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Functional monadic Heyting algebras

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Abstract. We show every monadic Heyting algebra is isomorphic to a functional monadic Heyting algebra. This solves a 1957 problem of Monteiro and Varsavsky [9].

1. Introduction

In 1955 Halmos [5] introduced monadic algebras as algebraic counterparts of the one-variable fragment of classical predicate calculus. A monadic algebra (abbreviated: MA) is a Boolean algebra B with an additional unary operation ∇ satisfying (i) $\nabla 0 = 0$, (ii) $a \leq \nabla a$, and (iii) $\nabla(\nabla a \wedge b) = \nabla a \wedge \nabla b$. Roughly, ∇ is the algebraic counterpart of the existential quantifier $\exists$.[1]

Primary examples of MAs arise by considering a complete Boolean algebra B, a set X, and defining the operation ∇ on the Boolean algebra B^X of all functions $f\colon X \to B$ by setting

$$(\nabla f)(x) = \bigvee_{y \in X} f(y).$$

We call a MA *functional* if it is a subalgebra of one arising in this manner from some B, X.[2]

In [5,6] Halmos provided a representation theorem for MAs by showing that every MA is isomorphic to a functional MA. Using the close connection between functional MAs and algebraic models of classical predicate calculus, Halmos' representation theorem shows MAs do serve as algebraic counterparts of the one-variable fragment of classical predicate calculus.

In 1957 Monteiro and Varsavsky [9] introduced monadic Heyting algebras to serve the same purpose for intuitionistic predicate calculus as Halmos' MAs for classical

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[1]In fact Halmos used the symbol $\exists$ rather than ∇. Our use of ∇ is for consistency with later notation introduced by Monteiro and Varsavsky.

[2]Halmos' definition of functional MAs is slightly different but equivalent to ours (see Remark 2.9 in Preliminaries). Our preference is motivated by the transparency of the above definition as well as its consistency with later terminology of Monteiro and Varsavsky.

predicate calculus. A monadic Heyting algebra (abbreviated: MHA) is a Heyting algebra H with two additional unary operations Δ, ∇ satisfying certain conditions listed in the preliminaries below. Note that two operations are required as the intuitionistic quantifiers $\forall, \exists$ are not inter-definable through the other intuitionistic connectives.

Primary examples of MHAs arise by considering a complete Heyting algebra H, a set X, and defining operations Δ, ∇ on the Heyting algebra H^X of all functions $f\colon X \to H$ by setting

$$(\Delta f)(x) = \bigwedge_{y \in X} f(y) \quad \text{and} \quad (\nabla f)(x) = \bigvee_{y \in X} f(y).$$

A MHA is called *functional* if it is a subalgebra of one arising in this manner from some H, X.

It is natural to seek a representation theorem for MHAs by asking whether every MHA is isomorphic to a functional MHA. Indeed, Monteiro and Varsavsky posed this as an open problem in their 1957 paper [9] after showing Halmos' techniques were not applicable to MHAs. This problem has also been considered in [8, 10, 12].

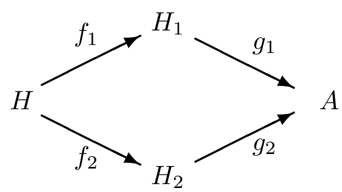
Here we provide a representation theorem for MHAs by showing that every MHA is isomorphic to a functional MHA. This solves the problem of Monteiro and Varsavsky. As our techniques are also applicable to MAs, we obtain a new proof, and slight strengthening, of Halmos' representation theorem. Additionally, as a by-product of our representation theorem, we obtain a new proof that MHAs are the algebraic counterparts of the one-variable fragment of intuitionistic predicate calculus. This was first proved by Bull [3], and later using different techniques by Ono and Suzuki [11].

