, or a coatom. We use the notation $\langle 1,2,3 \rangle$ to denote the one dimensional subspace (atom) spanned by the vector $(1,2,3)$. Throughout, a, b, c are the atoms of P given by:

$$a = \langle 1,0,0 \rangle, \quad b = \langle -1,-1,1 \rangle \quad \text{and} \quad c = \langle 0,1,0 \rangle.$$

It will be helpful for the reader to recall the usual geometric interpretation of P as the projective closure of the rational Euclidean plane. Atoms of P are points of this projective plane, coatoms are lines. An atom of the form $\langle x,y,1 \rangle$ corresponds to the point (x,y) of the Euclidean plane, one of the form $\langle x,y,0 \rangle$ to the point at infinity intersecting all lines of slope y/x. The x-axis is the two dimensional subspace of vectors whose second component vanishes, the y-axis the two-dimensional subspace of vectors whose first component vanishes, and the line at infinity is the two dimensional subspace of vectors whose third component vanishes.

It will be convenient to introduce the notation

$$f = (b \vee (a' \wedge c')) \wedge (a \vee c), \quad g = c \vee (b' \wedge c') \quad \text{and} \quad h = c \vee (a' \wedge c').$$

Definition 2.1. Define recursively elements p_n $(n \geq 0)$ of P by setting

$$\begin{aligned} p_0 &= a' \wedge b' \\ p_{n+1} &= (((p_n \vee f) \wedge g) \vee a) \wedge h \end{aligned}$$

Lemma 2.2. *For each* $n \geq 0$, $p_n = \langle 0, n+1, 1 \rangle$.

Proof. We first collect a few observations. As $\langle 0,1,1\rangle$ is an atom orthogonal to both a, b it follows that $a' \wedge b' = \langle 0,1,1\rangle$. Similarly $a' \wedge c' = \langle 0,0,1\rangle$ and $b' \wedge c' = \langle 1,0,1\rangle$. Geometrically, these are the point one unit up on the y-axis, the origin, and the point one unit over on the x-axis respectively. Also note $a \vee c$ is the two dimensional subspace of all vectors having trivial third component, geometrically the line at infinity. The two dimensional subspace $b \vee (a' \wedge c')$ consists of all vectors whose first two components agree, geometrically the line of slope one through the origin, and therefore f being the intersection of $b \vee (a' \wedge c')$ and $a \vee c$ is the point $\langle 1,1,0\rangle$ on the line at infinity that intersects every line of slope one. The two dimensional subspace $h = c \vee (a' \wedge c')$ consists of all vectors having trivial first component, geometrically the y-axis, and $g = c \vee (b' \wedge c')$ consists of all vectors whose first and third components agree, geometrically the vertical line one unit to the right of the y-axis.

We now establish our result by induction on n. For $n = 0$ we have noted above $a' \wedge b' = \langle 0,1,1\rangle$. Assume $p_n = \langle 0, n+1, 1\rangle$. Then $(p_n \vee f) \wedge g$ is the one dimensional subspace of vectors belonging to $\langle 0, n+1, 1\rangle \vee \langle 1,1,0\rangle$ whose first and third components agree, i.e., $\langle 1, n+2, 1\rangle$. Geometrically, $p_n \vee f$ is the line of slope one through p_n and $(p_n \vee f) \wedge g$ is the intersection of this line with the vertical line g one unit to the right of the y-axis. Our formula for p_{n+1} reduces to $((\langle 1, n+2, 1\rangle \vee a) \wedge h$ which is the one dimensional subspace of vectors in $\langle 1, n+2, 1\rangle \vee \langle 1,0,0\rangle$ with trivial first component, i.e., $\langle 0, n+2, 1\rangle$. Geometrically, $\langle 1, n+2, 1\rangle \vee a$ is the horizontal line through $\langle 1, n+2, 1\rangle$ and p_{n+1} is the intersection of this horizontal line with the y-axis. □

For our purposes the necessary fact is that the atoms p_n $(n \geq 0)$ belong to the subalgebra of P generated by a, b, c, and this is obvious from the definition of the p_n. However, the following result that a, b, c generate P clarifies the situation. A proof of this result can be given either by a brute force technique as in the previous lemma, or via the coordinatization theorem for projective geometries using the fact that the rationals have no proper subfields.