2. Preliminaries

A *Heyting algebra* is an algebra $(H, \wedge, \vee, \to, 0, 1)$ where $(H, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\to$ is a binary operation on H satisfying $x \leq a \to b$ iff $a \wedge x \leq b$. It is well known that these conditions can be written equationally, hence the class of all Heyting algebras forms a variety. A Heyting algebra homomorphism that is also one-one is called a *Heyting algebra embedding.*

Definition 2.1. A *V-formation of Heyting algebras* is a quintuplet (H, H_1, H_2, f_1, f_2) consisting of Heyting algebras H, H_1, H_2 and Heyting algebra embeddings $f_1\colon H \to H_1$ and $f_2\colon H \to H_2$. An *amalgam* of the V-formation is a triple (A, g_1, g_2)

consisting of a Heyting algebra A and Heyting algebra embeddings $g_1\colon H_1 \to A$ and $g_2\colon H_2 \to A$ satisfying $g_1 \circ f_1 = g_2 \circ f_2$.



The amalgam is a *superamalgam* if for every $a_1 \in H_1, a_2 \in H_2$ and $1 \leq i \neq j \leq 2$, $g_i(a_i) \leq g_j(a_j)$ implies there exists $a \in H$ with $g_i(a_i) \leq g_i f_i(a) = g_j f_j(a) \leq g_j(a_j)$.

Theorem 2.2. *Every V-formation of Heyting algebras has a superamalgam.*

This result was proved independently by Day [4] and Maksimova [7].

Definition 2.3. A *regular completion* of a Heyting algebra H is a pair $(\overline{H}, i)$ consisting of a complete Heyting algebra $\overline{H}$ and a Heyting algebra embedding $i\colon H \to \overline{H}$ such that i preserves all existing meets and joins from H.

Theorem 2.4. *Every Heyting algebra has a regular completion.*

Proof. It is well known that the MacNeille completion of any lattice preserves all existing meets and joins [1]. It is also well known that the MacNeille completion of a Heyting algebra is a Heyting algebra and that the associated embedding is a Heyting algebra embedding [1]. □

In order to define a MHA, we first introduce the notions of interior and closure operators on a Heyting algebra. An *interior operator* on a Heyting algebra H is a map $\Delta\colon H \to H$ such that (i) $\Delta 1 = 1$, (ii) $\Delta a \leq a$, (iii) $\Delta a = \Delta\Delta a$, and (iv) $\Delta(a \wedge b) = \Delta a \wedge \Delta b$. A *closure operator* on H is a map $\nabla\colon H \to H$ such that (i) $\nabla 0 = 0$, (ii) $a \leq \nabla a$, (iii) $\nabla a = \nabla\nabla a$, and (iv) $\nabla(a \vee b) = \nabla a \vee \nabla b$. We say an element $a \in H$ is *open* if $a = \Delta a$, and *closed* if $a = \nabla a$.

Definition 2.5. A *monadic Heyting algebra* (abbreviated: MHA) is a triple (H, Δ, ∇) where H is a Heyting algebra, Δ is an interior operator on H, ∇ is a closure operator on H, and the following conditions hold:

(i) $\Delta\nabla a = \nabla a$,
(ii) $\nabla\Delta a = \Delta a$,
(iii) $\nabla(\nabla a \wedge b) = \nabla a \wedge \nabla b$.

It is obvious from the definition that the class of all MHAs forms a variety. It is also worth mentioning that $\Delta(\Delta a \vee b) = \Delta a \vee \Delta b$, which is the dual to condition (iii), need not be satisfied in every MHA. The following well known result will be of use [9].