Proposition 2.3. *P is generated as an OML by a, b, c.*

3. The scaffold

Here we construct from the OML P introduced in the previous section a new OML called the scaffold S. Recall a, b, c, p_n $(n \geq 1)$ were atoms of P given by $a = \langle 1, 0, 0\rangle$, $b = \langle -1, -1, 1\rangle$, $c = \langle 0, 1, 0\rangle$, and $p_n = \langle 0, n+1, 1\rangle$ for each $n \geq 1$. We omit p_0 from our considerations solely to produce the pleasant diagram below. Note that

(1) p_n does not commute with c for any $n \geq 1$,
(2) p_m does not commute with p_n for any $m \neq n$.

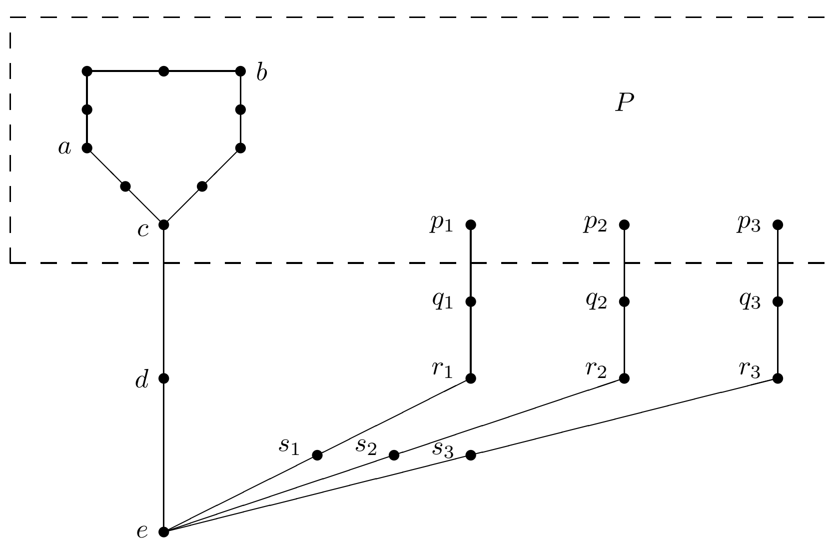
We next introduce new elements we will use to construct the scaffold S from P. Let $d, d', e, e', q_n, q'_n, r_n, r'_n, s_n, s'_n$ $(n \geq 1)$ be elements not occuring in P. Define a family of eight element Boolean algebras B, C_n, D_n $(n \geq 1)$ by setting

$$B = \{0, 1, c, c', d, d', e, e'\};$$

$$C_n = \{0, 1, e, e', s_n, s'_n, r_n, r'_n\};$$

$$D_n = \{0, 1, p_n, p'_n, q_n, q'_n, r_n, r'_n\}.$$

In each case we assume the un-primed letters are the atoms and the corresponding primed letters are their orthocomplements. We let $\mathcal{A}$ be the collection of all blocks of P and let $\mathcal{B} = \mathcal{A} \cup \{B\} \cup \{C_n, D_n \mid n \geq 1\}$.



We define the set $S = \bigcup \mathcal{B}$ to be the union of the Boolean algebras in $\mathcal{B}$ and define a relation $\leq$ on S to be the union of the partial orderings on the Boolean algebras in $\mathcal{B}$. Note that $\mathcal{B}$ is a family of eight element Boolean algebras, any two of which intersect either in $\{0, 1\}$ or in $\{0, x, x', 1\}$ where x is an atom of both. From this, it is well known [6, p. 42] that $\leq$ is a partial ordering on S and that the union of the orthocomplementations on the Boolean algebras in $\mathcal{B}$ yields an orthocomplementation on this partially ordered set S.

Proposition 3.1. *S is an OML and every maximal chain in S has four elements.*

Proof. The Greechie loop lemma [6, p. 42] yields S is an OML provided the family $\mathcal{B}$ has no 4-loops. As P is an OML, any 4-loop must necessarily use some member of $\mathcal{B}$ which is not a block of P. Assume we have some loop that uses B. Then this loop must go from B to C_n, D_n for some $n \geq 1$. Then as c does not commute with p_n this loop must use at least 5 Boolean algebras to close. A loop using Boolean algebras which are not blocks of P and not using B must use C_m, C_n for some $m \neq n$, and hence must also use D_m, D_n, and as p_m does not commute with p_n, the loop must use at least 6 Boolean algebras to close. This shows S is an OML. As every maximal chain in S contains the bounds and each element belongs to a chain with four elements, every maximal chain has at least four elements. But a chain with five elements would easily yield that a coatom of one Boolean algebra in $\mathcal{B}$ was an atom of another, and this is not the case. $\square$

Proposition 3.2. *S is generated as an OML by a, b, e.*

Proof. We first establish P is a subalgebra of S by showing joins in P agree with joins in S. Note first that each of the new Boolean algebras B, C_n, D_n intersects P either in $\{0, 1\}$ or in $\{0, 1, x, x'\}$ where x is an atom of P, so forming S introduces no new comparabilities between elements of P. Therefore the identical embedding of P into S is an order embedding. For x, y comparable elements of P surely the join of x, y in P agrees with the join of x, y in S. If x, y are incomparable elements of P with y a coatom of P, then x, y are incomparable elements of S with y a coatom of S, and in either setting x, y join to the unit. Finally if x, y are distinct atoms of P, then as P is a projective plane their join z in P is a coatom of P and hence of S. Then as x, y are incomparable in S and maximal chains in S have four elements, it follows that z is also the join of x, y in S. Thus P is a subalgebra of S. Let T be the subalgebra of S generated by a, b, e. As $a, e \leq c'$ it follows that $a \vee e = c'$ hence $c \in T$. As P is a subalgebra of S and P is generated by a, b, c it follows that P is a subalgebra of T. In particular $p_n \in T$ for each $n \geq 1$. But $r'_n = p_n \vee e$, $q'_n = p_n \vee r_n$ and $s'_n = r_n \vee e$. Thus $T = S$. $\square$

4. The construction

Let M be a countable OML in which each element can be expressed as a join of finitely many atoms. In this section we will construct from M and the scaffold S a new OML A which is three-generated and contains an isomorphic copy of M as a subalgebra.

As M is countable, the requirements placed on it ensure that M has countably many atoms. As we are working up to isomorphism, we may assume that the atoms of M are the elements q_n $(n \geq 1)$ of S, that the orthocomplements of these elements as taken in M are the elements q_n' $(n \geq 1)$ of S, that M and S have the same bounds, and that $S \cap M = \{q_n, q_n' \mid n \geq 1\} \cup \{0, 1\}$. Note, an atom of S that belongs to M is also an atom of M, and a coatom of S that belongs to M is the orthocomplement of an atom of M, hence a coatom of M. Define

$$A = S \cup M,$$
$$\leq \; = \; \leq_S \cup \leq_M,$$
$$' = {'_S} \cup {'_M}.$$

Note, as the orthocomplements on S and M agree on the intersection $S \cap M$ their union does define a map $'$ on $S \cup M$. Observe also that the partial orderings $\leq_S$ and $\leq_M$ need not agree on the intersection $S \cap M$.

Lemma 4.1. *$(A, \leq, {}')$ is an orthocomplemented poset.*

Proof. As $\leq_S$ and $\leq_M$ are reflexive, so also is $\leq$. Suppose $x, y \in A$ with $x \leq y$ and $y \leq x$. We must show $x = y$. This is obvious if $x \leq_S y$ and $y \leq_S x$ or if $x \leq_M y$ and $y \leq_M x$. So assume without loss of generality that $x \leq_S y$ and $y \leq_M x$, and note that this implies $x, y \in S \cap M$. If either x, y is a bound, the conclusion $x = y$ is obvious. So we need only consider the case that x is an atom of S and y is a coatom of S. Then as x, y also belong to M and x is an atom of M, the assumption $y \leq_M x$ yields $y = x$. Thus $\leq$ is antisymmetric. For transitivity, assume $x \leq y \leq z$. There are four cases: $x \leq_S y \leq_S z$, $x \leq_M y \leq_M z$, $x \leq_S y \leq_M z$, or $x \leq_M y \leq_S z$. The first two cases yield trivially $x \leq z$ as $\leq_S$ and $\leq_M$ are transitive. Once the third case is established the fourth follows as $x \leq_M y \leq_S z$ implies $z' \leq_S y' \leq_M x'$. Assume $x \leq_S y \leq_M z$. If any of x, y, z is a bound, or if two of x, y, z are equal, trivially $x \leq z$. Otherwise, as $x \leq_S y$ implies $x, y \in S$, we are left with the case that x is an atom of S and y is a coatom of S. But $y \leq_M z$ implies $y, z \in M$, thus the coatom y of S belongs to M, and therefore is a coatom of M. Then $y \leq_M z$ yields $y = z$. This establishes transitivity, and therefore that $\leq$ is a partial ordering.