Proposition 2.6. *For a MHA (H, Δ, ∇), an element $a \in H$ is open iff it is closed. Further, the set $B = \{a \in H \mid a = \Delta a\} = \{a \in H \mid a = \nabla a\}$ is closed under the operations $\wedge, \vee, \rightarrow, 0, 1, \Delta, \nabla$.*

Proof. Using the identities $\Delta\nabla a = \nabla a$ and $\nabla\Delta a = \Delta a$ one easily shows a is open iff a is closed. For, if a is open, then $\nabla a = \nabla\Delta a = \Delta a = a$, hence a is closed, and the converse is similar. As B is the set of closed elements, it follows from general properties of closure operators that B is closed under $\wedge, \vee, 0, 1$. To show B is closed under $\rightarrow$ take $a, b \in B$. We must show $\nabla(a \rightarrow b) = a \rightarrow b$. Clearly $\nabla(a \rightarrow b) \geq a \rightarrow b$ as ∇ is a closure operator. Using the identity $\nabla(\nabla a \wedge b) = \nabla a \wedge \nabla b$ and the fact $a = \nabla a$, $b = \nabla b$ we have $a \wedge \nabla(a \rightarrow b) = \nabla a \wedge \nabla(a \rightarrow b) = \nabla(\nabla a \wedge (a \rightarrow b)) = \nabla(a \wedge (a \rightarrow b)) \leq \nabla b = b$. It follows that $\nabla(a \rightarrow b) \leq a \rightarrow b$, hence equality. Finally, it is obvious that Δ, ∇ restrict to the identity map on B. □

Although we won't need it, one can show more. Given an element $a \in H$, there is a largest element in B beneath a, and a least element in B above a. Conversely, given a Heyting algebra H, a subalgebra B of H with the above properties uniquely determines a MHA structure on H in which B is the set of open, hence closed, elements [9].

Definition 2.7. For a complete Heyting algebra H and a set X, define operations Δ, ∇ on the Heyting algebra H^X of all functions $f\colon X \rightarrow H$ by setting

$$(\Delta f)(x) = \bigwedge_{y \in X} f(y) \quad \text{and} \quad (\nabla f)(x) = \bigvee_{y \in X} f(y).$$

The resulting algebra (H^X, Δ, ∇) is known to be a MHA and we call this the *full functional MHA* determined by H, X.

Definition 2.8. We say that a MHA is *functional* if it is a subalgebra of a full functional MHA.

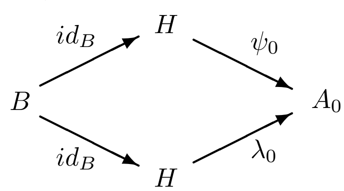
Remark 2.9. The above definition of functional MHAs is due to Monteiro and Varsavsky [9]. It is not a direct analog of the definition of functional MAs given by Halmos, but rather a simplification of Halmos' approach. Roughly, Halmos noted one could produce a MA by taking a subalgebra S of B^X even when B is not complete provided that for each $f \in S$ the range of f has a join in B and that the constant function evaluating to this join belongs to S. Using the well-known fact that the MacNeille completion of a Boolean algebra is Boolean [1], it follows that a MA is functional in Halmos' sense iff it is a subalgebra of a full functional MA. Similar results hold for MHAs as the MacNeille completion of a Heyting algebra is a Heyting algebra [1].

3. The main theorem

In this section we prove our main result that every MHA is isomorphic to a functional MHA. Throughout (H, Δ, ∇) is a MHA and $B = \{a \in H \mid a = \Delta a\} = \{a \in H \mid a = \nabla a\}$. By Proposition 2.6, B is a subalgebra of the Heyting algebra H.

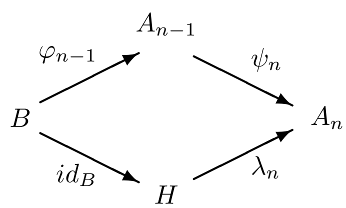
Definition 3.1. Define recursively for each $n \geq 0$ Heyting algebras A_n and Heyting algebra embeddings $\psi_n, \lambda_n, \varphi_n$ as follows.

For $n = 0$ let (A_0, ψ_0, λ_0) be a triple superamalgamating the V-formation (B, H, H, id_B, id_B) (see Theorem 2.2).