Clearly the common bounds $0, 1$ of S and M are the bounds of A. Also, as each of $'_S$ and $'_M$ are order inverting and period two it follows that $'$ is order inverting and period two. We need only show for any $x \in A$ that 0 is the only lower bound of x, x'. Assume y is a lower bound of both x, x'. Note, $y \leq_S x$ and $y \leq_S x'$ implies $y = 0$, also $y \leq_M x$ and $y \leq_M x'$ implies $y = 0$. So we may assume without loss of generality that $y \leq_S x$ and $y \leq_M x'$. Note this implies $x, y \in S \cap M$. If either (hence both) x, x' is a bound, clearly $y = 0$. Otherwise one of x, x' is an atom of S, hence also an atom of M. Then $y \neq 0$ yields either $y = x$ or $y = x'$. The first gives $x \leq_M x'$ hence $y = x = 0$, and the second gives $x' \leq_S x$ hence $y = x' = 0$. □

Lemma 4.2. *If x, y are elements of S, neither belonging to M, then $x \vee_S y$ is the least upper bound of x, y in A.*

Proof. Surely $x \vee_S y$ is an upper bound of x, y in A. Suppose $z \in A$ and $x, y \leq z$. As neither x, y belongs to M we must have $x \leq_S z$ and $y \leq_S z$ hence $x \vee_S y \leq_S z$ and therefore $x \vee_S y \leq z$. □

Thus far we have used relatively little of the way M intersects S. It is the following lemma where we use more detailed properties, including the specifics of the elements p_n $(n \geq 1)$ created in P.

Lemma 4.3. *With xC_Sy denoting that x, y commute in the OML S, we have:*

(1) *If $x, y \in M$ then $x \leq_S y$ implies $x \leq_M y$, hence $x \leq y$ implies $x \leq_M y$.*
(2) *If w, x, y, z are atoms of S with wC_Sx, xC_Sy, yC_Sz, and $w, z \in M$, then $w = z$.*

Proof. For the first statement assume $x, y \in M$ and $x \leq_S y$. Then x, y belong to the intersection $S \cap M$ which is equal to $\{q_n, q_n' \mid n \geq 1\} \cup \{0, 1\}$. As $x \leq_S y$, the definition of S yields that either one of x, y is a bound or $x = y$. For the second statement assume w, z are distinct atoms of S that belong to M. Then there are $m \neq n$ with $w = q_m$ and $z = q_n$. The only atoms of S that commute with q_m in the OML S are p_m, q_m, r_m. But none of p_m, q_m, r_m commutes with any of p_n, q_n, r_n in S. □

Lemma 4.4. *If $x, y \in M$ then $x \vee_M y$ is the least upper bound of x, y in A.*

Proof. Surely $x \vee_M y$ is an upper bound of x, y in A. Suppose $z \in A$ and that $x, y \leq z$. If $z \in M$ then by the first part of the previous lemma $x \leq_M z$ and $y \leq_M z$ hence $x \vee_M y \leq_M z$ and $x \vee_M y \leq z$. So assume z belongs to S but not M. Then $x \leq_S z$ and $y \leq_S z$ so $x, y \in S \cap M$. If either x, y is a bound or a coatom of S, and hence of M, or if $x = y$, the result follows easily. Otherwise x, y are distinct atoms of S belonging to M. Then $x \leq_S z$ and $y \leq_S z$ implies either the trivial case that $z = 1$ or that z is a coatom of S. Then z' is an atom of S and as $x, y \leq_S z$ we have xC_Sz', $z'C_Sz'$, $z'C_Sy$, contrary to the second part of the above lemma. □

Lemma 4.5. *If $x \in S - M$ and $y \in M$, then x, y have a least upper bound in A.*

Proof. If x is a coatom of S then the only upper bounds of x in A are $x, 1$ and therefore either x or 1 is the least upper bound of x, y depending on whether $y \leq x$. So assume x is an atom of S. Any upper bound of x, y must necessarily belong to S as $x \notin M$, and assuming y is non-zero, must therefore be either 1 or a coatom of S. We claim that if $y \neq 0$ there can be at most one coatom of S that is an upper bound of x, y. From this it follows that x, y have a least upper bound in A.