Then set $\varphi_0 = \psi_0|_B = \lambda_0|_B$.

For $n > 0$ let (A_n, ψ_n, λ_n) be a triple superamalgamating the V-formation $(B, A_{n-1}, H, \varphi_{n-1}, id_B)$.

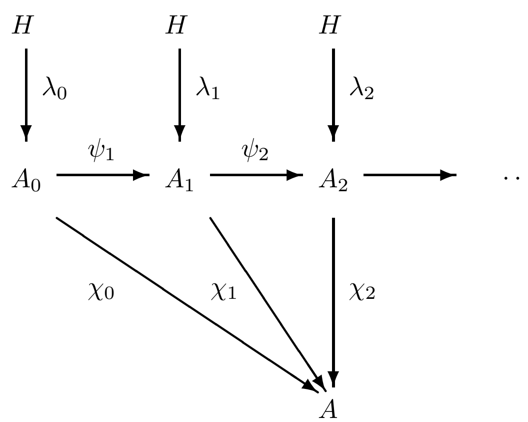


Then set $\varphi_n = \psi_n \circ \varphi_{n-1} = \lambda_n|_B$.

Definition 3.2. Consider the directed family generated by the diagram

$$A_0 \xrightarrow{\psi_1} A_1 \xrightarrow{\psi_2} A_2 \to \cdots$$

Let A, together with the maps $\chi_n \colon A_n \to A$, be the direct limit of this family.



Note, as each A_n is a Heyting algebra, A is a Heyting algebra, and as each ψ_n is a Heyting algebra embedding, each χ_n is a Heyting algebra embedding.

Lemma 3.3. *For any $a \in B$ and any $m, n \in \omega$, $\chi_m \circ \lambda_m(a) = \chi_n \circ \lambda_n(a)$.*

Proof. We show for every $n \in \omega$ that $\chi_n \circ \lambda_n(a) = \chi_{n+1} \circ \lambda_{n+1}(a)$. Note, by the definition of direct limit, $\chi_{n+1} \circ \psi_{n+1} = \chi_n$. Also, by Definition 3.1, $\varphi_n = \lambda_n|_B$ and $\psi_{n+1} \circ \varphi_n = \lambda_{n+1}|_B$. Therefore $\chi_n \circ \lambda_n(a) = \chi_n \circ \varphi_n(a) = \chi_{n+1} \circ \psi_{n+1} \circ \varphi_n(a) = \chi_{n+1} \circ \lambda_{n+1}(a)$. □

Lemma 3.4. *Suppose $x \in A_m$ and $a \in H$.*

(1) *If $\chi_m(x) \leq \chi_n \circ \lambda_n(a)$ for all $n \in \omega$, then $\chi_m(x) \leq \chi_n \circ \lambda_n(\Delta a)$ for all $n \in \omega$.*
(2) *If $\chi_n \circ \lambda_n(a) \leq \chi_m(x)$ for all $n \in \omega$, then $\chi_n \circ \lambda_n(\nabla a) \leq \chi_m(x)$ for all $n \in \omega$.*

Proof. (1) Under these assumptions $\chi_{m+1} \circ \psi_{m+1}(x) = \chi_m(x) \leq \chi_{m+1} \circ \lambda_{m+1}(a)$, and as χ_{m+1} is a Heyting algebra embedding, $\psi_{m+1}(x) \leq \lambda_{m+1}(a)$. By Definition 3.1 the triple $(A_{m+1}, \psi_{m+1}, \lambda_{m+1})$ superamalgamates the V-formation $(B, A_m, H, \varphi_m, id_B)$. Therefore, there exists $b \in B$ with $\psi_{m+1}(x) \leq \psi_{m+1} \circ \varphi_m(b) = \lambda_{m+1}(b) \leq \lambda_{m+1}(a)$. As λ_{m+1} is a Heyting algebra embedding, $b \leq a$. Hence, as $b \in B$, $b \leq \Delta a$. Also, as ψ_{m+1} is a Heyting algebra embedding and $\psi_{m+1}(x) \leq \psi_{m+1} \circ \varphi_m(b)$, we have $x \leq \varphi_m(b) \leq \varphi_m(\Delta a)$. Hence $\chi_m(x) \leq \chi_m \circ \varphi_m(\Delta a)$. As Δa belongs to B, the previous lemma yields $\chi_m(x) \leq \chi_n \circ \varphi_n(\Delta a)$ for all $n \in \omega$.