Suppose u, v are atoms of S with u' and v' upper bounds of x, y. We wish to show $u = v$. Let z be an atom of M with $z \leq y$. Then u', v' are also upper bounds of x, z. As $x \notin M$ we must have $x \leq_S u'$ and $x \leq_S v'$. If u, v are distinct, then as maximal chains in S have four elements, $x = u' \wedge_S v'$. If $z \leq_S u'$ and $z \leq_S v'$ then $z \leq_S u' \wedge_S v'$ hence $z \leq_S x$ yielding $z = x$ contrary to the assumption $x \notin M$. If $z \leq_S u'$ and $z \leq_M v'$ then $z, v \in M$. But $z \leq_S u'$ and $x \leq_S u', v$ give zC_Su, uC_Sx, xC_Sv. So by Lemma 4.3 $z = v$, contrary to z being an atom and $z \leq v'$. Finally, if $z \leq_M u'$ and $z \leq_M v'$ then $u', v' \in M$. But $x \leq_S u'$ and $x \leq_S v'$ then yield uC_Sx, xC_Sx, xC_Sv and Lemma 4.3 gives $u = v$. □

Proposition 4.6. *A is an OML, M is a subalgebra of A, and A is generated as an OML by the elements a, b, e.*

Proof. Consider two elements x, y of A. Lemma 4.2 shows x, y have a least upper bound in A if neither belongs to M, Lemma 4.4 shows they have a least upper bound in A if both belong to M, and Lemma 4.5 shows they have a least upper bound in A if exactly one belongs to M. Thus any two elements in A have a least upper bound, and as A is an orthocomplemented poset, it follows that any two elements in A have a greatest lower bound. Thus A is a lattice, hence an ortholattice. To show A is an OML it is sufficient to show there do not exist elements x, y with $x < y$ and $x \vee y' = 1$. For such x, y we would have either $x <_S y$ or $x <_M y$. In the first case $x \vee_S y'$ would be a non-trivial upper bound of x, y, and in the second case $x \vee_M y'$ would be a non-trivial upper bound of x, y. Thus A is an OML. Lemma 4.4 provides M is a subalgebra of A.

Consider the subalgebra of A generated by a, b, e. As $a, e \in S - M$, Lemma 4.2 provides $a \vee_S e$ is the join of a, e in A. But $a \vee_S e = c'$. Thus a, b, c are in the subalgebra of A generated by a, b, e. But P is disjoint from M, except for the bounds, and P is generated as an OML by a, b, c. Thus by Lemma 4.2 each element of P belongs to the subalgebra of A generated by a, b, e. Similarly as none of e or p_n, r_n $(n \geq 1)$ belong to M and $e \vee_S p_n = r_n'$, $r_n \vee_S p_n = q_n'$ and $r_n \vee_S e = s_n'$, it follows that S is a subset of the subalgebra of A generated by a, b, e. In particular each atom of M belongs to this subalgebra. Then as each element of M is a join of finitely many atoms and M is a subalgebra of A, it follows that each element of M also belongs to the subalgebra of A generated by a, b, e. Thus a, b, e generate A. □

Note, we have not asserted that S is a subalgebra of A. Indeed as M is a subalgebra of A, S will be a subalgebra of A only if any two atoms of M join to 1, i.e., if $M = MO_\omega$.

5. Creating atoms

We present here the result that every OML can be embedded into an OML in which each element is a join of two or fewer atoms. This useful result has an interesting history. It is based fundamentally on the coatom extension developed by Bruns and Kalmbach [3] to insert an atom into an OML beneath a given element. Soon after its discovery the coatom extension was simplified by Greechie [5] and presented in terms of his paste job. That this device could be iterated to embed an OML into an atomic OML seems not to have been widely recognized. Remarkably the first definite source to attribute this result is an unpublished manuscript of Schröder [7] which owes its simplified form to Roddy. I seem to have played the small role of noticing that every element in the resulting OML was a join of finitely many atoms, in fact of two atoms.