(2) is proved similarly. □

Lemma 3.5. *For any $a \in H$, $\{\chi_n \circ \lambda_n(a) \mid n \in \omega\}$ has both a greatest lower bound and a least upper bound in A. Further, for each $k \in \omega$,*

$$\bigwedge_{n\in\omega} \chi_n \circ \lambda_n(a) = \chi_k \circ \lambda_k(\Delta a) \text{ and } \bigvee_{n\in\omega} \chi_n \circ \lambda_n(a) = \chi_k \circ \lambda_k(\nabla a).$$

Proof. Let $k \in \omega$. For any $n \in \omega$, Lemma 3.3 provides $\chi_k \circ \lambda_k(\Delta a) = \chi_n \circ \lambda_n(\Delta a)$ since $\Delta a \in B$. But $\Delta a \leq a$, so $\chi_k \circ \lambda_k(\Delta a) \leq \chi_n \circ \lambda_n(a)$ for every $n \in \omega$. Thus $\chi_k \circ \lambda_k(\Delta a)$ is a lower bound of $\{\chi_n \circ \lambda_n(a) \mid n \in \omega\}$. Suppose $y \in A$ and y is a lower bound of $\{\chi_n \circ \lambda_n(a) \mid n \in \omega\}$. Then, by basic properties of direct limits, there is $m \in \omega$ and $x \in A_m$ with $y = \chi_m(x)$. By Lemma 3.4, $y = \chi_m(x) \leq \chi_n \circ \lambda_n(\Delta a)$ for all $n \in \omega$, and in particular, $y \leq \chi_k \circ \lambda_k(\Delta a)$. This proves $\bigwedge_{n\in\omega} \chi_n \circ \lambda_n(a) = \chi_k \circ \lambda_k(\Delta a)$. The other statements follow similarly. □

Theorem 3.6. (Main Theorem) *Every MHA is isomorphic to a functional MHA.*

Proof. Given a MHA (H, Δ, ∇) let $A_n, A, \chi_n, \lambda_n$ be as above. Let $\overline{A}$, together with the map $i\colon A \to \overline{A}$, be a regular completion of A (see Theorem 2.4). Note $\overline{A}$ is a Heyting algebra and i is a Heyting algebra embedding. Define the map $f\colon H \to (\overline{A})^\omega$ by putting

$$f(a)(n) = i \circ \chi_n \circ \lambda_n(a).$$

We prove f is a MHA embedding of the MHA (H, Δ, ∇) into the full functional MHA $((\overline{A})^\omega, \Delta, \nabla)$. For each $n \in \omega$, $i \circ \chi_n \circ \lambda_n$ is a composite of Heyting algebra embeddings, hence is a Heyting algebra embedding. Since $i \circ \chi_n \circ \lambda_n$ is a Heyting algebra homomorphism for each $n \in \omega$, it follows that f is a Heyting algebra homomorphism. As all, hence at least one, of the $i \circ \chi_n \circ \lambda_n$ is a Heyting algebra embedding, f is a Heyting algebra embedding.

It remains to show that f preserves the monadic operations Δ, ∇. Let $a \in H$ and $k \in \omega$. Then

$$\begin{aligned}(\Delta f(a))(k) &= \textstyle\bigwedge_{n\in\omega} f(a)(n)\\ &= \textstyle\bigwedge_{n\in\omega} i \circ \chi_n \circ \lambda_n(a)\\ &= i(\textstyle\bigwedge_{n\in\omega} \chi_n \circ \lambda_n(a))\\ &= i(\chi_k \circ \lambda_k(\Delta a))\\ &= f(\Delta a)(k).\end{aligned}$$

In the third line we have used the fact that $i\colon A \to \overline{A}$ preserves all existing meets in A, and the fact, provided by Lemma 3.5, that $\{\chi_n \circ \lambda_n(a) \mid n \in \omega\}$ has a meet in A. In the fourth line we use the fact, also provided by Lemma 3.5, that $\bigwedge_{n\in\omega} \chi_n \circ \lambda_n(a) = \chi_k \circ \lambda_k(\Delta a)$.

This shows $\Delta f(a) = f(\Delta a)$. One can similarly show $\nabla f(a) = f(\nabla a)$. □

4. Concluding remarks

In [5,6] Halmos proved every MA is isomorphic to a subalgebra of a full functional MA (B^X, ∇). However, the set X constructed in his proof was created from the set of endomorphisms of a Boolean algebra, and could be uncountable. Later proofs of this result by LeBlanc (described in [6]) and Varsavsky [13] also yielded uncountable sets. We can slightly extend this result as follows.

Theorem 4.1. *Every MA is isomorphic to a subalgebra of a full functional MA* (B^ω, ∇).

Proof. The proof of the main theorem relies on two facts; that every V-formation in the variety of Heyting algebras has a superamalgam, and that every Heyting algebra has a regular completion. Since it is well known that the variety of Boolean algebras has superamalgams [7] and regular completions [1], we can apply our method also to the variety of MAs. □

Our proof that every MHA is isomorphic to a subalgebra of a full functional MHA $(H^\omega, \Delta, \nabla)$ produces an infinite Heyting algebra H even when the original Heyting algebra is finite. The following result, first stated in [9], shows this is unavoidable. Let $(\mathbf{3}, \Delta, \nabla)$ be the MHA defined by setting the Heyting algebra $\mathbf{3}$ to be the three-element chain $\{0, a, 1\}$ and setting $\Delta x = 0$ for $x \neq 1$, $\nabla x = 1$ for $x \neq 0$, $\Delta 1 = 1$, $\nabla 0 = 0$.

Proposition 4.2. *If* (H^X, Δ, ∇) *is a full functional MHA and* $(\mathbf{3}, \Delta, \nabla)$ *is isomorphic to a subalgebra of* (H^X, Δ, ∇)*, then both* H *and* X *are infinite.*

Proof. Consider the identity $\neg\neg\Delta x = \Delta\neg\neg x$ where $\neg$ denotes pseudocomplement. This identity does not hold in $(\mathbf{3}, \Delta, \nabla)$ as $\neg\neg\Delta a = 0$ and $\Delta\neg\neg a = 1$. We show this identity holds in any full functional MHA (H^X, Δ, ∇) in which either H or X is finite. Suppose $f \in H^X$. As either H or X is finite, the range of f is a finite subset $\{a_1, \ldots, a_n\}$ of H. Then, the definition of Δ yields

$$(\neg\neg\Delta f)(x) = \neg\neg \bigwedge_{i=1}^{n} a_i = \bigwedge_{i=1}^{n} \neg\neg a_i = (\Delta\neg\neg f)(x),$$

as $\neg\neg \bigwedge_{i=1}^n a_i = \bigwedge_{i=1}^n \neg\neg a_i$ holds in every Heyting algebra.[3] □

We remark that our results will not apply to all subvarieties of MHAs. For convenience, we call a variety $\mathbf{V}$ of MHAs *functional* if every algebra in $\mathbf{V}$ is isomorphic to a subalgebra of a full functional MHA belonging to $\mathbf{V}$.