Theorem 5.1. *Every OML can be embedded into an OML in which each element is a join of two or fewer atoms.*

The proof follows by a sequence of lemmata.

Lemma 5.2. *Given an OML L and $x \in L$ there is an OML $L(x)$ such that* (i) $L \leq L(x)$*,* (ii) *each atom of L is an atom of $L(x)$, and* (iii) *x is a join of two or fewer atoms in $L(x)$.*

Proof. If x is either 0 or an atom of L simply set $L(x) = L$. Otherwise use Greechie's paste job to paste L and $([0, x'] \cup [x, 1]) \times 2$ along the sections $[0, x'] \cup [x, 1]$ and $([0, x'] \times \{0\}) \cup ([x, 1] \times \{1\})$. Then $(0, 1)$ and $(x, 0)$ are atoms of the result that join to the element $(x, 1)$ identified with x. □

Lemma 5.3. *Given an OML L there is an OML L^* such that* (i) $L \leq L^*$*,* (ii) *each atom of L is an atom of L^*, and* (iii) *each element of L is a join of two or fewer atoms of L^*.*

Proof. Let $(x_\alpha)_\kappa$ be an indexing over a cardinal κ of L. Define recursively $L_0 = L$, $L_{\alpha+1} = L_\alpha(x_\alpha)$, and $L_\alpha = (\bigcup_{\beta<\alpha} L_\alpha)(x_\alpha)$ for α a limit ordinal. Set $L^* = L_\kappa$. □

Lemma 5.4. *Given an OML L there is an OML $\hat{L}$ such that* (i) $L \leq \hat{L}$*, and* (ii) *each element of $\hat{L}$ is the join of two or fewer atoms of $\hat{L}$.*

Proof. Define recursively $L^0 = L$, $L^{n+1} = (L^n)^*$. Set $\hat{L} = \bigcup_n L^n$. □

Remark 5.5. As a union of κ sets of cardinality κ has cardinality κ, the resulting OML $\hat{L}$ is of the same cardinality of the original, provided the original is infinite.

Remark 5.6. For each $x \in \hat{L}$ that is neither zero nor an atom we can find two atoms a_x and b_x of $\hat{L}$ that join to x such that $\{a_x, b_x\} \cap \{a_y, b_y\} = \emptyset$ for $x \neq y$.

6. Conclusions

Theorem 6.1. *Every countable OML can be embedded into a three-generated OML.*

Proof. Let L be a countable OML. Using theorem 5.1 embed L into a countable OML M in which each element is a join of two or fewer atoms. From M and the scaffold S form A. By proposition 4.6 A is a three-generated OML containing M and hence L as a subalgebra. □

While our modification of Greechie's original techniques has allowed the above result, our techniques also yield Greechie's original result detailed below.

Theorem 6.2. *Every complete atomic OML with countably many atoms is a subalgebra of one completely generated (generated using the operations of infinite join and meet) by three elements.*

Proof. We only sketch the details. Given a complete atomic OML M with countably many atoms, use M and the scaffold S to produce A as above. The proof that A is an OML requires only that M is atomic. One can also show that M is a complete subalgebra of A. Then as every atomic OML is atomistic, i.e., in every atomic OML each element is the join of the atoms beneath it, A is generated using the operations of orthocomplementation and complete join and meet by a, b, e. □

The completeness in the above theorem is not necessary if one modifies the notion of generation to be closure under existing infinite joins and meets.

Theorem 6.3. *The free OML on countably many generators F_ω is a subalgebra of the free OML on three generators F_3.*

Proof. As F_ω is countable we can find a three-generated OML A containing F_ω as a subalgebra. Then there is an onto homomorphism $\varphi : F_3 \to A$. As F_ω is projective it is a subalgebra of F_3. □

We can also determine a free generating set of F_ω in terms of the free generators of F_3. In the following we abuse notation and consider the p_n defined in 2.1 to be ortholattice polynomials $p_n(a, b, c)$.

Theorem 6.4. *Let x, y, z be the free generators of F_3. For each $n \geq 1$ set*

$$u_n = p_n(x, y, x' \wedge z'), \quad v_n = u_n' \wedge (u_n \vee z) \quad \textit{and} \quad w_n = v_{4n-3} \vee v_{4n-1}.$$

Then the w_n $(n \geq 1)$ freely generate a subalgebra of F_3 isomorphic to F_ω.

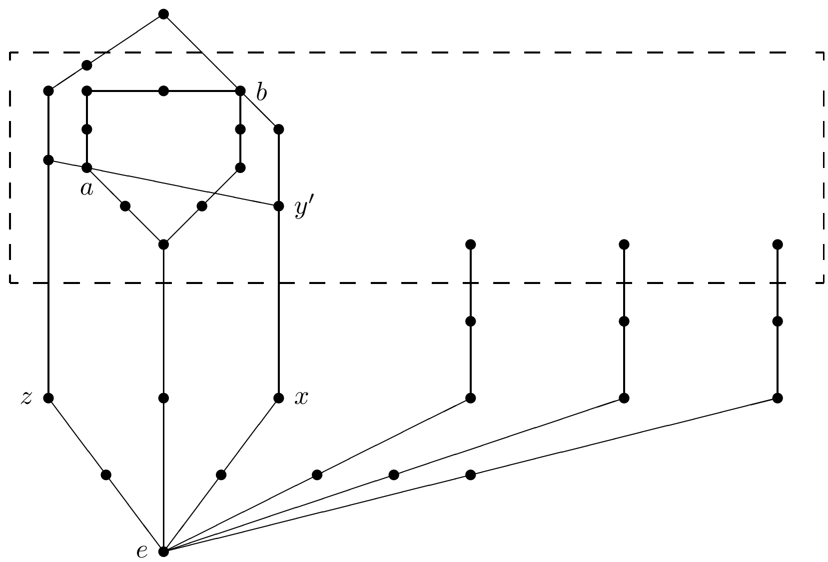
Proof. Embed F_ω into a countable OML M in which each nonzero element is a join of two or fewer atoms. By remark 5.6 to each element of M which is neither zero nor an atom we can associate two atoms of M that join to the given element, and furthermore this can be done in a way that never uses the same atom twice. When choosing an indexing $\{q_n \mid n \geq 1\}$ of the atoms of M we can and will make the choice so that q_1, q_3 join to one free generator of F_ω, q_5, q_7 to another, and so forth. In other words we require the elements $q_{4n-3} \vee q_{4n-1}$ $(n \geq 1)$ to comprise exactly the free generators of F_ω without repetition. This mechanism is chosen because we can devote infinitely many atoms of M to yield the generators, but we cannot use all the atoms of M for this task.

Construct from M and the scaffold S the OML A as before. Next define the homomorphism $\varphi\colon F_3 \to A$ to be the unique extension of the map taking the free generators x, y, z to a, b, e respectively. Note $\varphi(x' \wedge z') = a' \wedge e' = c$. It follows that $\varphi(u_n)$ is the element p_n of A, that $\varphi(v_n)$ is the element q_n of A, and therefore that the elements $\varphi(w_n)$ comprise exactly the free generators of F_ω without repetition.

Define the homomorphism $\psi\colon F_\omega \to F_3$ to be the unique extension of the map taking the generator $\varphi(w_n)$ to w_n. Then $\varphi \circ \psi$ is a homomorphism that restricts to the identity map on the generators of F_ω, hence $\varphi \circ \psi$ is the identity map on F_ω, and therefore ψ is an embedding of F_ω into F_3. As F_ω is isomorphic with its image under ψ, which is the subalgebra of F_3 generated by the w_n $(n \geq 1)$, our result follows. □

The modification to the scaffold S shown below allows us to embed any countable OML into an OML generated by three elements x, y, z with $x \leq y$. As a consequence one obtains that the free OML on countably many generators is a subalgebra of the freely presented OML on three generators x, y, z with $x \leq y$. We remark that a similar result on complete three generation was known to Greechie.

While we have given a free generating set of F_ω in terms of the free generators of F_3 we have not succeeded in characterizing such free generating sets. Any progress in this direction may be of significant use in developing the theory of free OMLs.



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