[3]It is the failure of $\neg\neg \bigwedge_I a_i = \bigwedge_I \neg\neg a_i$ for I infinite that allows one to falsify the identity $\neg\neg\Delta x = \Delta\neg\neg x$ in a functional MHA.

Theorem 4.3. *There are continuum many varieties of MHAs that are not functional.*

Proof. Let $\mathbf{V}$ be a variety of MHAs and $\alpha = \beta$ an identity in the language of Heyting algebras such that (i) for each $(H, \Delta, \nabla) \in \mathbf{V}$, $\alpha = \beta$ is satisfied in $B = \{x \in H \mid x = \Delta x\}$, and (ii) there is $(M, \Delta, \nabla) \in \mathbf{V}$ with $\alpha = \beta$ not satisfied in M. We claim $\mathbf{V}$ is not functional. Suppose (H^X, Δ, ∇) is a full functional MHA in $\mathbf{V}$. Then $B = \{f \in H^X \mid f = \Delta f\}$ is exactly the Heyting algebra of all constant functions from X to H, hence B is isomorphic to H. Thus H, and hence H^X, satisfies $\alpha = \beta$. Therefore (M, Δ, ∇) cannot be isomorphic to a subalgebra of (H^X, Δ, ∇). Hence $\mathbf{V}$ is not functional.

There are continuum many varieties of MHAs satisfying the identity $\Delta x = \neg\neg\Delta x$ and not contained in the variety of MAs [2]. Any such $\mathbf{V}$ satisfies conditions (i) and (ii) above with respect to the identity $x = \neg\neg x$. This yields the result. $\square$

Remark 4.4. In contrast to Theorem 4.3 there are partial positive results on functional varieties of MHAs. It is known from Maksimova [7] that there are exactly eight varieties of Heyting algebras with the property that every V-formation in the variety has superamalgam in the variety. These varieties are:

- $\mathbf{V}_1 =$ the trivial variety;
- $\mathbf{V}_2 =$ the variety of Boolean algebras;
- $\mathbf{V}_3 =$ the variety generated by the three-element chain $\mathbf{3}$;
- $\mathbf{V}_4 =$ the variety generated by all chains;
- $\mathbf{V}_5 =$ the variety generated by the algebra which is obtained by adjoining a new top to the four element Boolean algebra;
- $\mathbf{V}_6 =$ the variety generated by the algebras which are obtained by adjoining a new top to Boolean algebras;
- $\mathbf{V}_7 =$ the variety satisfying the identity $\neg x \vee \neg\neg x = 1$;
- $\mathbf{V}_8 =$ the variety of all Heyting algebras.

Most of these varieties are of interest in the study of intermediate logics. For example, $\mathbf{V}_4$ corresponds to the well-known logic of Gödel and Dummett, and $\mathbf{V}_7$ corresponds to the logic of the weak excluded middle. It may also be of interest to study MHAs (H, Δ, ∇) where H is restricted to lie in one of these eight varieties.

Suppose (H, Δ, ∇) is a MHA and $H \in \mathbf{V}_i$, $i = 1, \ldots, 8$. The algebras A_n, A constructed in Section 3 can also be chosen to lie in $\mathbf{V}_i$. In the proof of our main theorem, we required not only that the variety of Heyting algebras had superamalgams, but also that it admitted regular completions. This latter fact may not be available for the varieties $\mathbf{V}_i$ (we do not know the status of this question[4]).

[4] The authors have recently shown the only varieties of HAs that are closed under MacNeille completions are the trivial variety, the variety of Boolean algebras, and the variety of all HAs.

However, if we modify the proof of the main theorem and consider the map $g\colon H \to A^\omega$ defined by $g(a)(n) = \chi_n \circ \lambda_n(a)$, we can show that the image of g is a functional MHA in the sense of Halmos (see Remark 2.9), and its Heyting reduct again belongs to $\mathbf{V}_i$. In other words the variety of all monadic $\mathbf{V}_i$ algebras is functional in the sense of Halmos.

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